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A SLIGHT IMPROVEMENT TO GARAEV'S SUM PRODUCT ESTIMATE

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0. Introduction

Let A and B be two finite sets of integers. We let

$$A + B = \{a + b : a \in A, b \in B\}$$

and

$$AB = \{ab : a \in A, b \in B\}.$$

There have been many studies of the size of the sum and product sets for the case $A = B$, since Erdös and Szemerèdi made their well-known conjecture that

$$\max(|A + A|, |AA|) \geq C_\epsilon |A|^{2-\epsilon} \forall \epsilon > 0.$$

The conjecture is still open, and the best result to date is due to Solymosi [S], who showed that

$$\max(|A + A|, |AA|) \geq C_\epsilon |A|^{\frac{14}{11}-\epsilon}.$$

In the finite field setting this situation is much more complicated because the main tool, the Szemerèdi-Trotter incidence theorem, does not hold in the same generality. It is known, via the work in [BKT], that if A is a subset of F_p, the field of p elements with p prime, and if $p^\delta < |A| < p^{1-\delta}$, where $\delta > 0$, then one has the sum product estimate

$$\max(|A + A|, |AA|) \geq |A|^{1+\epsilon}$$

for some $\epsilon > 0$. This result has found many applications in combinatorial problems and exponential sum estimates (see e.g. [BKT], [BGK], [G2]). Recently, Garaev [G1] showed that when $|A| < p^{\frac{1}{2}}$, one has the estimate

$$\max(|A + A|, |AA|) \gtrapprox |A|^{\frac{15}{14}}.$$

By using Plünnecke's inequality in a slightly more sophisticated way, we improve this exponent to $\frac{14}{13}$. We believe that further improvements might be possible through aggressive use of the Ruzsa covering.

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1. Preliminaries

Throughout this paper A will denote a fixed set in the field F_p of p elements with p prime. For B, any set, we will denote its cardinality by $|B|$.

Whenever X and Y are quantities we will use

$$X \lesssim Y$$

to mean

$$X \leq CY,$$

where the constant C is universal (i.e. independent of p and A). The constant C may vary from line to line. We will use

$$X \lessapprox Y$$

to mean

$$X \leq C(\log |A|)^{\alpha} Y,$$

and $X \approx Y$ to mean $X \lessapprox Y$ and $Y \lessapprox X$, where C and α may vary from line to line but are universal.

We state some preliminary lemmas, mostly those stated by Garaev, but occasionally with different emphasis.

The first lemma is a consequence of the work of Glibichuk and Konyagin [GK].

Lemma 1.1. *Let $A_1 \subset F_p$ with $1 < |A_1| < p^{\frac{1}{2}}$. Then for any elements a_1, a_2, b_1, b_2 so that*

$$\frac{b_1 - b_2}{a_1 - a_2} + 1 \notin \frac{A_1 - A_1}{A_1 - A_1},$$

we have that for any $A' \subset A_1$ with $|A'| \gtrsim |A_1|$

$$|(a_1 - a_2)A' + (a_1 - a_2)A' + (b_1 - b_2)A'| \gtrsim |A_1|^2.$$

In particular such a_1, a_2, b_1, b_2 exist unless $\frac{A_1 - A_1}{A_1 - A_1} = F_p$. In the case $\frac{A_1 - A_1}{A_1 - A_1} = F_p$, we may find $a_1, a_2, b_1, b_2 \in A_1$ so that

$$|(a_1 - a_2)A_1 + (b_1 - b_2)A_1| \gtrsim |A_1|^2.$$

Sketch of the proof. If $\frac{A_1 - A_1}{A_1 - A_1} \neq F_p$, it is immediate that there exist $a_1, a_2, b_1, b_2 \in A_1$ with $1 + \frac{b_1 - b_2}{a_1 - a_2} \notin \frac{A_1 - A_1}{A_1 - A_1}$. This automatically implies

$$|(a_1 - a_2)A' + (a_1 - a_2)A' + (b_1 - b_2)A'| \gtrsim |A_1|^2.$$

(See [GK]. If $x \notin \frac{A_1 - A_1}{A_1 - A_1}$, then each element of $A_1 + xA_1$ has but one representative $a + xa'$.) On the other hand, if

