

WEIGHTED REPRODUCING KERNELS
AND TOEPLITZ OPERATORS
ON HARMONIC BERGMAN SPACES ON THE REAL BALL

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ABSTRACT. For the standard weighted Bergman spaces on the complex unit ball, the Berezin transform of a bounded continuous function tends to this function pointwise as the weight parameter tends to infinity. We show that this remains valid also in the context of harmonic Bergman spaces on the real unit ball of any dimension. This generalizes the recent result of C. Liu for the unit disc, as well as the original assertion concerning the holomorphic case. Along the way, we also obtain a formula for the corresponding weighted harmonic Bergman kernels.

1. INTRODUCTION

Let \mathbb{B}^n be the ball in \mathbb{R}^n , $n \geq 2$, and dz the Lebesgue measure on \mathbb{B}^n . For $\alpha > -1$, consider the measure

$$dA_\alpha(z) := c_\alpha(1 - |z|^2)^\alpha dz,$$

where

$$c_\alpha = \frac{\Gamma(\alpha + \frac{n}{2} + 1)}{\pi^{\frac{n}{2}} \Gamma(\alpha + 1)}$$

is chosen so as to make dA_α a probability measure. For simplicity, we will usually assume that α is an integer.

The harmonic Bergman space $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ consists, by definition, of all harmonic functions in $L^2(\mathbb{B}^n, dA_\alpha)$. It is known that point evaluation functionals are continuous on the harmonic Bergman space, so it possesses a reproducing kernel; i.e., there exists a function $R_\alpha(x, y)$ on $\mathbb{B}^n \times \mathbb{B}^n$, harmonic in each variable, such that

$$f(x) = \int_{\mathbb{B}^n} f(y) R_\alpha(x, y) dA_\alpha(y)$$

for each $f \in L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ and for each $x \in \mathbb{B}^n$.

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The Berezin transform of a bounded linear operator T on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the function $\tilde{T}^\alpha(z)$ on \mathbb{B}^n defined by

$$\tilde{T}^\alpha(z) = \frac{\langle T^{(\alpha)}R_{\alpha z}, R_{\alpha z} \rangle}{\langle R_{\alpha z}, R_{\alpha z} \rangle} = \frac{T^{(\alpha)}R_{\alpha z}(z)}{R_\alpha(z, z)},$$

where, for the sake of brevity, we have denoted $R_{\alpha z}(w) := R_\alpha(z, w)$.

Finally, for $f \in L^\infty(\mathbb{B}^n)$, the Toeplitz operator T_f with symbol f is the operator on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ defined by

$$T_f g = Q_\alpha(fg),$$

where $Q_\alpha : L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha) \rightarrow L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the orthogonal projection. That is,

$$T_f g(z) = \int_{\mathbb{B}^n} g(x)f(x)R_\alpha(z, x) dA_\alpha(x).$$

It was shown by C. Liu [7] that if $n = 2$ (so that \mathbb{B}^2 is just the unit disc in the complex plane \mathbb{C}), then for $f \in C(\mathbb{B}^n)$,

$$(1) \quad \tilde{T}_f^{(\alpha)} \rightarrow f \quad \text{uniformly, and}$$

$$(2) \quad \|T_f^{(\alpha)}\| \rightarrow \|f\|_\infty$$

as $\alpha \rightarrow \infty$.

This extends the same result known previously for Toeplitz operators on Bergman spaces of holomorphic functions, which finds important applications in mathematical physics (quantization on Kähler manifolds; see e.g. [4]).

The aim of the present paper is to generalize Liu’s result also to $n \geq 3$. To do this, we first establish a (reasonably) explicit formula for the kernels $R_\alpha(x, y)$; this is done in Section 2. Our main result (the generalization of (1) and (2)) is proved in Section 3.

We remark that we actually obtain a somewhat stronger result than (1); namely, we show that for any $f \in BC(\mathbb{B}^n) := C(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n)$ we also have

$$\tilde{T}_f^{(\alpha)}(z) \rightarrow f(z)$$

as $\alpha \rightarrow \infty$ for all $z \in \mathbb{B}^n$. This gives a new piece of information even for the original case $n = 2$.

