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A NEW CHARACTERIZATION OF ULTRASPHERICAL POLYNOMIALS

R. LASSER AND J. OBERMAIER

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Abstract. We characterize the class of ultraspherical polynomials in between all symmetric orthogonal polynomials on $[-1,1]$ via the special form of the representation of the derivatives $p'_{n+1}(x)$ by $p_k(x),\ \ k=0,...,n$.

Let $(p_n(x))_{n\in\mathbb{N}_0}$ be a symmetric orthonormal polynomial sequence with respect to a symmetric probability measure π with supp $\pi \subseteq [-1,1]$, that is, p_n is an algebraic polynomial with $\deg p_n = n$, $\int p_n p_k \, d\pi = \delta_{n,k}$ and p_n has a positive leading coefficient. It is well-known [2] that the $p_n(x)$ satisfy a recursion formula

$$x\,p_n(x) \;=\; A_{n+1}\,p_{n+1}(x) \;+\; A_n\,p_{n-1}(x) \qquad \text{for } n\in\mathbb{N}_0, \tag{1}$$

where $p_{-1}\equiv 0,\ \ p_0(x)=1$, and $A_n>0,\ n\geq 1$ are bounded. Since supp $\pi\subseteq[-1,1]$ the n simple zeroes of $p_n(x)$ are contained in $]-1,1[$. Hence $p_n(1)\neq 0$ for each $n\in\mathbb{N}_0$.

Conversely, by Favard's theorem [2] we know that a sequence of polynomials which fulfills (1) is an orthogonal polynomial sequence with respect to a symmetric measure on $\mathbb{R}$. Hence Favard's theorem is a characterization theorem for orthogonal polynomials. It is of great interest to characterize more special classes of orthogonal polynomials. A survey of such characterization theorems is given in [1]. One problem mentioned in [1] is to find all orthogonal polynomial sequences whose derivatives are special linear combinations of polynomials of the same system.
The purpose of this note keeps going in the same direction, namely to characterize the ultraspherical polynomials in between the class of symmetric orthonormal polynomials on $[-1,1]$ by means of the linear representation of $p'_{n+1}(x)$ by $p_k(x),\ \ k=0,...,n$.

We prefer to use another normalization of the polynomials $p_n(x)$. Let

$$R_n(x) \;= p_n(x)/p_n(1).$$

Then the recursion formula for $R_n(x)$ is

$$x\,R_n(x) \;=\; a_n\,R_{n+1}(x) + c_n\,R_{n-1}(x) \qquad \text{for } n\in\mathbb{N}_0, \tag{2}$$

where $R_{-1}\equiv 0$, $R_0(x)=1$ and $a_0=1$.

Since $R_{n+1}(1)=1$ yields $\lim_{x\to\infty} R_{n+1}(x)=\infty$ the leading coefficient of R_{n+1} has to be positive, that is, $a_n>0$ for all $n\geq 1$. Additionally $R_n(1)=1$ implies

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$a_n + c_n = 1$ for all $n \geq 1$. From the theory of orthogonal polynomials [2] it is known that $c_n a_{n-1} > 0$, which implies $c_n > 0$ for $n \geq 1$. Hence it holds that $0 < a_n, c_n < 1$ for $n \geq 1$.

The coefficients of (1) and (2) are related by

$$A_n \;=\; \sqrt{c_n a_{n-1}} \qquad \text{for } n \in \mathbb{N}.$$

Furthermore, $p_n(1) = \sqrt{h(n)}$, where $h(0) = 1$ and

$$h(n+1)c_{n+1} \;=\; a_n h(n) \qquad \text{for } n \in \mathbb{N}_0. \tag{3}$$

The ultraspherical polynomials $R_n^{(\alpha)}(x)$, $\alpha > -1$ (normalized such that $R_n^{(\alpha)}(1) = 1$) satisfy

$$\int_{-1}^{1} R_n^{(\alpha)}(x)\, R_k^{(\alpha)}(x)\, d\pi(x) \;=\; \delta_{n,k}\, \frac{1}{h(n)},$$

where $d\pi(x) = c_\alpha (1-x^2)^\alpha dx$, $c_\alpha = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}$. The recurrence coefficients are

$$a_n \;=\; \frac{n+2\alpha+1}{2n+2\alpha+1}, \qquad c_n \;=\; \frac{n}{2n+2\alpha+1}, \qquad n \in \mathbb{N}. \tag{4}$$

