PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 8, August 2008, Pages 2771–2783
S 0002-9939(08)08809-6
Article electronically published on April 14, 2008

ON THE BEST HÖLDER EXPONENT FOR TWO DIMENSIONAL ELLIPTIC EQUATIONS IN DIVERGENCE FORM

TONIA RICCIARDI

(Communicated by David S. Tartakoff)

ABSTRACT. We obtain an estimate for the Hölder continuity exponent for weak solutions to the following elliptic equation in divergence form:

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } \Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^2$ and, for every $x \in \Omega$, $A(x)$ is a symmetric matrix with bounded measurable coefficients. Such an estimate "interpolates" between the well-known estimate of Piccinini and Spagnolo in the isotropic case $A(x) = a(x)I$, where a is a bounded measurable function, and our previous result in the unit determinant case $\det A \equiv 1$. Furthermore, we show that our estimate is sharp. Indeed, for every $\tau \in [0,1]$ we construct coefficient matrices A_τ such that A_0 is isotropic and A_1 has unit determinant, and such that our estimate for A_τ reduces to an equality, for every $\tau \in [0,1]$.

1. Introduction and main results

Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution to the elliptic, divergence form equation with measurable coefficients,

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } \Omega, \tag{1}$$

where $A(x)$, $x \in \Omega$, is a 2×2 symmetric matrix satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \le \langle \xi, A(x)\xi \rangle \le \Lambda|\xi|^2, \tag{2}$$

for every $x \in \Omega$, for all $\xi \in \mathbb{R}^2$ and for some constants $0 < \lambda \le \Lambda$. By classical results of De Giorgi [1], Moser [5], and Nash [6], it is well-known that u is locally Hölder continuous in Ω. Namely, there exists a constant $\alpha \in (0,1)$ such that for every $K \Subset \Omega$ there exists $C(K) > 0$ such that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \le C(K) \qquad \forall x, y \in K,\ x \ne y. \tag{3}$$

Received by the editors November 25, 2005, and, in revised form, March 9, 2006.
2000 *Mathematics Subject Classification.* Primary 35J15.
Key words and phrases. Linear elliptic equation, measurable coefficients, Hölder regularity.
The author was supported in part by the INdAM-GNAMPA Project *Funzionali policonvessi e mappe quasiregolari* and by the MIUR National Project *Variational Methods and Nonlinear Differential Equations.*

The sharp estimate of α in terms of the ellipticity constant $L = \Lambda/\lambda$ was obtained by Piccinini and Spagnolo [7], who showed that

$$\alpha \geq L^{-1/2}. \tag{4}$$

Under additional assumptions on A, this estimate may be improved. For example, if A is isotropic, namely if $A(x) = a(x)I$ for some measurable function a satisfying $1 \leq a \leq L$, it was shown by Piccinini and Spagnolo [7] that

$$\alpha \geq \frac{4}{\pi} \arctan L^{-1/2}. \tag{5}$$

On the other hand, we showed in [8] that if A has unit determinant, namely if $\det A(x) = 1$ for all $x \in \Omega$, then

$$\alpha \geq \left(\sup_{S_\rho(x) \subset \Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \langle n, A\, n \rangle \right)^{-1}, \tag{6}$$

where $S_\rho(x)$ is the circle of radius ρ centered at x and n is the outward unit normal. See Iwaniec and Sbordone [3] for the relevance of the unit determinant case in the context of quasiconformal mappings. Our aim in this note is to obtain an estimate for α, in the case of *general* symmetric coefficient matrices A satisfying the ellipticity condition (2), which "unifies" the estimates (5)–(6). We shall obtain a formula which, despite its complicated form, is indeed attained on a family of coefficient matrices A_τ, $\tau \in [0, 1]$, such that A_0 is isotropic and A_1 has unit determinant. In fact, our main effort in this paper is to construct A_τ. More precisely, for every symmetric matrix valued function A satisfying (2), let

$$\alpha(A) = \sup \left\{ \alpha \in (0, 1) \ : \ \begin{array}{c} \text{property (3) holds for every} \\ \text{solution } u \in H^1_{\text{loc}}(\Omega) \text{ to (1)} \end{array} \right\}.$$

We prove the following results.

Theorem 1 (Estimate)**.** *Suppose A is symmetric and satisfies* (2)*. Then, $\alpha(A) \geq \beta(A)$, where*

$$\beta(A) = \left(\sup_{S_\rho(x) \subset \Omega} \frac{\frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \frac{\langle n, A\, n \rangle}{\sqrt{\det A}}}{\frac{4}{\pi} \arctan \left(\frac{\inf_{S_\rho(x)} \det A}{\sup_{S_\rho(x)} \det A} \right)^{1/4}} \right)^{-1}. \tag{7}$$

As already mentioned, Theorem 1 is sharp, in the following sense.

Theorem 2 (Sharpness)**.** *For every $\tau \in [0, 1]$ and for every $x \neq 0$, let $A_\tau = J K_\tau J^*$, where*

$$K_\tau(x) = \begin{cases} \mathrm{Id}_{\mathbb{R}^2}, & \text{if } \arg x \in [0, \frac{\pi}{1+M^{-\tau}}) \cup [\pi, \pi + \frac{\pi}{1+M^{-\tau}}), \\ \mathrm{diag}(M, M^{1-2\tau}), & \text{otherwise} \end{cases}$$

for some $M > 1$ and J is the rotation matrix defined by

$$J(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{8}$$

There exists $m_0 > 1$ such that the equality

$$\alpha(A_\tau) = \beta(A_\tau) = \frac{2}{\pi} \left(1 + M^{-\tau}\right) \arctan M^{-(1-\tau)/2}$$

holds for all $M \in (1, m_0^{1/\tau})$ if $\tau > 0$, and with no restriction on M if $\tau = 0$.