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2. REPRODUCING KERNEL

In this part we will find an explicit formula for the reproducing kernel of the space $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ of all harmonic functions on the unit ball in \mathbb{R}^n square-integrable with respect to the measure $dA_\alpha(y)$, when α is an integer. Let $\mathcal{H}_m(\mathbb{R}^n)$ denote the space of the harmonic polynomials on \mathbb{R}^n that are homogeneous of degree m . For each $x \in \mathbb{R}^n$, there exists a unique function $\mathcal{Z}_m(\cdot, x) \in \mathcal{H}_m(S)$ such that

$$(3) \quad p(x) = \int_S p(\xi) \overline{\mathcal{Z}_m(\xi, x)} d\sigma(\xi),$$

for all $p \in \mathcal{H}_m(S)$. The polynomial $\mathcal{Z}_m(\cdot, x)$ is called the zonal harmonic of degree m and pole x . See e.g. [3, p. 94]. It extends to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by setting

$$\mathcal{Z}_m(x, y) = |x|^m |y|^m \mathcal{Z}_m(x/|x|, y/|y|)$$

for $m > 0$, and for $m = 0$, $\mathcal{Z}_0 = 1$.

Passing to polar coordinates $z = r\xi$ ($r > 0, \xi \in S$), the Lebesgue measure becomes

$$dz = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1} dr d\sigma(\xi).$$

From (3), we have therefore for any $p \in \mathcal{H}_m$,

$$\begin{aligned} & \int_{\mathbb{B}^n} p(y) \mathcal{Z}_m(x, y) dA_\alpha(y) \\ &= \frac{2\Gamma(\frac{n}{2} + \alpha + 1)}{\alpha! \Gamma(\frac{n}{2})} \int_0^1 r^{n-1} (1-r^2)^\alpha \int_S p(r\xi) \mathcal{Z}_m(x, r\xi) d\sigma(\xi) dr \\ &= \frac{2\Gamma(\frac{n}{2} + \alpha + 1)}{\alpha! \Gamma(\frac{n}{2})} \int_0^1 r^{n+2m-1} (1-r^2)^\alpha \left(\int_S p(\xi) \mathcal{Z}_m(x, \xi) d\sigma(\xi) \right) dr \\ &= \frac{2\Gamma(\frac{n}{2} + \alpha + 1)}{\alpha! \Gamma(\frac{n}{2})} p(x) \int_0^1 r^{n+2m-1} (1-r^2)^\alpha dr \\ (4) \quad &= \frac{\Gamma(\frac{n}{2} + \alpha + 1) \alpha! (m + \frac{n}{2} - 1)!}{\alpha! \Gamma(\frac{n}{2}) (\alpha + m + \frac{n}{2})!} p(x) \end{aligned}$$

for each $x \in \mathbb{R}^n$.

Now recall that for any orthonormal basis $\{\varphi_j\}$ of $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$, the reproducing kernel R_α is given by

$$R_\alpha(x, \bar{y}) = \sum_{j=1}^\infty \varphi_j(x) \overline{\varphi_j(y)}.$$

(See e.g. [2].) Now each \mathcal{H}_m is a closed subspace of $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$, the spaces \mathcal{H}_m and \mathcal{H}_k are orthogonal if $m \neq k$, and the span of all $\mathcal{H}_m, m \geq 0$, is the whole $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$; in other words,

$$\bigoplus_{m=1}^\infty \mathcal{H}_m = L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha).$$

Thus if we choose a basis $\{\varphi_{mj}\}_{j=1}^{\dim \mathcal{H}_m}$ in each \mathcal{H}_m so that

$$\sum_{j=1}^{\dim \mathcal{H}_m} \varphi_j(x) \overline{\varphi_j(y)} := K_m(x, y)$$

is the reproducing kernel of \mathcal{H}_m , then $\bigcup_{m=0}^\infty \{\varphi_{mj}\}_{j=1}^{\dim \mathcal{H}_m}$ is a basis for the whole $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$; consequently,

$$(5) \quad K(x, y) = \sum_m^\infty K_m(x, y).$$

On the other hand, from (4) we get

$$K_m(x, y) = \frac{\Gamma(\frac{n}{2})(\alpha + m + \frac{n}{2})!}{\Gamma(\frac{n}{2} + \alpha + 1)(m + \frac{n}{2} - 1)!} \mathcal{Z}_m(x, y).$$

Thus we arrive at the following result.

