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TAUBERIAN TYPE THEOREM FOR OPERATORS WITH INTERPOLATION SPECTRUM FOR HÖLDER CLASSES

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ABSTRACT. We consider an invertible operator T on a Banach space X whose spectrum is an interpolating set for Hölder classes. We show that if $\|T^n\| = O(n^p)$, $p \geq 1$, $\|T^{-n}\| = O(w_n)$ with $n^q = o(w_n)$ $\forall q \in \mathbb{N}$ and $\sum_n 1/(n^{1-\alpha}(\log w_n)^{1+\alpha}) = +\infty$, then $\|T^{-n}\| = O(n^{p+s})$ for all $s > \frac{1}{2}$, assuming that $(w_n)_{n\geq 1}$ satisfies suitable regularity conditions. When X is a Hilbert space and $p = 0$ (i.e. T is a contraction), we show that under the same assumptions, T is unitary and this is sharp.

1. Introduction

In this note, we are interested in invertible operators T on a Banach space X with polynomial growth and whose spectrum, denoted by $\sigma(T)$, is a K-set. We study growth of the norms of the negative iterates of T. A closed set E of the unit circle $\mathbb{T}$ is said to be a K-set if there exists $c_E > 0$ such that for all arcs $L \subset \mathbb{T}$,

$$\sup_{\zeta \in E} d(\zeta, E) \geq c_E |L|,$$

where $|L|$ denotes the length of L. Dynkin [4] showed that K-sets are the interpolating sets for Hölder classes: if we denote by $\mathcal{A}(\mathbb{D})$ the disc algebra and set, for $s \in (0,1)$,

$$\Lambda^s = \{f \in \mathcal{C}(\mathbb{T}) : \|f\|_s = \|f\|_{\mathcal{C}(\mathbb{T})} + \sup_{h\neq 0, t\in\mathbb{R}} \frac{|f(e^{i(t+h)}) - f(e^{it})|}{|h|^s} < +\infty\}$$

and $A^s = \Lambda^s \cap \mathcal{A}(\mathbb{D})$, then E is K-set iff $\Lambda^s|E = A^s|E$.

We also need the following definition: let $w = (w_n)_{n\geq 1}$ be a sequence of positive real numbers; we say that w satisfies condition (R), and we write $w \in (R)$, if it satisfies:

(1) $(\log w_n)_{n\geq 1}$ is non-decreasing, and $(w_{n+1}/w_n)_{n\geq 1}$ is non-increasing;
(2) $n^q = o(w_n)$ for all $q \geq 0$;
(3) the sequence $(\log w_n/n^\beta)_{n\geq n_0}$ is non-increasing for some $\beta < 1/2$.

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Theorem 1.1. *Let $w \in (R)$ and let T be an operator on a Banach space X such that $\sigma(T)$ is a K-set, $\|T^n\| = O(n^p)$ for some $p \geq 1$ and $\|T^{-n}\| = O(w_n)$. If for all $\alpha \in (0,1)$,*

$$\sum_{n\geq 1} \frac{1}{n^{1-\alpha}(\log w_n)^{1+\alpha}} = +\infty, \tag{1}$$

then, for all $\varepsilon > 0$,

$$\|T^{-n}\| = O(n^{p+\frac{1}{2}+\varepsilon}), \qquad n \to +\infty.$$

In [5], Theorem 1.1 was obtained when norms of the negative powers of T satisfy the condition $\sum_{n\geq 1} 1/(n \log w_n) = +\infty$ instead of (1) and the spectrum was an arbitrary K-set. In [1], Theorem 1.1 was obtained when $\sigma(T) = E_\zeta$ is a perfect symmetric set with constant of ratio $\xi \in (0, 1/2)$ (special classes of K-sets) and under the condition $\|T^{-n}\| = O(e^{n^\beta})$ with $\beta < |\log 2\zeta|/|\log 2\zeta^2|$. Theorem 1.1 extends results of [1, 5]. For contractions on a Hilbert space we improve Theorem 1.1 to obtain the following result.

