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REGULARITY FOR THE NAVIER–STOKES EQUATIONS WITH SLIP BOUNDARY CONDITION

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ABSTRACT. For the Navier-Stokes equations with slip boundary conditions, we obtain the pressure in terms of the velocity. Based on the representation, we consider the relationship in the sense of regularity between the Navier-Stokes equations in the whole space and those in the half space with slip boundary data.

1. INTRODUCTION

In this paper we consider Navier-Stokes equations with slip boundary condition. Consider the Navier-Stokes equations in $\mathbb{R}^3_+ \times (0,T)$:

$$\begin{array}{rcl} \operatorname{div} \mathbf{u} & = & 0 \\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p & = & 0 \end{array} \quad \text{in } \mathbb{R}^3_+ \times (0,T) \tag{1.1}$$

for some positive $T > 0$ satisfying the initial boundary condition;

$$\partial_3 u_1 = \partial_3 u_2 = u_3 = 0 \text{ on } x_3 = 0, \qquad \mathbf{u}(\mathbf{x}) \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty, \tag{1.2}$$

and $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) = (u_{0,1}(\mathbf{x}), u_{0,2}(\mathbf{x}), u_{0,3}(\mathbf{x}))$, where $\mathbf{u}_0 \in L^2(\mathbb{R}^3_+)$ with $\operatorname{div}\mathbf{u}_0 = 0$ in $\mathbb{R}^3_+$. The existence of the suitable weak solution $\mathbf{u}$ in $L^2(0,T;H^1_0(\mathbb{R}^3_+))$ was mentioned in [7] for the domain $\mathbb{R}^3$ or a bounded domain, in [20] and [1] for the half space, and [11] for the exterior domain. The suitably weak solution $(\mathbf{u}, p)$ satisfies Navier-Stokes equations in the sense of distribution, with the properties of global energy inequality

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3_+} \le \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}$$

and localized energy inequality:

$$2\iint |\nabla\mathbf{u}|^2\phi \le \int \left[|u|^2(\phi_t + \Delta\phi) + (|\mathbf{u}|^2 + 2p)\mathbf{u}\cdot\nabla\phi\right]$$

for each real-valued $\phi \in C_0^\infty(\mathbb{R}^3_+ \times (0,T))$ with $\phi \ge 0$.

The slip boundary condition has been considered in terms of the study of the free boundary problem by several mathematicians (see [24], [25], [12], [17], [13], [19],

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[23], etc.), and recently Beirao da Veiga [2, 3, 4, 5] has considered its regularity. In particular, in [4] he derived the regularity result for the weak solutions of (1.1) and (1.2), which is an extension of the regularity result in $\mathbb{R}^3$ by P. Constantin and C. Fefferman [10]. (See also [6].)

In this work, we represent the pressure by the velocity as a first step in the half space $\mathbb{R}^3_+$. Moreover, we show that the extension of $(\mathbf{u}, p)$ by (even, even, odd, even) reflection will give a solution of the Navier-Stokes equation in the whole space. Then we will conclude that almost every regularity result obtained for the solution of the Navier-Stokes equations in the whole space hold for the solution of the Navier-Stokes equations with slip boundary conditions in the half space. We emphasize that our observation will give another, but simpler, reasoning for the regularity result in [4].

2. Main results

Proposition 2.1. *Suppose* $\mathbf{u} = (u_1, u_2, u_3)$, p *is a solution of the incompressible Navier-Stokes equation* (1.1) *with slip boundary condition* (1.2) *in the half space* $\mathbb{R}^3_+$ *with initial data* $\mathbf{u}_0 = (u_{0,1}, u_{0,2}, u_{0,3})$, *where* $\nabla \cdot \mathbf{u}_0 = 0$ *in* $\mathbb{R}^3_+$, $\partial_3 u_{0,1} = \partial_3 u_{0,2} = u_{0,3} = 0$ *on* $x_3 = 0$, *and* $\lim_{|\mathbf{x}| \to \infty} \mathbf{u}_0 = 0$. *Then* p *has the following representation. For almost all time* $t \in (0, T)$