Proposition 2.1.

$$R_\alpha(x, y) = \sum_{m=0}^\infty \frac{\Gamma(\frac{n}{2})(\alpha + m + \frac{n}{2})!}{\Gamma(\frac{n}{2} + \alpha + 1)(m + \frac{n}{2} - 1)!} \mathcal{Z}_m(x, y).$$

It should be noted that this result is definitely not new, see e.g. [5, p. 32 (1)], or [6, Section 2], or [9, Proposition 3], but it is convenient to recall it here.

For α an integer, the last sum can be summed explicitly. Recall [3, p. 178] that the usual Poisson kernel

$$P(x, y) = \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{\frac{n}{2}}}$$

is equal to

$$P(x, y) = \sum_{m=0}^\infty \mathcal{Z}_m(x, y),$$

for $x, y \in B$. Hence, for α an integer ($\alpha = 0, 1, 2, \dots$) we have

$$\begin{aligned} \left(t \frac{d}{dt} + \alpha\right) P(tx, y) &= \sum_{m=0}^\infty \left(t \frac{d}{dt} + \alpha\right) \mathcal{Z}_m(tx, y) \\ &= \sum_{m=0}^\infty \left(t \frac{d}{dt} + \alpha\right) t^m \mathcal{Z}_m(x, y) \\ &= \sum_{m=0}^\infty (m + \alpha) t^m \mathcal{Z}_m(x, y). \end{aligned}$$

By iteration it follows that

$$\begin{aligned} \left(t \frac{d}{dt} + \frac{n}{2}\right) \dots \left(t \frac{d}{dt} + \frac{n}{2} + \alpha\right) P(tx, y) \\ = \sum_{m=0}^\infty \left(m + \frac{n}{2}\right) \dots \left(m + \frac{n}{2} + \alpha\right) t^m \mathcal{Z}_m(x, y). \end{aligned}$$

Consequently,

$$\begin{aligned} \left(t \frac{d}{dt} + \frac{n}{2}\right) \dots \left(t \frac{d}{dt} + \frac{n}{2} + \alpha\right) P(tx, y) \Big|_{t=1} \\ = \sum_{m=0}^\infty \left(m + \frac{n}{2}\right) \dots \left(m + \frac{n}{2} + \alpha\right) t^m \mathcal{Z}_m(x, y) \\ = \sum_{m=0}^\infty \frac{(m + \frac{n}{2} + \alpha)!}{(m + \frac{n}{2} - 1)!} \mathcal{Z}_m(x, y) \\ = \frac{2\Gamma(\frac{n}{2} + \alpha + 1)}{\alpha! \Gamma(\frac{n}{2})} \alpha! R_\alpha(x, y), \end{aligned}$$

so we get the following formula.

Proposition 2.2. *Let $x, y \in B$. Then*

$$(6) \quad R_\alpha(x, y) = \frac{\Gamma(\frac{n}{2})}{2\Gamma(\frac{n}{2} + \alpha + 1)} \mathcal{D}_{\alpha+1} P(tx, y) \Big|_{t=1},$$

where

$$\mathcal{D}_{\alpha+1} = \left(t \frac{d}{dt} + \frac{n}{2}\right) \left(t \frac{d}{dt} + \frac{n}{2} + 1\right) \cdots \left(t \frac{d}{dt} + \frac{n}{2} + \alpha\right).$$

Remark. The formula (6) has apparently also appeared in [5, p. 32 (1)], though in the somewhat different form

$$R_{\alpha}(x, y) = \left[\rho^{1-n} \left(\frac{\partial}{\partial \rho} \right)^{\alpha+1} \rho^{n+\alpha} P(x, \rho^2 y') \right]_{\rho=\sqrt{|y|}}.$$

3. MAIN RESULT

Recall that the Möbius transformation φ_z is the smooth map of \mathbb{B}^n onto itself defined for each $z \in \mathbb{B}^n$ by

$$(7) \quad \varphi_z(w) = \frac{|w-z|^2 z - (1-|z|^2)(w-z)}{1-2\langle w, z \rangle + |w|^2|z|^2},$$

where $\langle w, z \rangle := w_1 z_1 + \cdots + w_n z_n$ denotes the usual scalar product in \mathbb{R}^n . In the next lemma, we summarize the properties of the mapping φ_z . The proofs can be found e.g. in [1].