Theorem 1.2. *Let $w \in (R)$ and let T be an invertible contraction on a Hilbert space X, such that $\sigma(T)$ is a and $\|T^{-n}\| = O(w_n)$. If condition (1) is satisfied for all $\alpha \in (0,1)$, then T is unitary.*

On the other hand, if there exists $\alpha \in (0,1)$ such that

$$\sum_{n\geq 1} \frac{1}{n^{1-\alpha}(\log w_n)^{1+\alpha}} < +\infty, \tag{2}$$

then there exists an invertible contraction on a Hilbert space T such that $\sigma(T)$ is a K-set, $\|T^{-n}\| = O(w_n)$ and $\|T^{-n}\| \underset{n\to+\infty}{\to} +\infty$.

Theorem 1.2 is not valid for contractions on general Banach spaces. Indeed, Esterle constructed in [7] a contraction T on a Banach space such that $\sigma(T)$ is a K-set (a perfect symmetric set with constant of ratio ζ such that $1/\zeta$ is not a Pisot number) and $\|T^{-n}\| \to +\infty$. Observe also that Theorem 1.2 is not valid when $\sigma(T)$ is a null measure set (see [10]). Similar results of Tauberian type were obtained in [1, 2, 5, 6, 7, 8, 10, 14].

2. Proofs

2.1. **Hausdorff measure of K-sets.** A non-decreasing continuous function on $[0, +\infty)$ such that $h(0) = 0$ is said to be a Hausdorff function, and the h-measure of Hausdorff of a closed set $E \subset \mathbb{T}$ is defined by

$$H_h(E) = \lim_{t\to 0} \inf \sum_i h(|\Delta_i|),$$

where the infimum is taken over all the coverings (Δ_i) of E by arcs of $\mathbb{T}$ with length $|\Delta_i| \leq t$. Dynkin showed in [4] that if E is a K-set, then there exists $\alpha_E > 0$ such that

$$\int_0^1 \frac{|E_t|}{t^{1+\alpha_E}} dt < +\infty,$$

where

$$E_t = \{\zeta \in \mathbb{T} : d(\zeta, E) \leq t\}, \qquad t > 0,$$

$|E_t|$ denotes the length of E_t and $\alpha_E \geq \log(1/(1-c_E))/\log(2/(1-c_E))$. Note that a K-set is a Beurling–Carleson set since

$$\int_0^1 \frac{|E_t|}{t} dt < +\infty.$$

Shapiro gave in [12] a complete characterisation of Beurling–Carleson sets of null h-Hausdorff measure: he showed that $H_h(E) = 0$ for all Beurling–Carleson sets E if and only if $\int_0^1 dt/h(t) = +\infty$. Let $(\zeta_n)_{\geq 1}$ be a sequence of real numbers such that $0 < \zeta_n < 1/2$. We set

$$E_{(\zeta_n)} = \Big\{ \exp\big[2i\pi \sum_{n\geq 1} \varepsilon_n \zeta_1 \cdots \zeta_n (1-\zeta_n)\big], \varepsilon_n = 0 \text{ or } 1 \Big\}.$$

When $\zeta_n = \zeta$ for all n, E_ζ is the perfect symmetric set of constant ratio ζ ($E_{1/3}$ is the usual Cantor triadic) and E_ζ is a K-set of Hausdorff dimension $d_E = |\log 2\zeta|/|\log 2\zeta^2|$ (see [9]). When $\limsup_{n\to\infty} \zeta_n < 1/2$, Esterle showed in [7] (Proposition 2.5) that $E_{(\zeta_n)}$ is still also a K-set. The following lemma gives a complete description of a K-set of null h-Hausdorff measure.

