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ON ENDOMORPHISM RINGS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let R be a local complete ring. For an R-module M the canonical ring map $R \to \mathrm{End}_R(M)$ is in general neither injective nor surjective; we show that it is bijective for every local cohomology module $M := H_I^h(R)$ if $H_I^l(R) = 0$ for every $l \neq h$ (= height(I)) (I an ideal of R); furthermore the same holds for the Matlis dual of such a module. As an application we prove new criteria for an ideal to be a set-theoretic complete intersection.

1. Introduction

For an ideal I of a local ring (R, m) we denote the n-th local cohomology functor with support in I by H_I^n and the (contravariant) Matlis dual functor by D; i.e., $D(M) = \mathrm{Hom}_R(M, E)$ for any R-module M, where $E := E_R(R/M)$ is a fixed R-injective hull of the residue field R/m.

Let R be a (always commutative, unitary) ring and M an R-module. Consider the canonical map $\mu_M : R \to \mathrm{End}_R(M)$ that maps $r \in R$ to multiplication by r on M; it is a homomorphism of (associative) R-algebras. In general, μ_M is neither injective nor surjective. In section 2 we will show that, if R is Noetherian local complete and I an ideal of R such that $H_I^l(R) = 0$ for every $l \neq h$ (= height(I)), then $\mu_{H_I^h(R)}$ is bijective. In particular, the endomorphism ring of the R-module $H_I^h(R)$ is commutative and $\mathrm{Ann}_R(H_I^h(R)) = 0$.

The proof of this result uses a generalization of Theorem 3.2 from [11], which says that, for a special class of Noetherian local complete rings R, it is true that $D(H_I^h(D(H_I^h(R))))$ is either zero or isomorphic to R if $H_I^l(R) = 0$ for every $l > h = \mathrm{height}(I)$; the generalization is due to Khashyarmanesh ([14, Corollary 2.6]) and says that $D(H_I^h(D(H_I^h(R)))) \cong R$ for every Noetherian local complete ring and every ideal $I \subseteq R$ such that $H_I^l(R) = 0$ for $l \neq h = \mathrm{height}(I)$.

We also show in section 2 that $\mu_{D(H_I^h(R))}$ is an isomorphism if $H_I^l(R) = 0$ for every $l \neq h$.

Recently there was some work on Matlis duals of local cohomology modules (e.g. [6, 7, 10, 11, 12]). In [10, Corollary 1.1.4] the following was proved: If, for some $h \in \mathbb{N}$, $H_I^l(R) = 0$ for all $l > h$ and $\underline{x} = x_1, \ldots, x_h \in I$ is an R-regular sequence,

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then one has the following equivalence:

$$\sqrt{I} = \sqrt{\underline{x}R} \iff \underline{x} \text{ is a } D(H_I^h(R)) \text{ regular sequence.}$$

In section 3 we extend this equivalence:

Theorem. *Let (R, m) be a Noetherian local complete ring and I an ideal of R such that $H_I^l(R) = 0$ for every $l > h := \operatorname{height}(I) \geq 1$; let $\underline{x} = x_1, \ldots, x_h \in I$ be an R-regular sequence. Set $D := D(H_I^h(R))$. The following statements are equivalent:*

(i) $\sqrt{I} = \sqrt{\underline{x}R}$; in particular, I is a set-theoretic complete intersection.

(ii) $\underline{x}$ is a D-regular sequence.

(iii) The canonical map $D/\underline{x}D \to H_{\underline{x}R}^h(D)$ (coming from $H_{\underline{x}R}^h(D) = \varinjlim_{l\in\mathbb{N}} D/\underline{x}^l D$) is injective.

(iv) The canonical map $\{r \in R | \forall_{l\in\mathbb{N}} \exists_{s\in\mathbb{N}}\, r \cdot I^s \subseteq \underline{x}^l R\} \to \Gamma_I(R/\underline{x}R)$ is surjective.

The equivalence of (ii) and (iii) is inspired by a result of Marley and Rogers ([18, Prop. 2.3], which is a version of (ii) $\iff$ (iii) for (arbitrary) finitely generated modules D.

2. Endomorphism rings

Definition 2.1. (i) Let R be a ring and M an R-module. The map

$$R \to \operatorname{End}_R(M), \quad r \mapsto \text{ multiplication by } r \text{ on } M$$

is an R-algebra homomorphism and will be denoted by μ_M.