$$p(\mathbf{x}, t) = \frac{-\delta_{ij}}{3}(u_i^* u_j^*)(\mathbf{x}, t) + \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{\partial^2}{\partial y_i \partial y_j}\frac{1}{|\mathbf{x}-\mathbf{y}|}[(u_i^* u_j^*)(\mathbf{y}, t)]d\mathbf{y} \tag{2.1}$$

in the sense of distributions, where δ_{ij} *is the Kronecker delta function. Here,* $\mathbf{u}^*(\mathbf{y}) = \mathbf{u}(\mathbf{y})$ *for* $y_3 > 0$, $u_1^*(\mathbf{y}, t) = u_1(\mathbf{y}^*, t), u_2^*(\mathbf{y}, t) = u_2(\mathbf{y}^*, t), u_3^*(\mathbf{y}, t) = -u_3(\mathbf{y}^*, t)$ *for* $y_3 < 0$, *and* $\mathbf{y}^* = (y_1, y_2, -y_3)$.

Proof. Let $Q = \mathbb{R}^3_+ \cup \{x_3 = 0\}$, $Q_T = Q \times [0, T)$. Without loss of generality, assume $\mathbf{u}_0 \in C^1(Q)$ and $\mathbf{u} \in L^2(0, T; C^1(Q))$ considering the existence space $L^2(0, T; H^1(Q))$.

Set $\mathbf{f}(\mathbf{x}, t) = -[(\mathbf{u} \cdot \nabla)\mathbf{u}](\mathbf{x}, t)$ for $x_3 \geq 0$. Define $\mathbf{f}^* = (f_1^*, f_2^*, f_3^*)$ by

$$f_1^*(\mathbf{x}, t) = \begin{cases} f_1(\mathbf{x}, t), & \text{if } x_3 \geq 0, \\ f_1(\mathbf{x}^*, t), & \text{if } x_3 < 0, \end{cases}$$

$$f_2^*(\mathbf{x}, t) = \begin{cases} f_2(\mathbf{x}, t), & \text{if } x_3 \geq 0, \\ f_2(\mathbf{x}^*, t), & \text{if } x_3 < 0, \end{cases}$$

$$f_3^*(\mathbf{x}, t) = \begin{cases} f_3(\mathbf{x}, t), & \text{if } x_3 \geq 0, \\ -f_3(\mathbf{x}^*, t), & \text{if } x_3 < 0. \end{cases}$$

Here $\mathbf{x}^*$ is the even-even-odd extension of $\mathbf{x}$ like $\mathbf{y}^*$. Since $u_3 = 0$ on $x_3 = 0$, one has $\partial_1 u_3 = \partial_2 u_3 = 0$ on $x_3 = 0$, therefore $f_3 = 0$. By noting $\partial_3 u_{0,1} = \partial_3 u_{0,2} = u_{0,3} = 0$, and $f_3 = 0$ on $x_3 = 0$, it follows that $\mathbf{u}_0^* \in C(\mathbb{R}^3)$ and $\mathbf{f}^* \in C(\mathbb{R}^3)$.

Observe that $f_1^*(\mathbf{x}, t) = -[(\mathbf{u}^* \cdot \nabla)u_1^*](\mathbf{x}, t)$, $f_2^*(\mathbf{x}, t) = -[(\mathbf{u}^* \cdot \nabla)u_2^*](\mathbf{x}, t)$, and $f_3^*(\mathbf{x}, t) = -[(\mathbf{u}^* \cdot \nabla)u_3^*](\mathbf{x}, t)$ for $x_3 < 0$. Hence, $\mathbf{f}^*(\mathbf{x}, t) = -\text{div}\,(\mathbf{u} \otimes \mathbf{u})(\mathbf{x}, t)$ for $x_3 > 0$ and $\mathbf{f}^*(\mathbf{x}, t) = -\text{div}\,(\mathbf{u}^* \otimes \mathbf{u}^*)(\mathbf{x}, t)$ for all $x_3 < 0$. Since $\partial_3 u_1 = \partial_3 u_2 = u_3 = 0$ on $x_3 = 0$, it follows that

$$\mathbf{f}^*(\mathbf{x}, t) = -\text{div}(\mathbf{u}^* \otimes \mathbf{u}^*)(\mathbf{x}, t)$$

in the sense of distributions.