Lemma 2.1. *Let h be a Hausdorff function such that $h(t)/t$ is strictly decreasing. Then the following two conditions are equivalent.*

(i) For all K-sets E, $H_h(E) = 0$.
(ii) For all $\alpha \in (0,1)$,

$$\int_0^1 \frac{dt}{t^\alpha h(t)} = +\infty.$$

Proof. $(ii) \Rightarrow (i)$. Suppose that there exists a K-set E such that $H_h(E) = c > 0$. For all $t > 0$, E_t is a disjoint union of arcs Δ_i with $|\Delta_i| \geq 2t$: $E_t = \bigcup_{1\leq i\leq N} \Delta_i$, and so

$$c \leq \sum_{1\leq i\leq N} h(|\Delta_i|) \leq \sum_{1\leq i\leq N} \frac{h(|\Delta_i|)}{|\Delta_i|} |\Delta_i| \leq \frac{h(2t)}{2t} |E_t|. \tag{3}$$

Since E is a K-set, there exists $\alpha \in (0,1)$ such that $\int_0^1 |E_t|/t^{1+\alpha} dt < +\infty$, and we deduce from (3) that

$$\int_0^1 \frac{dt}{t^\alpha h(t)} < +\infty.$$

$(i) \Rightarrow (ii)$. Suppose that there exists $\alpha \in (0,1)$ such that

$$\int_0^1 \frac{dt}{t^\alpha h(t)} < +\infty.$$

We will construct a K-set E satisfying $H_h(E) > 0$. In order to do that, we define $(\lambda_n)_{n\geq 0}$ by $\lambda_0 = 1$ and $h(\lambda_n) = 2^{-n}$, $n \geq 1$. Let $E = E_{(\zeta_n)}$ be the perfect symmetric set associated with $(\zeta_n)_{n\geq 0} := (\lambda_n/\lambda_{n-1})_{n\geq 1}$. The set E is as described in [9], $E = \bigcap_{n\geq 0} E_n$, where E_n is a disjoint union of 2^n closed arcs $E_{i,n}$ with

$|E_{i,n}| = 2\pi(\zeta_1 \cdots \zeta_n) = 2\pi\lambda_n$, $1 \le i \le 2^n$. For all $N \ge 0$,

$$\begin{aligned}
+\infty > (1-\alpha)\int_0^1 \frac{dt}{t^\alpha h(t)} &= (1-\alpha)\int_0^{\lambda_{N+1}} \frac{dt}{t^\alpha h(t)} + (1-\alpha)\sum_{0\le n\le N}\int_{\lambda_{n+1}}^{\lambda_n} \frac{dt}{t^\alpha h(t)}\\
&\ge 2^{N+1}\lambda_{N+1}^{1-\alpha} + \sum_{0\le n\le N} 2^n(\lambda_n^{1-\alpha} - \lambda_{n+1}^{1-\alpha})\\
&\ge 2^{N+1}\lambda_{N+1}^{1-\alpha} + \sum_{1\le n\le N} 2^{n-1}\lambda_n^{1-\alpha} + 1 - 2^N\lambda_{N+1}^{1-\alpha} \ge \sum_{1\le n\le N} 2^{n-1}\lambda_n^{1-\alpha}.
\end{aligned}$$

Hence $\sum_{n\ge 1} 2^{n-1}\lambda_n^{1-\alpha} < +\infty$ and so

$$\limsup_{n\to\infty} \zeta_n = \limsup_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} \le \frac{1}{2^{1/(1-\alpha)}}.$$

The perfect symmetric set $E = E_{(\zeta_n)}$ is a K-set and $H_h(E) = \lim_{n\to\infty} 2^n h(\lambda_n) = 1$. □

2.2. **Hyperfunctions supported by a K-set.** A hyperfunction on $\mathbb{T}$ is a holomorphic function on $\mathbb{C}\backslash\mathbb{T}$ vanishing at infinity. We denote by $\mathcal{H}(\mathbb{T})$ the set of all hyperfunctions. The support of a hyperfunction $\psi \in \mathcal{H}(\mathbb{T})$, denoted by $\operatorname{supp}\psi$, is the smallest closed set $E \subset \mathbb{T}$ such that ψ can be analytically extended on $\mathbb{C}\backslash E$. For a closed set $E \subset \mathbb{T}$, we set $\mathcal{H}(E) = \{\psi \in \mathcal{H}(T) : \operatorname{supp}\psi \subset E\}$. The Taylor coefficients of ψ are given by

$$\begin{cases} \psi^+(z) := \psi_{|\mathbb{D}}(z) = \sum_{n\ge 1} \widetilde{\psi}_n z^{n-1}, & |z| < 1,\\ \psi^-(z) := \psi_{|\mathbb{C}\backslash\overline{\mathbb{D}}}(z) = -\sum_{n\le 0} \widetilde{\psi}_n z^{n-1}, & |z| > 1.\end{cases}$$

