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ON THE COFINITENESS OF LOCAL COHOMOLOGY MODULES

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Abstract. In this note we show that if I is an ideal of a Noetherian ring R and M is a finitely generated R-module, then for any minimax submodule N of $H_I^t(M)$ the R-module $\mathrm{Hom}_R(R/I, H_I^t(M)/N)$ is finitely generated, whenever the modules $H_I^0(M)$, $H_I^1(M)$, ..., $H_I^{t-1}(M)$ are minimax. As a consequence, it follows that the associated primes of $H_I^t(M)/N$ are finite. This generalizes the main result of Brodmann and Lashgari (2000).

1. Introduction

It is well known that for a Noetherian ring R, an ideal I of R, and a finitely generated R-module M, the local cohomology modules $H_I^i(M)$ are not always finitely generated. On the other hand if $(R, \mathfrak{m})$ is a local ring with residue field k, then the local cohomology modules $H_{\mathfrak{m}}^i(M)$ are Artinian, so $\mathrm{Hom}_R(k, H_{\mathfrak{m}}^i(M))$ is finitely generated. Taking this fact, Grothendieck [6] conjectured the following:

If R is a Noetherian ring, then for any ideal I of R and any finitely generated R-module M, the module $\mathrm{Hom}_R(R/I, H_I^i(M))$ is finitely generated.

Here, $H_I^j(M)$ denotes the j^{th} local cohomology module of M with support in I. This conjecture is false in general. In fact, Hartshorne [7] gave the following counterexample:

Let k be a field and let $R = k[x, y, z, u]/(xy - zu)$. Set $I = (x, u)$. Then

$$\mathrm{Hom}_R(R/I, H_I^2(R))$$

is not finitely generated.

On the other hand, an important problem in commutative algebra is determining when the set of associated primes of the i^{th} local cohomology modules $H_I^i(M)$ of M with support in I is finite (see [9, Problem 4]). A. Singh [18] and M. Katzman [11] have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see, [1], [2], [8], [10], [12], [14]. In particular, Brodmann and Lashgari [1, Theorem 2.2] showed that, if for a finitely generated R-module M and an integer t, the local cohomology modules $H_I^0(M), H_I^1(M), ..., H_I^{t-1}(M)$ are finitely generated, then $\mathrm{Ass}_R(H_I^t(M)/N)$ is finite

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for every finitely generated submodule N of $H^t_I(M)$. For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik's interesting paper [13]. This paper is concerned with what might be considered a generalization of the above mentioned result of Brodmann and Lashgari to the class of minimax modules. More precisely, we shall show that:

Theorem 1.1. *Let R be a Noetherian ring and I an ideal of R. Let M be a non-zero finitely generated R-module and let $t \geq 0$ be an integer such that $H^i_I(M)$ is a minimax R-module for all $i < t$. Then, for every minimax submodule N of $H^t_I(M)$, the R-module $\operatorname{Hom}_R(R/I, H^t_I(M)/N)$ is finitely generated.*

Recall that a module is called *minimax module* when it has a finitely generated submodule, such that the quotient by it is an Artinian module [20].

One of our tools for proving Theorem 1.1 is the following:

Lemma 1.2. *Let I denote an ideal of a Noetherian ring R and let M be a non-zero finitely generated R-module. Let $t \geq 0$ be an integer such that $H^i_I(M)$ is I-cofinite and minimax R-module for all $i < t$. Then $\operatorname{Hom}_R(R/I, H^t_I(M))$ is a finitely generated R-module.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, M will be a non-zero finitely generated R-module, and I will be an ideal of R. The i^{th} local cohomology module of M with support in I is defined by

$$H^i_I(M) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [5] or [3] for the basic properties of local cohomology.

2. The results

Let us, first, recall the important concept of an I-cofinite of an R-module N with respect to an ideal I of R, introduced by Hartshorne in [7]. For any ideal I of R, Hartshorne defined a module N to be I-cofinite, if $\operatorname{Supp}(N) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated module for all i.