Now we construct $(\mathbf{v}, q)$ as a solution of the Stokes system in $\mathbb{R}^3$,

$$\begin{array}{rcl} \operatorname{div} \mathbf{v} &=& 0 \\ \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla q &=& \mathbf{f}^* \end{array} \quad \text{in } \mathbb{R}^3 \times (0,T),$$

with initial data $\mathbf{v}(\mathbf{x}, 0) = \mathbf{u}_0^*(\mathbf{x})$ and infinity condition $\mathbf{v}(\mathbf{x}, t) \to \mathbf{0}$ as $|\mathbf{x}| \to \infty$.

Then q satisfies the Laplace equation $\Delta q(\mathbf{x}, t) = \operatorname{div} \mathbf{f}^*(\mathbf{x}, t)$ in $\mathbb{R}^3 \times (0,T)$. We try to find q integrable. By integral representation, q is expressed by

$$\begin{aligned} q(\mathbf{x}, t) &= -\frac{3}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|} \partial_j f_j^*(\mathbf{y}, t) d\mathbf{y} \\ &= \frac{-\delta_{ij}}{3} (u_i^* u_j^*)(\mathbf{x}, t) + \frac{3}{4\pi} \int_{\mathbb{R}^3} \partial_i \partial_j \frac{1}{|\mathbf{x}-\mathbf{y}|} (u_i^* u_j^*)(\mathbf{y}, t) d\mathbf{y}, \end{aligned}$$

where δ_{ij} is the Kronecker delta function.

Our aim is to show $\mathbf{u} \equiv \mathbf{v}$ and $p \equiv q + c_0$ in the half space $\mathbb{R}^3_+ \times (0,T)$. Set $W_1(\mathbf{x}, t) = v_1(\mathbf{x}, t) - v_1(\mathbf{x}^*, t), W_2(\mathbf{x}, t) = v_2(\mathbf{x}, t) - v_2(\mathbf{x}^*, t)$, $W_3(\mathbf{x}, t) = v_3(\mathbf{x}, t) + v_3(\mathbf{x}^*, t)$ and $Q(\mathbf{x}, t) = q(\mathbf{x}, t) - q(\mathbf{x}^*, t)$.

Then $\mathbf{W}$ and Q satisfy

$$\begin{array}{rcl} \operatorname{div} \mathbf{W} &=& 0 \\ \mathbf{W}_t - \nu\Delta\mathbf{W} + \nabla Q &=& 0 \end{array} \quad \text{in } \mathbb{R}^3 \times (0,T)$$

and $\mathbf{W}(\mathbf{x}, 0) = \mathbf{0}$, $W(\mathbf{x}, t) \to \mathbf{0}$ as $|\mathbf{x}| \to 0$. By the uniqueness of the Stokes solution, we conclude that $\mathbf{W} \equiv 0$. Therefore, $v_1(\mathbf{x}, t) = v_1(\mathbf{x}^*, t), v_2(\mathbf{x}, t) = v_2(\mathbf{x}^*, t)$ and $v_3(\mathbf{x}, t) = -v_3(\mathbf{x}^*, t)$. From the above identity, we deduce that $(\partial_3 v_1)(x_1, x_2, 0, t) = (\partial_3 v_2)(x_1, x_2, 0, t) = v_3(x_1, x_2, 0, t) = 0$.

Now, it only remains to show $\mathbf{v} \equiv \mathbf{u}$ and $p = q + c_0$ for $x_3 > 0$. Let $\mathbf{U} = \mathbf{u} - \mathbf{v}$, $P = p - q$. Then $\mathbf{U}(\mathbf{x}, 0) = \mathbf{0}$, $(\partial_3 U_1)(x_1, x_2, 0, t) = (\partial_3 U_2)(x_1, x_2, 0, t) = U_3(x_1, x_2, 0, t) = 0$ and $\mathbf{U}(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to 0$. Moreover, $\mathbf{U}$ and P satisfy

$$\begin{array}{rcl} \operatorname{div} \mathbf{U} &=& 0 \\ \mathbf{U}_t - \nu\Delta\mathbf{U} + \nabla P &=& 0 \end{array} \quad \text{in } \mathbb{R}^3_+ \times (0,T).$$

From the uniqueness of the Stokes solution, we again conclude that $\mathbf{U} \equiv 0$ in $x_3 > 0$. This again implies that $P \equiv \mathit{const}$ and that $p = q + c_0$ for a constant c_0. Since we try to find q integrable, we may ignore the constant c_0. □

Remark. If $p \in L^r(\mathbb{R}^3_+)$, then the constant c_0 is zero, so that we have $\|p\|_{L^r} \leq c\|\mathbf{u}\|^2_{L^{2r}}$ for $1 < r < \infty$ for a constant c depending only on r.