The derivatives $R'_{n+1}(x)$ (and $p'_{n+1}(x)$) are polynomials of degree n. Hence

$$R'_{n+1}(x) \;=\; \sum_{k=0}^{n} d_{n,k}\, R_k(x),$$

with linearization coefficients $d_{n,k}$. From [5, (7.32.5)] we obtain for ultraspherical polynomials $(R_n^{(\alpha)})'(x) = \frac{n(n+2\alpha+1)}{2+2\alpha} R_{n-1}^{(\alpha+1)}(x)$. In that case we can calculate $d_{n,k}$ from the so-called connection coefficients which connect $R_n^{(\alpha+1)}(x)$ and $R_k^{(\alpha)}(x)$; see [3, (9.1.2)]. We shall show that for $m \in \mathbb{N}_0$

$$(R_{2m+1}^{(\alpha)})'(x) \;=\; \gamma_{2m+1} \sum_{k=0}^{m} R_{2k}^{(\alpha)}(x)\, h(2k) \qquad \text{and} \tag{5}$$

$$(R_{2m+2}^{(\alpha)})'(x) \;=\; \gamma_{2m+2} \sum_{k=0}^{m} R_{2k+1}^{(\alpha)}(x)\, h(2k+1), \tag{6}$$

where $\gamma_n = \frac{(n+2\alpha+1)\, n!}{(2+2\alpha)(2\alpha+3)_{n-1}}$, $n \in \mathbb{N}$. More important, we shall prove that this special form of the linearization coefficients $d_{n,k}$ in (5) and (6) characterizes the ultraspherical polynomials in between the symmetric orthogonal polynomials on $[-1,1]$. For the orthonormal versions the formula (5) and (6) write as

$$(p_{2m+1}^{(\alpha)})'(x) \;=\; p_{2m+1}^{(\alpha)}(1)\, \gamma_{2m+1} \sum_{k=0}^{m} p_{2k}^{(\alpha)}(x)\, p_{2k}^{(\alpha)}(1) \qquad \text{and} \tag{7}$$

$$(p_{2m+2}^{(\alpha)})'(x) \;=\; p_{2m+2}^{(\alpha)}(1)\, \gamma_{2m+2} \sum_{k=0}^{m} p_{2k+1}^{(\alpha)}(x)\, p_{2k+1}^{(\alpha)}(1). \tag{8}$$

Hence, (7) and (8) characterize the ultraspherical polynomials in between the symmetric orthonormal polynomials on $[-1,1]$.

In order to formalize the calculation denote by c_{00} the linear space of finite sequences and by

$$\hat{f}(x) \;=\; \sum_{k=0}^{\infty} f(k) R_k(x) h(k) \;=\; \sum_{k=0}^{\infty} f(k) p_k(x) p_k(1)$$

the Fourier expansion of $f \in c_{00}$. Further put $\epsilon_n \in c_{00}$ with

$$\epsilon_n(k) \;=\; \frac{1}{h(k)}\, \delta_{n,k} \qquad \text{for } k \in \mathbb{N}_0,$$

and define the convolution $\epsilon_1 * f$ by

$$\epsilon_1 * f(k) \;=\; a_k f(k+1) + c_k f(k-1) \qquad \text{for } k \in \mathbb{N}_0,$$

where we set $c_0 = 0$. The notion of convolution is motivated by the theory of polynomial hypergroups; compare [4]. Of course, here we do not suppose that the orthogonal polynomials $R_n(x)$ induce a polynomial hypergroup.