Lemma 3.1. *For every $z \in \mathbb{B}^n$, φ_z has the following properties:*

- (1) $\varphi_z(0) = z$ and $\varphi_z(z) = 0$,
- (2) φ_z is an involution, i.e. $\varphi_z \circ \varphi_z = id$, the identity mapping,
- (3) the identity

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{1-2\langle w, z \rangle + |w|^2|z|^2}$$

holds for every $z, w \in \mathbb{B}^n$,

- (4) the identity

$$|\varphi'_z(w)| = \frac{1-|z|^2}{1-2\langle w, z \rangle + |w|^2|z|^2}$$

holds for every $z, w \in \mathbb{B}^n$.

Theorem 3.2. *If $f \in BC(\mathbb{B}^n)$, the space of all bounded continuous functions on \mathbb{B}^n , then for each $z \in \mathbb{B}^n$,*

$$\tilde{T}_f^{(\alpha)}(z) \rightarrow f(z)$$

as $\alpha \rightarrow \infty$ through the integers.

For $f \in C(\overline{\mathbb{B}^n})$, the convergence is even uniform on \mathbb{B}^n .

Proof. By the definitions,

$$\begin{aligned} \tilde{T}_f^{(\alpha)}(z) &= \frac{\langle T_f^{\alpha} R_{\alpha z}, R_{\alpha z} \rangle}{R_{\alpha}(z, z)} = \frac{\langle Q_{\alpha} f R_{\alpha z}, R_{\alpha z} \rangle}{R_{\alpha}(z, z)} = \frac{\langle f R_{\alpha z}, R_{\alpha z} \rangle}{R_{\alpha}(z, z)} \\ &= \int_{\mathbb{B}^n} \frac{f(w) R_{\alpha z}(w) \overline{R_{\alpha z}(w)}}{R_{\alpha}(z, z)} dA_{\alpha}(w) \\ &= \int_{\mathbb{B}^n} f(w) \frac{|R_{\alpha}(z, w)|^2}{R_{\alpha}(z, z)} dA_{\alpha}(w). \end{aligned}$$

Also, by the reproducing property,

$$\int_{\mathbb{B}^n} f(z) \frac{|R_{\alpha}(z, w)|^2}{R_{\alpha}(z, z)} dA_{\alpha}(w) = f(z) \frac{\langle R_{\alpha z}, R_{\alpha z} \rangle}{R_{\alpha}(z, z)} = f(z).$$

Hence

$$\begin{aligned} \tilde{T}_f^{(\alpha)}(z) - f(z) &= \int_{\mathbb{B}^n} (f(w) - f(z)) \frac{|R_\alpha(z, w)|^2}{R_\alpha(z, z)} dA_\alpha(w) \\ &= \int_{|\varphi_z(w)| \leq \delta} + \int_{|\varphi_z(w)| > \delta}, \end{aligned}$$

for any $0 < \delta < 1$.

By the continuity of f , we may, for each fixed z and $\varepsilon > 0$, choose $\delta > 0$ so small that

$$(8) \quad |f(w) - f(z)| < \varepsilon$$

whenever $|\varphi_z(w)| < \delta$. Then

$$\int_{|\varphi_z(w)| \leq \delta} \leq \varepsilon \underbrace{\int \frac{|R_\alpha(z, w)|^2}{R_\alpha(z, z)} dA_\alpha(w)}_1 = \varepsilon.$$

To estimate the second integral we use the following lemma:

Lemma 3.3. *There exist constants c and C , depending only on α and n , such that for all $z, w \in \mathbb{B}^n$,*

$$(9) \quad |R_\alpha(z, w)| \leq \frac{C}{[z, w]^{\frac{n+\alpha}{2}}}$$

and

$$(10) \quad \frac{c}{[z, z]^{\frac{n+\alpha}{2}}} \leq R_\alpha(z, z) \leq \frac{C}{[z, z]^{\frac{n+\alpha}{2}}}.$$