In [20] H. Zöschinger introduced the interesting class of minimax modules, and he has in [20, 21] given many equivalent conditions for a module to be minimax. The R-module N is said to be a *minimax module*, if there is a finitely generated submodule L of N, such that N/L is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules. It was shown by T. Zink [19] and by E. Enochs [4] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

The following lemma is needed in the proof of main theorem.

Lemma 2.1. *Let R be a Noetherian ring and let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence of R-modules. Then N is minimax if and only if N' and N'' are both minimax.*

Proof. We may suppose for the proof that N' is a submodule of N and that $N'' = N/N'$. If N is minimax, then it easily follows from the definition that N' and N/N' are minimax. Now, suppose that N' and N/N' are minimax. Then there exists a finitely generated submodule T of N', such that N'/T is Artinian. Let $M' = N'/T$ and $M = N/T$. Then we obtain the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0,$$

where M' is Artinian and M/M' is minimax (note that $M/M' \cong N/N'$). Now, since M/M' is minimax it follows from the definition that there is a finitely generated submodule L/M' of M/M' such that M/L is Artinian. As L/M' is finitely generated, it follows that $L = M' + K$ for some finitely generated submodule K of L. Then it follows from $L/K \cong M'/K \cap M'$ that L/K is an Artinian R-module. Therefore the exact sequence

$$0 \longrightarrow L/K \longrightarrow M/K \longrightarrow M/L \longrightarrow 0$$

implies that M/K is Artinian. Consequently M is a minimax module. Since $M = N/T$, it follows that $K = S/T$ for some submodule S of N. As T and K are finitely generated, we deduce that S is also finitely generated. Now because

$$N/S \cong (N/T)/(S/T) = M/K$$

is Artinian, we get from definition that N is minimax and the claim is true. □

The following lemma plays a key role in the proof of the main theorem.

Lemma 2.2. *Let R be a Noetherian ring, M a non-zero finitely generated R-module and I an ideal of R. Let t be a non-negative integer, such that $H^i_I(M)$ is I-cofinite minimax for all $i < t$. Then, the R-module* $\operatorname{Hom}_R(R/I, H^t_I(M))$ *is finitely generated. In particular, the set* $\operatorname{Ass}_R(H^t_I(M))$ *is finite.*

Proof. We use induction on t. When $t = 0$ there is nothing to prove. So suppose that $t > 0$ and that the result has been proved for smaller values of t. It follows from [3, Corollary 2.1.7] that $H^i_I(M) \cong H^i_I(M/\Gamma_I(M))$ for all $i > 0$. Also, by [3, Lemma 2.1.2], $M/\Gamma_I(M)$ is an I-torsion-free R-module. Hence we can (and do) assume that M is an I-torsion-free R-module. Then, in view of [3, Lemma 2.1.1], the ideal I contains an element x which is M-regular. The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^i_I(M) \xrightarrow{x} H^i_I(M) \longrightarrow H^i_I(M/xM) \longrightarrow H^{i+1}_I(M) \xrightarrow{x} H^{i+1}_I(M) \longrightarrow \cdots.$$

Therefore we deduce that the sequence

$$0 \longrightarrow H^{i-1}_I(M)/xH^{i-1}_I(M) \longrightarrow H^{i-1}_I(M/xM) \longrightarrow (0 :_{H^i_I(M)} x) \longrightarrow 0$$

is exact. Also, by [16, Corollary 4.4], Lemma 2.1 and the hypothesis, $H^{i-1}_I(M/xM)$ is I-cofinite minimax for all $i < t$, so that, by the inductive hypothesis,

$$\operatorname{Hom}_R(R/I, H^{t-1}_I(M/xM))$$

is finitely generated. On the other hand, the exact sequence

$$0 \longrightarrow H^{t-1}_I(M)/xH^{t-1}_I(M) \longrightarrow H^{t-1}_I(M/xM) \longrightarrow (0 :_{H^t_I(M)} x) \longrightarrow 0$$

induces the exact sequence

$$\operatorname{Hom}_R(R/I, H^{t-1}_I(M/xM)) \longrightarrow \operatorname{Hom}_R(R/I, 0 :_{H^t_I(M)} x) \longrightarrow \operatorname{Ext}^1_R(R/I, H^{t-1}_I(M)/xH^{t-1}_I(M)).$$