We set

$$\mathcal{H}^2_w(\mathbb{T}) = \Big\{\psi \in \mathcal{H}(\mathbb{T}) : \sup_{n\ge 1} \frac{|\widetilde{\psi}_n|}{w_n} < +\infty \text{ and } \sum_{n\le 0} |\widetilde{\psi}_n|^2 < \infty\Big\}$$

and $\mathcal{H}^2_w(E) = \mathcal{H}^2_w(\mathbb{T}) \cap \mathcal{H}(E)$. We will need the following lemma, which follows from a result of Hruscev [11].

Lemma 2.2. *Let $w \in (R)$. The following conditions are equivalent.*

(i) *For all K-sets E, we have $\mathcal{H}^2_w(E) = \{0\}$.*
(ii) *For all $\alpha \in (0,1)$, condition (1) is satisfied.*

Proof. Define $\mathcal{F}_h(E)$ for a Hausdorff function h by

$$\mathcal{F}_h(E) = \Big\{\psi \in \mathcal{H}(E) : |\psi^+(z)| = O\big(\exp\frac{h(1-|z|)}{1-|z|}\big) \text{ and } \psi^- \in \mathrm{H}^2(\mathbb{C}\backslash\overline{\mathbb{D}})\Big\}.$$

We set $h_w(t) = t\log\sup_{n\ge 1}(1-t)^n w_n$. According to Lemma 5.2 of [3], the function h_w is a Hausdorff function, $h_w(t)/t$ is strictly decreasing and

$$\int_0^1 \frac{dt}{t^\alpha h_w(t)} \asymp \sum_{n\ge 1} \frac{[(n+1)/\log w_{n+1}]^\alpha - [n/\log w_n]^\alpha}{\log w_n}.$$

Since $\big(\log w_n/\sqrt{n}\big)_{n\ge 0}$ is non-increasing and $\big(\log w_n\big)_{n\ge 0}$ is non-decreasing,

$$\Big(\frac{\sqrt{n}}{\log w_n}\Big)^\alpha((n+1)^{\alpha/2} - n^{\alpha/2}) \le \Big[\frac{n+1}{\log w_{n+1}}\Big]^\alpha - \Big[\frac{n}{\log w_n}\Big]^\alpha \le \frac{(n+1)^\alpha - n^\alpha}{(\log w_n)^\alpha}.$$

So

$$\int_0^1 \frac{dt}{t^\alpha h_w(t)} \asymp \sum_{n\geq 1} \frac{1}{n^{1-\alpha}(\log w_n)^{1+\alpha}}. \tag{4}$$

Hence $\mathcal{F}_{h_w}(E) \subset \mathcal{H}^2_w(E) \subset \mathcal{F}_{2h_w}(E)$ (see [10]). Theorem 9.1 of [11] shows that $\mathcal{F}_h(E) = \{0\}$ iff $H_h(E) = 0$. The lemma is proved. □