Here, for the sake of brevity, we have introduced the notation

$$[z, w] = 1 - 2\langle z, w \rangle + |z|^2|w|^2.$$

Postponing the proof of the lemma for a moment, using (9) and (10) we can estimate the integral over $|\varphi_z(w)| \geq \delta$ by

$$\begin{aligned} \int_{|\varphi_z(w)| > \delta} &\leq 2\|f\|_\infty \int_{|\varphi_z(w)| > \delta} \frac{|R_\alpha(z, w)|^2}{R_\alpha(z, z)} dA_\alpha(w) \\ &\leq 2\|f\|_\infty \frac{C^2}{c} \int_{|\varphi_z(w)| > \delta} \frac{(1 - |z|^2)^{n+\alpha}}{[z, w]^{n+\alpha}} dA_\alpha(w) \\ &= 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|\varphi_z(w)| > \delta} \frac{(1 - |z|^2)^{n+\alpha} (1 - |w|^2)^{n+\alpha}}{[z, w]^{n+\alpha}} \frac{dw}{(1 - |w|^2)^n} \\ &= 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|\varphi_z(w)| > \delta} (1 - |\varphi_z(w)|^2)^{n+\alpha} \frac{dw}{(1 - |w|^2)^n}. \end{aligned}$$

Note that the measure

$$d\lambda(x) = \frac{dx}{(1 - |x|^2)^n}$$

is invariant on \mathbb{B}^n , in the sense that

$$d\lambda(\varphi_a(x)) = \frac{|\varphi'_a(x)|^n dx}{(1 - |\varphi_a(x)|^2)^n} = d\lambda(x),$$

by Lemma 3.1, parts (3) and (4).

Hence, making the change of variable $\varphi_z(w) = x$, we can continue with

$$\begin{aligned} \int_{|\varphi_z(w)| > \delta} &\leq 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|x| > \delta} (1 - |x|^2)^{n+\alpha} d\lambda(x) \\ &= 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|x| > \delta} (1 - |x|^2)^\alpha dx \\ &\leq 2\|f\|_\infty \frac{C^2}{c} c_\alpha (1 - \delta^2)^\alpha. \end{aligned}$$

Since $c_\alpha \sim \alpha^{n/2}$ as $\alpha \rightarrow \infty$, the right-hand side tends to zero as $\alpha \rightarrow \infty$.

Hence

$$(11) \quad \lim_{\alpha \rightarrow \infty} \left(\int_{|\varphi_z(w)| \leq \delta} + \int_{|\varphi_z(w)| > \delta} \right) \leq \varepsilon.$$

If f is not only bounded and continuous on the ball but even continuous on its closure, then for each $\varepsilon > 0$ we can choose $\delta > 0$ so that (8) holds for all $z \in \mathbb{B}^n$ simultaneously, by uniform continuity. This completes the proof of Theorem 3.2. \square

The proof of (9) actually occurs in [8, Lemma 3.1] taking $t = 0, s = \alpha$ in the operator $Q_{s,t}$ there. For $|y| = 1$, see also [6, Lemma 2.7]. A simple proof of (10) can be found in [9, Proposition 4.1]. For α an integer, it is possible to give another proof using the formula (2.2). Since this might be useful for other applications we include it for completeness.

Proof of Lemma 3.3. Consider the sets of functions

$$\begin{aligned} A_{\beta,r} &:= \left\{ \frac{p(t,z,w)}{[tz,w]^{\beta/2}} \left(\frac{1-t^2|z|^2|w|^2}{[tz,w]^{1/2}} \right)^r : p \text{ a polynomial} \right\}, \\ A_\beta &:= \left\{ \frac{p(t,z,w)}{[tz,w]^{\beta/2}} q \left(\frac{1-t^2|z|^2|w|^2}{[tz,w]^{1/2}} \right) : p, q \text{ polynomials} \right\}. \end{aligned}$$