(ii) Let R be a local ring and M an R-module. M has a canonical embedding

$$M \to D(D(M)) = D^2(M), \quad m \mapsto (\varphi \mapsto \varphi(m))$$

into its bidual; this map will be denoted by ι_M. We will consider M as a submodule of $D^2(M)$ via ι_M.

(iii) Let (R, m) be a Noetherian local ring and $\underline{x} = x_1, \ldots, x_h$ a sequence of elements of R. For every R-module M there is a canonical map

$$M/\underline{x}M \overset{\iota_{M,\underline{x}}}{\to} H_{\underline{x}R}^h(M)$$

(coming from the description $H_{\underline{x}R}^h(M) = \varinjlim_{l\in\mathbb{N}} M/(x_1^l, \ldots, x_h^l)M$, where the transition maps are induced by multiplication by $x_1 \cdot \cdots \cdot x_h$).

Theorem 2.2. *Let (R, m) be a Noetherian local complete ring and I an ideal of R such that $H_I^l(R) = 0$ for every $l \neq h$ (h is then necessarily the height of I). Set $H := H_I^h(R)$.*

(i) $\operatorname{Hom}_R(H, \iota_H) : \operatorname{End}_R(H) \to \operatorname{Hom}_R(H, D^2(H))$ is an isomorphism.

(ii) There is a canonical isomorphism

$$\gamma_H : \operatorname{Hom}_R(H, D^2(H)) \to D(H_I^h(D(H))).$$

(iii) $\mu_H : R \to \operatorname{End}_R(H)$ is an isomorphism of R-algebras. Consequently there is a canonical isomorphism

$$\gamma_H \circ \operatorname{Hom}_R(H, \iota_H) \circ \mu_H : R \to D(H_I^h(D(H))).$$

Proof. (i) It is clear that $\operatorname{Hom}_R(H, \iota_H)$ is injective. To show surjectivity, let $\varphi \in \operatorname{Hom}_R(H, D^2(H))$ be arbitrary; let $x \in H$ be arbitrary and $n \in \mathbb{N}$ such that $I^n \cdot x = 0$. This implies $I^n \cdot \varphi(x) = 0$; i.e.

$$\varphi(x) \in (0 :_{D^2(H)} I^n) = D^2((0 :_H I^n)) = (0 :_H I^n) \subseteq H$$

(the first equality follows from exactness of D; for the second equality we remark that $(0 :_H I^n)$ is finitely generated, as the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(R/I^n, H_I^q(R)) \Rightarrow \mathrm{Ext}_R^{p+q}(R/I^n, R)$$

shows $(0 :_H I^n) = \mathrm{Ext}_R^h(R/I^n, R))$. This means that the image of φ is contained in $H \subseteq D^2(H)$, which was precisely what we had to show.

(ii) Hom-Tensor adjointness shows

$$\mathrm{Hom}_R(H, D^2(H)) = D(H \otimes_R D(H)).$$

On the other hand, our hypotheses imply $H_I^l = 0$ for every $l > h$; in particular, H_I^h is right exact, we get

$$H \otimes_R D(H) = H_I^h(R) \otimes_R D(H) = H_I^h(D(H))$$

and statement (ii) is now clear.

(iii) [14, Corollary 2.6] implies that there exists an isomorphism of R-modules

$$D(H_I^h(D(H))) \cong R.$$

Therefore, (i) and (ii) show that the R-module $\mathrm{End}_R(H)$ is free of rank one. Fix any isomorphism $R \cong \mathrm{End}_R(H)$ and let $\psi \in \mathrm{End}_R(H)$ be the element corresponding to $1 \in R$. In particular there exists a (unique) $x \in R$ such that $\mathrm{id}_H = x \circ \psi$, where x is multiplication by x on H. This implies $\varphi = x \circ \psi \circ \varphi$ for every $\varphi \in \mathrm{End}_R(H)$; in particular, multiplication by x is surjective on $\mathrm{End}_R(H) \cong R$, x is a unit in R and μ_H is bijective. □

Remark 2.3. The hypothesis of the previous theorem is fulfilled if I is perfect and R is regular and has positive characteristic. This was shown in [19, Prop. 4.1].