Remark 2.3. Denote by $\mathcal{A}(\mathbb{D})$ the disk algebra, denote by $\mathcal{A}^p(\mathbb{D})$ the algebra of all functions f such that $f^{(k)} \in \mathcal{A}(\mathbb{D})$, $0 \leq k \leq p$, and let $\mathcal{A}^\infty(\mathbb{D}) = \bigcap_{p\geq 1} \mathcal{A}^p(\mathbb{D})$. First observe that a K-set E is a Beurling–Carleson set, and so there exists $f \in \mathcal{A}^\infty(\mathbb{D})$ with $f^{(n)}|E = 0$ (see [13]). Now set

$$\mathcal{H}_{w,p}(\mathbb{T}) = \big\{\psi \in \mathcal{H}(\mathbb{T}) : \sup_{n\geq 1} \frac{|\widetilde{\psi}_n|}{w_n} < +\infty \text{ and } \sup_{n\leq 0} \frac{|\widetilde{\psi}_n|}{(1+|n|)^p} < +\infty\big\}$$

and set $\mathcal{H}_{w,p}(E) = \mathcal{H}_{w,p}(\mathbb{T}) \cap \mathcal{H}(E)$. If $f \in \mathcal{A}^\infty(\mathbb{D})$ and $\psi \in \mathcal{H}_{w,p}(\mathbb{T})$, we define the hyperfunction $f.\psi$ whose Taylor coefficients are given by

$$\widetilde{f.\psi}_n = \sum_{m\in\mathbb{Z}} \widehat{f}(n)\widetilde{\psi}_{n-m}, \qquad n \in \mathbb{Z}. \tag{5}$$

If $\psi \in \mathcal{H}_{w,p}(E)$ and $f^{(n)}|E = 0$, then $f.\psi \in \mathcal{H}^2_w(E)$ (see [5], Proposition 2.1). Hence, if condition (ii) of the lemma is satisfied, then for all K-sets E and for all $p \geq 0$, $\psi \in \mathcal{H}_{w,p}(E)$, $f \in A^\infty(\mathbb{D})$ with $f^{(n)}|E = 0$ we have $f.\psi = 0$.

2.3. **Proofs of Theorem 1.1 and Theorem 1.2.** Suppose that condition (1) is satisfied. Letting $x \in X$ and $l \in X^*$, we set

$$\phi(z) = \langle (T - zI)^{-1}x, l\rangle, \qquad z \notin \sigma(T).$$

We have $\phi \in \mathcal{H}_{w,p}(\sigma(T))$ ($p = 0$ for Theorem 1.2). Consider an outer function $f \in \mathcal{A}^\infty(\mathbb{D})$ such that $f^{(m)}|\sigma(T) = 0$ for all $m \geq 0$. A standard computation of (5) gives that

$$f.\phi(z) = \langle (T - zI)^{-1}f(T)x, l\rangle, \qquad z \notin \sigma(T).$$

According to Remark 2.3, $f.\phi = 0$, and so $f(T) = 0$. The conclusion follows from the proof of Theorem 4.1 of [5] (see also [2]) for Theorem 1.1, and from the proof of Theorem 6.4 of [6] for Theorem 1.2.

Now suppose that condition (2) is satisfied for some $\alpha \in (0,1)$. Set $\widetilde{w}_n = w_n^{1/2}$. Then $\widetilde{w}$ satisfies (R) and (2). According to (4), we have $\int_0^1 dt/(t^\alpha h_{\widetilde{w}}(t)) < +\infty$, where $h_{\widetilde{w}}(t) = t \log \sup_n (1-t)\widetilde{w}_n$ is a Hausdorff function and $h_{\widetilde{w}}(t)/t$ is strictly decreasing. Lemma 1 and Frostman's Theorem [9] give the existence of a K-set E and a singular measure μ supported by E which modulus of continuity satisfies $\rho_\mu(t) = O(h_{\widetilde{w}}(t))$. Let S_μ be the singular inner function associated with μ. Consider the operator $T : \mathrm{H}^2 \ominus S_\mu\mathrm{H}^2 \to \mathrm{H}^2 \ominus S_\mu\mathrm{H}^2$ defined by $Tg = \mathrm{P}_\mu(zg)$, where P_μ is the orthogonal projection on $\mathrm{H}^2 \ominus S_\mu\mathrm{H}^2$. Then T is an invertible contraction with spectrum E, $\|T^{-n}\| = O(w_n)$ and $\|T^{-n}\| \to \infty$ (see [10] for more details).

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