Define recursively $\kappa_n \in c_{00}$ by $\kappa_0 \equiv 0,\;\; \kappa_1 = \epsilon_0$ and

$$\kappa_{n+1} \;=\; \frac{1}{a_n}\left(\epsilon_n + \epsilon_1 * \kappa_n - c_n \kappa_{n-1}\right) \qquad \text{for } n \in \mathbb{N}. \tag{9}$$

It is easy to check that supp $\kappa_n \subseteq \{0, \ldots, n\}$. Furthermore, most importantly it holds that

$$\hat{\epsilon}_n(x) \;=\; R_n(x) \quad \text{and} \quad \hat{\kappa}_n(x) \;=\; R_n'(x) \qquad \text{for } n \in \mathbb{N}_0. \tag{10}$$

To verify the latter identity first notify that by (3) we have

$$\begin{aligned} x\hat{f}(x) \quad &= \quad \sum_{k=0}^{\infty} f(k) x R_k(x) h(k) \\ &= \quad \sum_{k=0}^{\infty} f(k)(a_k R_{k+1}(x) + c_k R_{k-1}(x)) h(k) \\ &= \quad \sum_{k=1}^{\infty} c_k f(k-1) R_k(x) h_k \;+\; \sum_{k=0}^{\infty} a_k f(k+1) R_k(x) h_k \\ &= \quad (\epsilon_1 * f)^{\wedge}(x) \end{aligned}$$

for $f \in c_{00}$. Differentiating the recursion formula (2) yields

$$R_{n+1}'(x) \;=\; \frac{1}{a_n}(R_n(x) + x R_n'(x) - c_n R_{n-1}'(x)).$$

Therefore the second identity in (10) easily follows by induction.

Lemma 1. *Let $(R_n(x))_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence defined by* (2) *with $a_n > 0$ and $a_n + c_n = 1$ for all $n \in \mathbb{N}$. If the recursion coefficients c_n satisfy*

$$c_{n+1} \;=\; \frac{(n+1)\, c_n}{2c_n + n} \qquad \textit{for } n \in \mathbb{N}, \tag{11}$$

then $(R_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of ultraspherical polynomials. More precisely $R_n(x) = R_n^{(\alpha)}(x)$, where $\alpha = \frac{1}{2}\,(\frac{1}{c_1} - 3)$. Conversely, the recursion coefficients c_n of $R_n^{(\alpha)}(x)$ satisfy (11) *with $c_1 = \frac{1}{2\alpha+3}$.*

Proof. Beginning with $c_1 = \frac{1}{2\alpha+3}$ a simple induction shows that the c_n defined by (11) yield exactly the c_n of (4). Conversely, for the coefficients c_n of (4) the identity (11) holds true. □

The next theorem is the main result and gives a characterization of ultraspherical polynomials.

Theorem 1. *Let $(R_n(x))_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence defined by (2) with $a_n > 0$ and $a_n + c_n = 1$ for all $n \in \mathbb{N}$, and $h(n)$ defined by (3). The following three conditions are equivalent:*

(i) *It holds that*

$$\begin{aligned} R'_{2m+1}(x) &= \sigma_{2m+1}\sum_{j=0}^{m} R_{2j}(x)\,h(2j) \quad \textit{and} \\ R'_{2m+2}(x) &= \sigma_{2m+2}\sum_{j=0}^{m} R_{2j+1}(x)\,h(2j+1) \quad \textit{for } m \in \mathbb{N}_0, \end{aligned}$$

where σ_{2m+1} and σ_{2m+2} are constants.

(ii) $c_{n+1} = \frac{(n+1)c_n}{2c_n+n}$ *for all $n \in \mathbb{N}$.*

(iii) *$(R_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of ultraspherical polynomials.*

The constants in (i) *are given by $\sigma_{n+1} = \frac{n+1}{a_n h(n)}$ for all $n \in \mathbb{N}_0$.*

Proof. By (10) we have $\sigma_{n+1} = \kappa_{n+1}(n)$ for all $n \in \mathbb{N}_0$. Moreover, we easily derive

$$\kappa_{n+1}(n) = \frac{1}{a_n}\left(\frac{1}{h(n)} + c_n\kappa_n(n-1)\right) \qquad \text{for } n \in \mathbb{N}.$$

Applying induction this identity implies

$$\sigma_{n+1} = \kappa_{n+1}(n) = \frac{n+1}{a_n h(n)} \qquad \text{for } n \in \mathbb{N}_0. \tag{12}$$

In fact, $\kappa_1(0) = 1 = \frac{1}{a_0 h(0)}$, and supposing $\kappa_n(n-1) = \frac{n}{a_{n-1}h(n-1)} = \frac{n}{c_n h(n)}$, we obtain

$$\kappa_{n+1}(n) = \frac{1}{a_n}\left(\frac{1}{h(n)} + \frac{n}{h(n)}\right) = \frac{n+1}{a_n h(n)}.$$

We first prove (i)⇒(ii).