Corollary 2.4. *In the situation of Theorem* 2.2 *the endomorphism ring of* $H_I^h(R)$ *is canonically isomorphic to* R*; in particular, it is commutative and* $\mathrm{Ann}_R(H_I^h(R)) = 0$ *holds.*

Let (R, m) be a Noetherian local ring and M an R-module. Consider the sequence of R-modules

$$R \to \mathrm{End}_R(D(M)) \to \mathrm{End}_R(D^2(M)) \to \mathrm{Hom}_R(M, D^2(M)),$$

where the first map is $\mu_{D(M)}$, the second is given by $\alpha \mapsto D(\alpha)$ and the third is restriction to $M \subseteq D^2(M)$. The composition of the second and third is always injective:

Lemma 2.5. *Let* (R, m) *be a Noetherian local ring and* M *an* R*-module. The* R*-linear map*

$$\mathrm{End}_R(D(M)) \to \mathrm{Hom}_R(M, D^2(M)), \quad \varphi \mapsto (m \mapsto (D(\varphi))(\iota_M(m)))$$

is injective.

Proof. This is straightforward: Let φ be in the kernel of the above map; this means that for all $m \in M$ and for all $\psi \in D(M)$ one has $\varphi(\psi)(m) = 0$, i.e. $\varphi = 0$. □

We apply this injectivity in the case where the local ring (R, m) is complete, $M := H := H_I^h(R)$ and I is an ideal of R such that $H_I^l(R) = 0$ for every $l \neq h$; we get R-linear maps

$$\begin{array}{rcl} R \to \mathrm{End}_R(D(H)) \to \mathrm{End}_R(D^2(H)) & \to & \mathrm{Hom}_R(H, D^2(H)) \\ & \stackrel{2.2(i)}{=} & \mathrm{End}_R(H) \\ & \stackrel{2.2(iii)}{=} & R. \end{array}$$

The composition of all these maps is clearly id_R. Thus, the injectivity statement from Lemma 2.5 shows:

Theorem 2.6. *Let (R, m) be a Noetherian local complete ring and I an ideal of R such that $H_I^l(R) = 0$ for every $l \neq h$. Then the canonical map*

$$\mu_{D(H_I^h(R))} : R \to \mathrm{End}_R(D(H_I^h(R)))$$

is an isomorphism of R-algebras.

3. Complete intersections and local cohomology

We need a couple of lemmata and remarks before we can prove Theorem 3.7, which is the main result of this section:

Remark 3.1. Let (R, m) be a Noetherian local ring, I an ideal of R and M an R-module such that

$$\mathrm{Supp}_R(M) \subseteq V(I)$$

(where $V(I) = \{p \in \mathrm{Spec}(R) | p \supseteq I\}$). Let $\hat{}$ denote I-adic completion. Then the natural map

$$D(M) \to \widehat{D(M)}$$

is an isomorphism; in particular, $\bigcap_{l \in \mathbb{N}} I^l \cdot D(M) = 0$.

Proof. We have to show that the canonical map

$$D(M) \to \varprojlim_{l \in \mathbb{N}} (D(M)/I^l D(M))$$

is bijective, but one has

$$\begin{array}{rcl} D(M) & = & D(\Gamma_I(M)) \\ & = & D(\varinjlim_{l \in \mathbb{N}} \mathrm{Hom}_R(R/I^l, M)) \\ & = & \varprojlim_{l \in \mathbb{N}} D(\mathrm{Hom}_R(R/I^l, M)) \\ & = & \varprojlim_{l \in \mathbb{N}} D(M)/I^l D(M), \end{array}$$

and it is easy to see that this is the canonical map $D(M) \to \widehat{D(M)}$. □

Let (R, m) be a Noetherian local ring and $\underline{x} = x_1, \ldots, x_h$ a sequence of elements of R. Marley and Rogers have shown ([18, Proposition 2.3]) that, for finitely generated M, $\iota_{M,\underline{x}}$ is injective iff $\underline{x}$ is an M-regular sequence. In this context, note that the proof of the following lemma is strongly based on their proof; our additional ingredient is Remark 3.1.

Lemma 3.2. *Let (R,m) be a Noetherian local ring, I an ideal of R, $n,h \in \mathbb{N}$, $\underline{x} = x_1,\dots,x_h \in I$ an arbitrary sequence and N an R-module. Set $H := H^n_I(N)$ and $D := D(H)$. The following two statements are equivalent:*

(i) For every $i = 1,\dots,h$, multiplication by x_i on $D/(x_1,\dots,x_{i-1})D$ is injective (i.e., $\underline{x}$ is a D-quasiregular sequence).