$$\frac{A_1 - A_1}{A_1 - A_1} = F_p,$$

then one can find $a_1, a_2, b_1, b_2 \in A_1$ so that $\frac{a_1 - a_2}{b_1 - b_2}$ has at most $|A_1|^2$ representatives as $\frac{a_3 - a_4}{b_3 - b_4}$ with $a_3, a_4, b_3, b_4 \in A_1$, which implies that $|A_1 + \frac{a_1 - a_2}{b_1 - b_2} A_1|$ is large. Again, for more details see [GK]. □

The following two lemmas, quoted by Garaev, are due to Ruzsa and may be found in [TV]. The first is usually referred to as Rusza's triangle inequality. The second is a form of Plünnecke's inequality.

Lemma 1.2. *For any subsets X, Y, Z of F_p where X is nonempty, we have*

$$|Y - Z| \leq \frac{|Y - X||X - Z|}{|X|}.$$

Lemma 1.3. *Let $X, B_1, \ldots, B_k$ be any subsets of F_p with*

$$|X + B_i| \leq \alpha_i |X|,$$

for i ranging from 1 to k. Then there exists $X_1 \subset X$ with

$$|X_1 + B_1 + \cdots + B_k| \leq \alpha_1 \ldots \alpha_k |X_1|. \tag{1.1}$$

We record a number of corollaries. The first two can be found in [TV]. We first became aware of the last one in the paper of Garaev [G1].

Corollary 1.4. *Let $X, B_1, \ldots, B_k$ be any subsets of F_p. Then*

$$|B_1 + \cdots + B_k| \leq \frac{|X + B_1| \ldots |X + B_k|}{|X|^{k-1}}.$$

Proof. Simply bound $|B_1 + \cdots + B_k|$ by $|X_1 + B_1 + \cdots + B_k|$ and $|X_1|$ by $|X|$. □

Corollary 1.4 is somewhat wasteful in that X_1 is unlikely to be both a singleton element and a set with the same cardinality as X. By applying Lemma 1.3 iteratively, we obtain the following corollary.

Corollary 1.5. *Let $X, B_1, \ldots, B_k$ be any subsets of F_p. Then there is $X' \subset X$ with $|X'| > \frac{1}{2}|X|$ so that*

$$|X' + B_1 + \cdots + B_k| \lesssim \frac{|X + B_1| \ldots |X + B_k|}{|X|^{k-1}}.$$

Proof. Observe that for any $Y \subset X$ with $|Y| \geq \frac{|X|}{2}$, we have that

$$\frac{|Y + B_i|}{|Y|} \lesssim \frac{|X + B_i|}{|X|}.$$

Now recursively apply Lemma 1.3. That is, first apply it to $X, B_1, \ldots, B_k$ obtaining a set X_1 satisfying

$$|X_1 + B_1 + \cdots + B_k| \lesssim \frac{|X + B_1| \ldots |X + B_k|}{|X|^k} |X_1|.$$

If $|X_1| > \frac{1}{2}|X|$, then stop and let $X' = X_1$. Otherwise apply Lemma 1.3 to $X \backslash X_1, B_1, \ldots, B_k$. Proceeding recursively if $|X_1 \cup \cdots \cup X_{j-1}| > \frac{1}{2}|X|$, set

$$X' = X_1 \cup \cdots \cup X_{j-1};$$

otherwise obtain the inequality

$$|X_j + B_1 + \cdots + B_k| \lesssim \frac{|X + B_1| \ldots |X + B_k|}{|X|^k} |X_j|.$$

Summing all the inequalities we obtained before stopping gives us the desired result. □

Corollary 1.6. *Let $A \subset F_p$ and let $a, b \in A$. Then we have the inequalities*

$$|aA + bA| \leq \frac{|A+A|^2}{|aA \cap bA|}$$

and

$$|aA - bA| \leq \frac{|A+A|^2}{|aA \cap bA|}.$$

Proof. To get the first inequality, apply Corollary 1.4 with $k = 2$, $B_1 = aA$, $B_2 = bA$, and $X = aA \cap bA$.