During the proof of Proposition 2.1, we also observe that $\mathbf{u}^*$ and p^* satisfy the Navier-Stokes equations in the whole space with initial velocity $\mathbf{u}_0^*$, and we state it as a corollary.

Corollary 2.2. *Expand* $\mathbf{u}$, p *to the whole space by*

$$(\mathbf{u}^*, p^*)(\mathbf{y}, t) = \begin{cases} (\mathbf{u}, p)(\mathbf{y}, t) & \text{for } y_3 > 0, \\ (u_1, u_2, -u_3, p)(\mathbf{y}^*, t) & \text{for } y_3 < 0, \end{cases}$$

where $\mathbf{y}^* = (y_1, y_2, -y_3)$. *Then* $\mathbf{u}^*$ *and* p^* *satisfy the Navier-Stokes equations in the whole space with initial velocity* $\mathbf{u}_0^*$, *where*

$$\mathbf{u}_0^*(\mathbf{y}, t) = \begin{cases} \mathbf{u}_0(\mathbf{y}) & \text{for } y_3 > 0, \\ (u_{0,1}, u_{0,2}, -u_{0,3})(\mathbf{y}^*) & \text{for } y_3 < 0. \end{cases}$$

Therefore, most regularity results for the Navier-Stokes equations in the whole space are also applicable to those in the half space as follows.

Theorem 2.3. *Suppose* $(\mathbf{u}, p)$ *is a suitably weak solution of the Navier-Stokes equations* (1.1) *in the half space with slip boundary* (1.2). *There is* $\epsilon_0 > 0$*, and* C_k *such that*

$$\int_{Q_1^+} |\mathbf{u}|^3 + |p|^{3/2} dxdt < \epsilon_0$$

implies that

$$\sup_{Q_{1/2}^+} |\nabla^k \mathbf{u}| \le C_k.$$

Proof. Let us recall that $\mathbf{u}^*, p^*$ defined in Corollary 2.2 satisfies Navier-Stokes equations in the whole space. Hence by Proposition 1 in [7] we have that there is ϵ_1 and C_k such that

$$\int_{Q_1} |\mathbf{u}^*|^3 + |p^*|^{3/2} dxdt < \epsilon_1$$

implies

$$\sup_{Q_{1/2}} |\nabla^k \mathbf{u}^*| \le C_k.$$

Observe that

$$\int_{Q_1} |\mathbf{u}^*|^3 + |p^*|^{3/2} dxdt = 2 \int_{Q_1^+} |\mathbf{u}|^3 + |p|^{3/2} dxdt.$$

Hence if we take $\epsilon_0 = \frac{1}{2}\epsilon_1$, then we complete the proof. (See also [15] and [20].) □

Remark. By the same argument as in Theorem 2.3, the following results also hold for the solutions of the Navier-Stokes equations (1.1) in the half space with boundary condition (1.2):

1. The only self-similar solution of the Navier-Stokes equation (1.1) with slip boundary condition (1.2) is equal to zero. (See [18] and [26].)
2. If $\mathbf{u} \in L^{q,s}(Q_1^+)$, $\frac{3}{q} + \frac{2}{s} \le 1$, $q \ge 3$, then $\mathbf{u}$ is smooth in $Q_{1/2}^+$. (See [1], [9], [14], [20] and [21].)
3. Let $\omega = \nabla \times \mathbf{u}$. Let us denote by $\theta(x, y, t)$ the angle between the vorticity ω at two distinct points x and y at time t. Assume that $|\sin\theta(x, y, t)| \le c|x - y|^\beta$, $\beta \in [0, 1/2]$, in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Moreover, suppose that $\omega \in L^2(0, T; L^r)$, where $r = \frac{3}{\beta + t}$. Then $\mathbf{u}$ is regular. (See [10], [6] and [8].)

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