(ii) $D/\underline{x}D \overset{\iota_{D,\underline{x}}}{\to} H^h_{\underline{x}R}(D)$ is injective.

Proof. (i) $\Rightarrow$ (ii): The finite case is well known; [20, Prop. 5.2.1] is a reference for the general case (note that (ii) holds trivially if $D/\underline{x}D = 0$).

(ii) $\Rightarrow$ (i): By induction on h: $h = 1$: Set $x = x_1$ and let $\alpha \in D$ be such that $x\alpha = 0$. We have to show $\alpha = 0$. α represents an element of $\ker(\iota_{D,\underline{x}})$; therefore, by assumption, $\alpha \in xD$. Choose $\alpha_1 \in D$ such that $\alpha = x\alpha_1$. We conclude $x^2\alpha_1 = 0$. Again, α_1 represents an element of $\ker(\iota_{D,\underline{x}})$ and so there exists $\alpha_2 \in D$ such that $\alpha_1 = x\alpha_2$. Continuing in this way, we get

$$\alpha \in \bigcap_{k\in\mathbb{N}} x^k D$$

and then $\alpha = 0$, by Remark 3.1. $h > 1$: First of all we prove injectivity of

$$D/(x_1,\dots,x_{h-1})D \overset{\iota_{D,x_1,\dots,x_{h-1}}}{\to} H^{h-1}_{(x_1,\dots,x_{h-1})R}(D);$$

to do so, let $\alpha \in \ker(\iota_{D,x_1,\dots,x_{h-1}})$ be arbitrary. We show $\alpha \in (x_1,\dots,x_{h-1})D + x_h^k D$ for every $k \in \mathbb{N}$ by induction on k: $k = 0$ is trivial, we assume $k > 0$ and we write $\alpha = \omega + x_h^k\beta$ for some $\omega \in (x_1,\dots,x_{h-1})D, \beta \in D$. By our choice of α there exists $t \in \mathbb{N}$ such that

$$(x_1 \cdot \dots \cdot x_{h-1})^t x_h^k \beta \in (x_1^{t+1},\dots,x_{h-1}^{t+1})D$$

and hence

$$(x_1 \cdot \dots \cdot x_h)^{t+k} \beta \in (x_1^{t+k+1},\dots,x_{h-1}^{t+k+1})D.$$

But $\iota_{D,\underline{x}}$ is injective; we conclude $\beta \in (x_1,\dots,x_h)D$ and our induction on k is finished:

$$\alpha \in \bigcap_{k\in\mathbb{N}} ((x_1,\dots,x_{h-1})D + x_h^k D).$$

The R-module

$$D/(x_1,\dots,x_{h-1})D = D(\mathrm{Hom}_R(R/(x_1,\dots,x_{h-1})R, H))$$

is x_hR-adically separated by Remark 3.1. This means

$$\bigcap_{k\in\mathbb{N}} ((x_1,\dots,x_{h-1})D + x_h^k D) = (x_1,\dots,x_{h-1})D$$

and the stated injectivity of $\iota_{D,x_1,\dots,x_{h-1}}$ follows. The induction hypothesis shows that $x_1,\dots,x_{h-1}$ is D-quasiregular; we have to show that multiplication by x_h on $D/(x_1,\dots,x_{h-1})D$ is injective. Let $\alpha \in D$ be such that $x_h\alpha \in (x_1,\dots,x_{h-1})D$. We state

$$\forall_{k\in\mathbb{N}} \alpha \in (x_1,\dots,x_{h-1})D + x_h^k D$$

and prove this statement by induction on k. We may assume $k > 0$ and write $\alpha = \omega + x_h^k\beta$ for some $\omega \in (x_1,\dots,x_{h-1})D, \beta \in D$. From $x_h\alpha = x_h\omega + x_h^{k+1}\beta$ we conclude $x_h^{k+1}\beta \in (x_1,\dots,x_{h-1})D$. Therefore,

$$(x_1^{k+1} \cdot \dots \cdot x_h^{k+1})\beta \in (x_1^{k+2},\dots,x_{h-1}^{k+2})D.$$

But $\iota_{D,\underline{x}}$ is injective and so $\beta \in (x_1, \ldots, x_h)D$, and induction on k is finished:

$$\alpha \in \bigcap_{k\in\mathbb{N}}((x_1, \ldots, x_{h-1})D + x_h D) \overset{3.1}{=} (x_1, \ldots, x_{h-1})D$$