To get the second inequality, apply Lemma 1.2 with $Y = aA$, $Z = -bA$ and $X = -(aA \cap bA)$. □

2. Modified Garaev's inequality

In this section, we slightly modify Garaev's argument to obtain

Theorem 2.1. *Let $A \subset F_p$ with $|A| < p^{\frac{1}{2}}$; then*

$$\max(|AA|, |A+A|) \gtrapprox |A|^{\frac{14}{13}}.$$

Proof. Following Garaev, we observe that

$$\sum_{a \in A} \sum_{b \in A} |aA \cap bA| \geq \frac{|A|^4}{|AA|}.$$

Therefore, we can find an element $b_0 \in A$, a subset $A_1 \subset A$ and a number N satisfying

$$|b_0 A \cap aA| \approx N,$$

for every $a \in A_1$. Further

$$N \gtrapprox \frac{|A|^2}{|AA|} \tag{2.1}$$

and

$$|A_1| N \gtrapprox \frac{|A|^3}{|AA|}. \tag{2.2}$$

Now there are two cases. In the first case, we have

$$\frac{A_1 - A_1}{A_1 - A_1} = F_p.$$

If so, applying Lemma 1.1, we can find $a_1, a_2, b_1, b_2 \in A_1$ so that

$$|A_1|^2 \lesssim |(a_1 - a_2)A_1 + (b_1 - b_2)A_1| \leq |a_1 A - a_2 A + b_1 A - b_2 A|.$$

Apply Corollary 1.4 with $k = 4$, and with $B_1 = a_1 A$, $B_2 = -a_2 A$, $B_3 = b_1 A$, $B_4 = -b_2 A$, and $X = b_0 A$. Then we apply Corollary 1.6 to bound above $|X + B_j|$. This yields

$$|A_1|^2 \lesssim \frac{|A+A|^8}{N^4 |A|^3}$$

or

$$|A_1|^2 N^4 |A|^3 \lessapprox |A+A|^8.$$

Applying (2.2), we get

$$N^2 |A|^9 \lessapprox |A+A|^8 |AA|^2, \tag{2.3}$$

and applying (2.1), we get

$$|A|^{13} \lessapprox |A+A|^8|AA|^4. \tag{2.4}$$

The estimate (2.4) implies that

$$\max(|A+A|,|AA|) \gtrapprox |A|^{\frac{13}{12}} \gtrapprox |A|^{\frac{14}{13}},$$

so that we have more than we need in this case.

Thus we are left with the case that

$$\frac{A_1 - A_1}{A_1 - A_1} \neq F_p.$$

Thus we can find a_1, a_2, b_1, b_2 so that for any refinement $A' \subset A_1$ with $|A'| \gtrsim |A_1|$, we have

$$|A_1|^2 \lesssim |(a_1-a_2)A' + (a_1-a_2)A' + (b_1-b_2)A'|.$$

Now we apply Corollary 1.5, choosing A' so that

$$|(a_1-a_2)A' + (a_1-a_2)A_1 + (b_1-b_2)A_1| \lesssim \frac{|A+A||(a_1-a_2)A_1 + (b_1-b_2)A_1|}{|A_1|}.$$

This is where we have improved on Garaev's original argument.

Then, as in the first case, estimating

$$|(a_1-a_2)A_1 + (b_1-b_2)A_1| \leq |a_1A - a_2A + b_1A - b_2A|$$

and applying Corollary 1.4 with $X = b_0A$ and Corollary 1.6, we obtain

$$|A_1|^3 N^4 |A|^3 \lesssim |A+A|^9.$$

Applying (2.2), we get

$$N|A|^{12} \lessapprox |A+A|^9|AA|^3. \tag{2.5}$$

Now applying (2.1), we get

$$|A|^{14} \lessapprox |A+A|^9|AA|^4. \tag{2.6}$$

Inequality (2.6) proves the theorem. □

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