Let $n \geq 2$. Taking into account (9) and (10) the assumption made in (i) implies

$$\begin{aligned} \sigma_{n+1} = \kappa_{n+1}(n) &= \kappa_{n+1}(n-2) \\ &= \frac{\epsilon_n(n-2) + c_{n-2}\kappa_n(n-3) + a_{n-2}\kappa_n(n-1) - c_n\kappa_{n-1}(n-2)}{a_n} \\ &= \frac{(c_{n-2} + a_{n-2})\kappa_n(n-1) - c_n\kappa_{n-1}(n-2)}{a_n} = \frac{\sigma_n - c_n\sigma_{n-1}}{a_n}. \end{aligned}$$

Hence, by (12) and (3) we get

$$n+1 = h(n)\left(\frac{n}{c_n h(n)} - \frac{c_n(n-1)}{c_{n-1}h(n-1)}\right) = \frac{n}{c_n} - \frac{(n-1)a_{n-1}}{c_{n-1}},$$

which gives

$$c_n = \frac{nc_{n-1}}{2c_{n-1} + (n-1)} \qquad \text{for } n \geq 2.$$

The implication (ii)⇒(iii) follows by Lemma 1.

Suppose that $R_n(x) = R_n^{(\alpha)}(x)$, $\alpha > -1$. If we show that

$$\kappa_{2m+1}(2j) = \frac{2m+1}{a_{2m}h(2m)}, \quad \kappa_{2m+1}(2j+1) \;=\; 0 \qquad \text{and}$$

$$\kappa_{2m+2}(2j) = 0, \quad \kappa_{2m+2}(2j+1) \;=\; \frac{2m+2}{a_{2m+1}h(2m+1)} \qquad \text{for } j = 0, ..., m, \tag{13}$$

then (iii)⇒(ii) holds. Especially $\sigma_{n+1} = \frac{n+1}{a_n h(n)} = \frac{(n+2\alpha+2)(n+1)!}{(2+2\alpha)(2\alpha+3)_n}$.

To prove (13) we use induction on m. The calculations are based on (9).

We start with $\kappa_1(0) = 1$, $\kappa_2(0) = 0$, $\kappa_2(1) = \frac{2}{a_1 h(1)}$. Further, using (11) we get

$$\begin{aligned}
\kappa_3(0) &= \frac{1}{a_2}\left(\kappa_2(1) - c_2\right) = \frac{2c_1 - a_1c_2}{a_1a_2} \\
&= \frac{2c_2c_1 + c_2 - a_1c_2}{a_1a_2} = \frac{3c_1c_2}{a_1a_2} = \frac{3}{a_2h(2)}, \\
\kappa_3(1) &= 0, \qquad \kappa_3(2) = \frac{3}{a_2h(2)}.
\end{aligned}$$

Finally, $\kappa_3(2) = \kappa_3(0)$ and (11) yield

$$\begin{aligned}
\kappa_4(0) &= 0, \qquad \kappa_4(1) = \frac{1}{a_3}\left(\kappa_3(2) - c_3\kappa_2(1)\right) = \frac{4c_1c_2c_3}{a_1a_2a_3} = \frac{4}{a_3h(3)}, \\
\kappa_4(2) &= 0, \qquad \kappa_4(3) = \frac{4}{a_3h(3)}.
\end{aligned}$$