Since by [16, Corollary 4.4] $H^{t-1}_I(M)/xH^{t-1}_I(M)$ is I-cofinite minimax, it follows from the above exact sequence that $\operatorname{Hom}_R(R/I, 0 :_{H^t_I(M)} x)$ is finitely generated. Now, as $x \in I$ we have

$$\operatorname{Hom}_R(R/I, 0 :_{H^t_I(M)} x) \cong \operatorname{Hom}_R(R/I \otimes_R R/(x), H^t_I(M)) \cong \operatorname{Hom}_R(R/I, H^t_I(M)),$$

and so $\operatorname{Hom}_R(R/I, H^t_I(M))$ is finitely generated. This completes the inductive step. $\square$

The following result will be used for the generalization of the main result of Nhan and Brodmann-Lashgari.

Theorem 2.3. *Let R be a Noetherian ring, M a non-zero finitely generated R-module and I an ideal of R. Let t be a non-negative integer such that $H^i_I(M)$ is minimax for all $i < t$. Then the R-module $\operatorname{Hom}_R(R/I, H^t_I(M))$ is finitely generated. In particular the set $\operatorname{Ass}_R(H^t_I(M))$ is finite.*

Proof. In view of Lemma 2.2 it is enough for us to show that $H^i_I(M)$ is I-cofinite for all $i < t$. We proceed by induction on i. The case $i = 0$ is obvious as $H^0_I(M)$ is finitely generated. So, let $i > 0$, and the result has been proved for smaller values of i. By the inductive assumption, $H^j_I(M)$ is I-cofinite for $j = 0, 1, ..., i-1$. Hence by Lemma 2.2 and assumption, $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated. Therefore by [16, Proposition 4.3], $H^i_I(M)$ is I-cofinite. Hence we get that $H^i_I(M)$ is I-cofinite minimax for all $i < t$. Now the assertion follows from Lemma 2.2. $\square$

Nhan, in [17, Proposition 5.5], established the following corollary in the case where R is local. The following result provides a slight generalization of [17, Proposition 5.5] and [1, Theorem 2.2].

Corollary 2.4. *Let $obj(\mathscr{N})$ (resp. $obj(\mathscr{A})$) denote the category of all Noetherian (resp. Artinian) R-modules and R-homomorphisms. Let t be a non-negative integer such that $H^i_I(M) \in obj(\mathscr{N}) \bigcup obj(\mathscr{A})$ for all $i < t$. Then the R-module $\operatorname{Hom}_R(R/I, H^t_I(M))$ is finitely generated. In particular the set $\operatorname{Ass}_R(H^t_I(M))$ is finite.*

Proof. Apply Theorem 2.3 and the fact that the class of minimax modules includes all Noetherian and Artinian modules. $\square$

We are now ready to state and prove the main theorem.

Theorem 2.5. *Let R be a Noetherian ring, M a non-zero finitely generated R-module and I an ideal of R. Let t be a non-negative integer such that $H^i_I(M)$ is minimax for all $i < t$ and let N be a minimax submodule of $H^t_I(M)$. Then the R-module $\operatorname{Hom}_R(R/I, H^t_I(M)/N)$ is finitely generated. In particular, the set $\operatorname{Ass}_R(H^t_I(M)/N)$ is finite.*

Proof. In view of Theorem 2.3, $\operatorname{Hom}_R(R/I, H^t_I(M))$ is finitely generated. On the other hand, according to Melkersson [16, Proposition 4.3], N is I-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow H^t_I(M) \longrightarrow H^t_I(M)/N \longrightarrow 0$$

induces the following exact sequence:

$$\operatorname{Hom}_R(R/I, H^t_I(M)) \longrightarrow \operatorname{Hom}_R(R/I, H^t_I(M)/N) \longrightarrow \operatorname{Ext}^1_R(R/I, N).$$

Consequently $\operatorname{Hom}_R(R/I, H^t_I(M)/N)$ is finitely generated, as required. $\square$

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