(note that the last equality has been explained above in a similar situation). $\square$

Let (R, m) be a Noetherian local complete ring, I an ideal of R, $h \in \mathbb{N}$; assume that $\underline{x} = x_1, \ldots, x_h \in I$ is an R-regular sequence. It follows from the Grothendieck spectral sequence belonging to the composed functors $\Gamma_I \circ \Gamma_{\underline{x}R}$ that

$$H^h_I(R) = \Gamma_I(H^h_{\underline{x}R}(R)) \subseteq H^h_{\underline{x}R}(R).$$

By applying the functors D, $H^h_{\underline{x}R}$ and then D again, we get a monomorphism (because D is exact and $H^h_{\underline{x}R}$ is right exact)

$$D(H^h_{\underline{x}R}(D(H^h_I(R)))) \hookrightarrow D(H^h_{\underline{x}R}(D(H^h_{\underline{x}R}(R)))).$$

Because of [14, Corollary 2.6], there is an isomorphism

$$D(H^h_{\underline{x}R}(D(H^h_{\underline{x}R}(R)))) \cong R.$$

Clearly, this isomorphism is unique up to a unit of R and so we may consider $D(H^h_{\underline{x}R}(D(H^h_I(R))))$ as an ideal of R (alternatively we use Theorem 2.2 and have a canonical isomorphism $D(H^h_{\underline{x}R}(D(H^h_{\underline{x}R}(R)))) = R$; the resulting ideal $J_{\underline{x},I}$ is the same in both cases).

Definition 3.3. In the above situation, set

$$J_{\underline{x},I} := D(H^h_{\underline{x}R}(D(H^h_I(R))))$$

and consider $J_{\underline{x},I}$ as an ideal of R.

Remark 3.4. Though the definition of $J_{\underline{x},I}$ is quite abstract, it also has the following concrete description: Because of the right exactness of $H^h_{\underline{x}R}$,

$$D(H^h_{\underline{x}R}(D(H^h_I(R)))) = D(H^h_{\underline{x}R}(R) \otimes_R D(H^h_I(R)))$$

and by Hom-Tensor adjointness, the latter module is

$$\mathrm{Hom}_R(H^h_{\underline{x}R}(R), D^2(H^h_I(R))).$$

Now the arguments from the proof of Theorem 2.2 (i) show

$$\mathrm{Hom}_R(H^h_{\underline{x}R}(R), D^2(H^h_I(R))) = \mathrm{Hom}_R(H^h_{\underline{x}R}(R), H^h_I(R)).$$

But $H^h_I(R) = \Gamma_I(H^h_{\underline{x}R}(R))$ and we get

$$J_{\underline{x},I} = \{\varphi \in \mathrm{End}_R(H^h_{\underline{x}R}(R)) | \operatorname{im}(\varphi) \subseteq \Gamma_I(H^h_{\underline{x}R}(R))\}.$$

We have $\mathrm{End}_R(H^h_{\underline{x}R}(R)) \overset{\text{can}}{=} R$ and thus

$$J_{\underline{x},I} = \{r \in R \mid r \cdot H^h_{\underline{x}R}(R) \subseteq \Gamma_I(H^h_{\underline{x}R}(R)).$$

Using the description $H^h_{\underline{x}R}(R) = \bigcup_{l\in\mathbb{N}} R/\underline{x}^l R$ (where $\underline{x}^l = x_1^l, \ldots, x_h^l$), we conclude

$$J_{\underline{x},I} = \{r \in R \mid \forall_{l\in\mathbb{N}} \exists_{s\in\mathbb{N}}\ r \cdot I^s \subseteq \underline{x}^l R\} = \bigcap_{l\in\mathbb{N}}(\underline{x}^l R : \langle I \rangle).$$

Therefore, if we restrict the canonical map $R \to R/\underline{x}R$ to $J_{\underline{x},I}$, we get a canonical map from $J_{\underline{x},I}$ to $\Gamma_I(R/\underline{x}R)$:

Definition 3.5. In the above situation, the canonical map

$$J_{\underline{x},I} \to \Gamma_I(R/\underline{x}R)$$

is denoted by $j_{\underline{x},I}$.