Differentiation yields

$$\begin{aligned} (12) \quad t \frac{d}{dt} A_{\beta,r} &= t \frac{d}{dt} \left(\frac{p(t,z,w)(1-t^2|z|^2|w|^2)^r}{[tz,w]^{\frac{\beta+r}{2}}} \right) \\ &= \frac{t \frac{\partial p}{\partial t}(t,z,w) \cdot (1-t^2|z|^2|w|^2)}{[tz,w]^{\frac{\beta+r}{2}}} \cdot (1-t^2|z|^2|w|^2)^{r-1} \\ &\quad + \frac{t \cdot r \cdot p(t,z,w)(-2t|z|^2|w|^2)}{[tz,w]^{\frac{\beta+r}{2}}} \cdot (1-t^2|z|^2|w|^2)^{r-1} \\ &\quad - \frac{\frac{\beta+r}{2} \cdot p(t,z,w)(1-t^2|z|^2|w|^2)^r \cdot t \cdot (-2\langle z,w \rangle + 2t|z|^2|w|^2)}{[tz,w]^{\frac{\beta+r}{2}+1}}. \end{aligned}$$

Since

$$\begin{aligned} -2t\langle z,w \rangle + 2t^2|z|^2|w|^2 &= [tz,w] - 1 - t^2|z|^2|w|^2 + 2t^2|z|^2|w|^2 \\ &= [tz,w] - 1 + t^2|z|^2|w|^2 = -(1-t^2|z|^2|w|^2) + [tz,w], \end{aligned}$$

the last term in (12) can be rewritten as

$$\begin{aligned}
 & - \frac{\frac{\beta+r}{2} \cdot p(t, z, w) \cdot (1 - t^2|z|^2|w|^2)}{[tz, w]^{\frac{\beta+r}{2}}} \cdot (1 - t^2|z|^2|w|^2)^{r-1} \\
 & + \frac{\frac{\beta+r}{2} \cdot p(t, z, w) \cdot (1 - t^2|z|^2|w|^2)^{r+1}}{[tz, w]^{\frac{\beta+r+2}{2}}}.
 \end{aligned}$$

Thus we get

$$\begin{aligned}
 t \frac{d}{dt} A_{\beta,r} & \in A_{\beta+1,r-1} + A_{\beta+1,r+1}, & r > 0, \\
 t \frac{d}{dt} A_{\beta,0} & \in A_{\beta+1,0} + A_{\beta+1,1}, & r = 0.
 \end{aligned}$$

For arbitrary $c \in \mathbb{R}$ we thus get

$$\left(t \frac{d}{dt} + c \right) A_{\beta,r} \in A_{\beta} + A_{\beta+1}$$

and

$$\mathcal{D}_{\alpha+1} A_{\beta,r} \in A_{\beta} + \dots + A_{\beta+\alpha+1}.$$

Applying this to $\frac{1-t^2|z|^2|w|^2}{[tz, w]^{n/2}} \in A_{n-1,1}$ we thus obtain by (2.2)

$$(13) \quad R_{\alpha}(z, w) \in A_{n-1} + \dots + A_{n+\alpha} \Big|_{t=1}.$$

Now

$$\begin{aligned}
 [z, w] & = 1 - 2\langle z, w \rangle + |z|^2|w|^2 \geq 1 - 2|z||w| + |z|^2|w|^2 \\
 & = (1 - |z||w|)^2 > 0
 \end{aligned}$$

and

$$[z, w] \leq 1 + 2|z||w| + |z|^2|w|^2 = (1 + |z||w|)^2 \leq 2^2,$$

so

$$0 \leq \frac{1 - |z|^2|w|^2}{[z, w]^{1/2}} \leq \frac{1 - |z|^2|w|^2}{1 - |z||w|} = 1 + |z||w| \leq 2,$$

and thus every function from A_{β} has for $t = 1$ the form

$$\frac{\text{something bounded on } \mathbb{B}^n}{[z, w]^{\beta/2}}.$$

From (13) we thus get

$$R_{\alpha}(z, w) = \frac{\text{something bounded on } \mathbb{B}^n}{[z, w]^{(n+\alpha)/2}},$$

which proves (9).