Now suppose that (13) holds for $m-1$. It follows by (11)

$$\begin{aligned}
\kappa_{2m+1}(0) &= \frac{\kappa_{2m}(1) - c_{2m}\kappa_{2m-1}(0)}{a_{2m}} \\
&= \frac{2m}{a_{2m}a_{2m-1}h(2m-1)} - \frac{c_{2m}(2m-1)}{a_{2m}a_{2m-2}h(2m-2)} \\
&= \frac{2m}{a_{2m}c_{2m}h(2m)} - \frac{c_{2m}a_{2m-1}(2m-1)}{a_{2m}c_{2m-1}c_{2m}h(2m)} \\
&= \frac{1}{a_{2m}h(2m)}\ \frac{2mc_{2m-1} - c_{2m}(1-c_{2m-1})(2m-1)}{c_{2m}c_{2m-1}} = \frac{2m+1}{a_{2m}h(2m)}
\end{aligned}$$

and therefore

$$\begin{aligned}
\kappa_{2m+1}(2j) &= \frac{a_{2j}\kappa_{2m}(2j+1) + c_{2j}\kappa_{2m}(2j-1) - c_{2m}\kappa_{2m-1}(2j)}{a_{2m}} \\
&= \frac{\kappa_{2m}(1) - c_{2m}\kappa_{2m-1}(0)}{a_{2m}} = \frac{2m+1}{a_{2m}h(2m)} \qquad \text{for } j = 1, ..., m-1.
\end{aligned}$$

Finally

$$\begin{aligned}
\kappa_{2m+1}(2m) &= \frac{1}{a_{2m}h(2m)} + \frac{c_{2m}\kappa_{2m}(2m-1)}{a_{2m}} \\
&= \frac{1}{a_{2m}h(2m)} + \frac{c_{2m}2m}{a_{2m}a_{2m-1}h(2m-1)} = \frac{2m+1}{a_{2m}h(2m)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\kappa_{2m+1}(2j+1) &= \frac{a_{2j+1}\kappa_{2m}(2j+2) + c_{2j+1}\kappa_{2m}(2j) - c_{2m}\kappa_{2m-1}(2j+1)}{a_{2m}} \\
&= 0 \qquad \text{for } j = 0, ..., m-1.
\end{aligned}$$

Just in the same way one derives from the induction hypothesis

$$\kappa_{2m+2}(2j+1) = \frac{2m+2}{a_{2m+1}h(2m+1)}$$

$$\kappa_{2m+2}(2j) = 0 \quad \text{for } j = 0, ..., m. \qquad \square$$

Finally, using the relationship between (1) and (2) discussed at the beginning we get the following corollary.

Corollary 1. *Let* $(p_n(x))_{n\in\mathbb{N}_0}$ *be a symmetric orthonormal polynomial sequence with respect to a symmetric probability measure with supp* $\pi \subseteq [-1,1]$. *Then the following two conditions are equivalent:*

(i)

$$\begin{aligned} p'_{2m+1}(x) &= \sigma_{2m+1}\, p_{2m+1}(1) \sum_{k=0}^{m} p_{2k}(x)\, p_{2k}(1), \\ p'_{2m+2}(x) &= \sigma_{2m+2}\, p_{2m+2}(1) \sum_{k=0}^{m} p_{2k+1}(x)\, p_{2k+1}(1), \qquad m \in \mathbb{N}_0, \end{aligned}$$

where σ_{2m+1} *and* σ_{2m+2} *are constants*

(ii) $(p_n(x))_{n\in\mathbb{N}_0}$ *belongs to the class of ultraspherical polynomials.*

The constants in (i) *are given by* $\sigma_{n+1} = \frac{n+1}{a_n h(n)}$ *for all* $n \in \mathbb{N}_0$.

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Helmholtz Zentrum München, German Research Center for Environmental Health, Institute of Biomathematics and Biometry, Ingolstädter Landstrasse 1, 85764 Neuherberg, Germany; and Münich University of Technology, Centre of Mathematics, 85748 Garching, Germany

E-mail address: `lasser@helmholtz-muenchen.de`

Helmholtz Zentrum München, German Research Center for Environmental Health, Institute of Biomathematics and Biometry, Ingolstädter Landstrasse 1, 85764 Neuherberg, Germany

E-mail address: `josef.obermaier@helmholtz-muenchen.de`