Remark 3.6. Let (R,m) be a Noetherian local complete ring. Let I be an ideal of R, $h \in \mathbb{N}$ and

$$\underline{x} = x_1, \dots, x_h \subseteq I$$

an R-regular sequence. Then

$$I \subseteq \sqrt{\underline{x}R + \mathrm{Ann}_R(J_{\underline{x},I})}.$$

In particular, if R is a domain and $\sqrt{\underline{x}R} \subsetneq \sqrt{I}$, then $J_{\underline{x},R} = 0$.

Proof. We use the description of $J_{\underline{x},R}$ from Remark 3.4. For the first statement we have to show

$$V(\mathrm{Ann}_R(J_{\underline{x},I})) \cap V(\underline{x}R) \subseteq V(I);$$

i.e., for every $r \in J_{\underline{x},R}$ we have to show

$$V(\mathrm{Ann}_R(r)) \cap V(\underline{x}R) \subseteq V(I).$$

Let $r \in J_{\underline{x},I}$ be arbitrary; by Remark 3.4, $J_{\underline{x},I} = \{r \in R \mid \forall_{l\in\mathbb{N}}\exists_{s\in\mathbb{N}}\ r \cdot I^s \subseteq \underline{x}^l R\}$. For every $p \in V(\underline{x}R) \setminus V(I)$ we get

$$r \cdot R_p \subseteq \bigcap_{l\in\mathbb{N}} \underline{x}^l R_p \subseteq \bigcap_{l\in\mathbb{N}} p^l R_p = 0;$$

i.e., $\mathrm{Ann}_R(r) \not\subseteq p$ and the first statement is proven. The second statement follows immediately from the first. □

Theorem 3.7. *Let (R,m) be a Noetherian local complete ring and I an ideal of R such that $H_I^l(R) = 0$ for every $l > h := \mathrm{height}(I) \geq 1$; let $\underline{x} = x_1, \dots, x_h \in I$ be an R-regular sequence (clearly, this implies $H_I^l(R) = 0$ for every $l \neq h$) . Set $D := D(H_I^h(R))$. The following statements are equivalent:*

(i) $\sqrt{I} = \sqrt{\underline{x}R}$; in particular, I is a set-theoretic complete intersection.
(ii) $\underline{x}$ is a D-regular sequence.
(iii) $D/\underline{x}D \stackrel{\iota_{D,\underline{x}}}{\to} H^h_{\underline{x}R}(D)$ is injective.
(iv) $j_{\underline{x},I}$ is surjective.
(v) $J_{\underline{x},I} = R$.

Proof. (i) $\iff$ (ii) was shown (for more general R) in [10, Cor. 1.1.4].

(ii) $\iff$ (iii) is a special case of Lemma 3.2 (note that $D/\underline{x}D = D(\mathrm{Hom}_R(R/\underline{x}R, H_I^h(R))) \neq 0$).

(iii) $\iff$ (iv): By definition, $D(H^h_{\underline{x}R}(D)) = J_{\underline{x},I}$. We have

$$D(D/\underline{x}D) = D(D(\mathrm{Hom}_R(R/\underline{x}R, H_I^h(R)))).$$

But $H^h_I(R) = \Gamma_I(H^h_{\underline{x}R}(R))$ and, therefore,

$$\begin{aligned} \mathrm{Hom}_R(R/\underline{x}R, H^h_I(R)) &= \Gamma_I(\mathrm{Hom}_R(R/\underline{x}, H^h_{\underline{x}R}(R))) \\ &= \Gamma_I(\mathrm{Ext}^h_R(R/\underline{x}R, R)) \\ &= \Gamma_I(R/\underline{x}R) \end{aligned}$$

(for the second and the third equality use the fact that $\underline{x}$ is an R-regular sequence). The latter module is finitely generated. We get

$$D(D(\mathrm{Hom}_R(R/\underline{x}R, H^h_I(R)))) = \Gamma_I(R/\underline{x}R).$$

Thus $D(\iota_{D,\underline{x}})$ is a map $J_{\underline{x},I} \to \Gamma_I(R/\underline{x}R)$; it is straightforward to see that it is in fact $j_{\underline{x},I}$ (to do so one should start with the description $H^h_{\underline{x}R}(D) = \varinjlim_{l\in\mathbb{N}}(D/\underline{x}^l D)$. $\square$

Question 3.8. In the situation of Definition 3.3, when exactly is $J_{\underline{x},I} = 0$?

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