To prove the second half of the lemma, note that for $z = w$,

$$\begin{aligned}
 R_{\alpha}(z, z) & = \left(t \frac{d}{dt} + n \right) \dots \left(t \frac{d}{dt} + n + \alpha \right) \frac{1 - t^2|z|^4}{(1 - t|z|^2)^n} \\
 & = \left(t \frac{d}{dt} + n \right) \dots \left(t \frac{d}{dt} + n + \alpha \right) \frac{1 + t|z|^2}{(1 - t|z|^2)^{n-1}}.
 \end{aligned}$$

Now again, for any polynomial p ,

$$\begin{aligned} \left(t \frac{d}{dt} + c\right) \frac{p(t, |z|^2)}{(1 - t|z|^2)^\beta} &= c \frac{p(t, |z|^2)}{(1 - t|z|^2)^\beta} + \frac{t \frac{\partial}{\partial t} p(t, |z|^2)}{(1 - t|z|^2)^\beta} + \frac{tp(t, |z|^2)\beta|z|^2}{(1 - t|z|^2)^{\beta+1}} \\ &= \frac{(cp(t, |z|^2) + t \frac{\partial}{\partial t} p(t, |z|^2))(1 - t|z|^2)}{(1 - t|z|^2)^{\beta+1}} \\ &\quad + \frac{tp(t, |z|^2)\beta|z|^2}{(1 - t|z|^2)^{\beta+1}} \\ &= \frac{p^*(t, |z|^2)}{(1 - t|z|^2)^{\beta+1}}, \end{aligned}$$

where p^* is also a polynomial, which further satisfies $p^*(1, 1) = \beta p(1, 1)$. By iteration and taking $|z| = 1, t = 1$, we get

$$R_\alpha(z, z) = \frac{P(|z|^2)}{(1 - |z|^2)^{n+\alpha}},$$

where P is a polynomial satisfying

$$(14) \quad P(1) = 2 \frac{(n + \alpha - 1)!}{(n - 2)!}.$$

Thus $R_\alpha(z, z)(1 - |z|^2)^{n+\alpha}$ is a positive function inside \mathbb{B}^n , which has a nonzero finite limit (14) at the boundary. Consequently, it has positive and finite lower and upper bounds c and C , respectively. This concludes the proof of Lemma 3.3. \square

Corollary 3.4. For any $f \in C(\overline{\mathbb{B}^n})$,

$$\|T_f^{(\alpha)}\| \longrightarrow \|f\|_\infty \quad \text{as } \alpha \rightarrow \infty.$$

Proof. From the Schwarz inequality we get

$$\begin{aligned} |\tilde{T}_f^{(\alpha)}(z)| &= \frac{|\langle T_f^{(\alpha)} R_{\alpha z}, R_{\alpha z} \rangle|}{R_\alpha(z, z)} = \frac{|\langle f R_{\alpha z}, R_{\alpha z} \rangle|}{R_\alpha(z, z)} \leq \frac{\|f R_{\alpha z}\|_{L^2} \|R_{\alpha z}\|_{L^2}}{\|R_{\alpha z}\|_{L^2}^2} \\ &\leq \frac{\|f\|_\infty \|R_{\alpha z}\|_{L^2}^2}{\|R_{\alpha z}\|_{L^2}^2} = \|f\|_\infty. \end{aligned}$$

Taking the supremum over all z gives

$$(15) \quad \|\tilde{T}_f^{(\alpha)}\|_\infty \leq \|f\|_\infty.$$

On the other hand, by Theorem 3.2,

$$\tilde{T}_f^{(\alpha)}(z) \longrightarrow f(z),$$

for all $z \in \mathbb{B}^n$. As $\|\tilde{T}_f^{(\alpha)}\|_\infty \geq |\tilde{T}_f^{(\alpha)}(z)|$ and $|\tilde{T}_f^{(\alpha)}(z)| \rightarrow |f(z)|$, thus

$$\liminf_{\alpha \rightarrow \infty} \|\tilde{T}_f^{(\alpha)}\| \geq |f(z)|.$$

Taking again the supremum over all $z \in \mathbb{B}^n$ yields

$$(16) \quad \liminf_{\alpha \rightarrow \infty} \|\tilde{T}_f^{(\alpha)}\|_\infty \geq \sup_{z \in \mathbb{B}^n} |f(z)| = \|f\|_\infty.$$

From (15) and (16) we therefore have

$$\lim_{\alpha \rightarrow \infty} \|\tilde{T}_f^{(\alpha)}\| = \|f\|_\infty,$$

which proves the “deformation estimate” (2). \square

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