It is readily seen that estimate (7) coincides with (5) when A is isotropic, and with (6) when $\det A \equiv 1$. It is also clear that, for the special family of matrices A_τ constructed in Theorem 2, (7) is strictly better than (4). However, for general matrix valued functions A satisfying (2), estimate (7) may not improve (4).

An estimate which is always better than (4) is provided by the following theorem.

Theorem 3. *In the assumptions of Theorem* 1*, we have* $\alpha(A) \geq \beta^*(A)$*, where*

$$\beta^*(A) = \left(\sup_{S_\rho(x)\subset\Omega} \inf_{\varphi,\psi\in\mathcal{B}_{x,\rho}} \sqrt{\frac{\max\varphi}{\min\psi}} \frac{\frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \frac{\langle n, An\rangle\sqrt{\psi}}{\sqrt{\det A\varphi}}}{\frac{4}{\pi}\arctan\sqrt{\frac{\min\sqrt{\det A/\varphi\psi}}{\max\sqrt{\det A/\varphi\psi}}}} \right)^{-1} \tag{9}$$

and where $\mathcal{B}_{x,\rho}$ *is the set of positive measurable functions defined on* $S_\rho(x)$ *which are bounded above and below away from zero.*

We note that by choosing

$$\varphi(t) = \langle e^{it}, A(x+\rho e^{it})e^{it}\rangle, \qquad \psi(t) = \frac{\det A(x+\rho e^{it})}{\langle e^{it}, A(x+\rho e^{it})e^{it}\rangle}$$

we have

$$\max\varphi \leq \Lambda, \qquad \min\psi \geq \lambda,$$

where λ, Λ are the constants in (2). Therefore,

$$\frac{\max\varphi}{\min\psi} \leq \frac{\Lambda}{\lambda} = L,$$

and (9) indeed improves (4).

Notation. Throughout this paper, for all $x \in \mathbb{R}^2$ and for all $\rho > 0$, $B_\rho(x)$ denotes the ball of radius ρ centered at x and $S_\rho(x) = \partial B_\rho(x)$. We denote $d_x = \mathrm{dist}(x, \partial\Omega)$. For every curve γ we denote by $|\gamma|$ the length of γ. For every measurable function f we denote by $\inf f$ and $\sup f$ the essential lower bound and the essential upper bound of f, respectively.

2. Proofs of Theorem 1 and Theorem 3

The proof of Theorem 1 relies on an argument of Piccinini and Spagnolo [7], together with an estimate for the best constant in the weighted Wirtinger inequality (10) below, namely estimate (12) below, obtained in [9]. The proof of Theorem 3 is obtained in a similar way, by using a refined version of (12); see Lemma 3 below. Let

$$\mathcal{B} = \{a \in L^\infty(\mathbb{R}) \ : \ a \text{ is } 2\pi\text{-periodic and } \inf a > 0\},$$

and for every $a, b \in \mathcal{B}$ let $C(a,b) > 0$ denote the best constant in the following weighted Wirtinger type inequality:

$$\int_0^{2\pi} a\, w^2 \leq C(a,b) \int_0^{2\pi} b w'^2, \tag{10}$$

where $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ is 2π-periodic and satisfies the constraint

$$\int_0^{2\pi} aw = 0. \tag{11}$$

Lemma 1 ([9]). *Suppose $a, b \in \mathcal{B}$. Then,*

$$C(a,b) \le \left(\frac{\frac{1}{2\pi}\int_0^{2\pi} \sqrt{ab^{-1}}}{\frac{4}{\pi}\arctan\left(\frac{\inf ab}{\sup ab}\right)^{1/4}} \right)^2. \tag{12}$$

We note that Lemma 1 reduces to the sharp Wirtinger inequality of Piccinini and Spagnolo [7] when $a = b$. Estimate (12) has been recently extended in [2] to the case $a, b^{-1}, \sqrt{ab^{-1}} \in L^1$ and $0 < \inf(ab) \le \sup(ab) < +\infty$.

In order to proceed, for every fixed $x \in \Omega$ and for every $\rho \in (0, d_x)$, we denote by $y = x + \rho e^{it}$ the polar coordinate transformation centered at x. We denote

$$\overline{\nabla} u = \left(u_\rho, \frac{u_t}{\rho}\right).$$

Then,

$$\nabla u = J(t)\overline{\nabla} u, \tag{13}$$

where J is the rotation matrix defined in (8).

Lemma 2. *For every matrix A satisfying* (2) *and for every $x \in \Omega$, $0 < \rho < d_x$, let*

$$C_A(x,\rho) = C\left(\langle e^{it}, A(x+\rho e^{it})e^{it}\rangle, \frac{\det A(x+\rho e^{it})}{\langle e^{it}, A(x+\rho e^{it})e^{it}\rangle}\right)$$

denote the best constant in (10)–(11) *with*

$$a(t) = \langle e^{it}, A(x+\rho e^{it})e^{it}\rangle, \qquad b(t) = \frac{\det A(x+\rho e^{it})}{\langle e^{it}, A(x+\rho e^{it})e^{it}\rangle}.$$

Then, $\alpha(A) \ge \beta_0(A)$, where

$$\beta_0(A) = \left(\sup_{x\in\Omega, 0<\rho<d_x} C_A(x,\rho)^{1/2}\right)^{-1}.$$

Proof. We show that for every $u \in H^1_{\mathrm{loc}}(\Omega)$ solution to (1) there holds

$$\sup_{0<\rho<d_x} \rho^{-2\beta_0(A)} \int_{B_\rho(x)} \langle \nabla u, A\nabla u\rangle < +\infty, \tag{14}$$

for every $x \in \Omega$. Once estimate (14) is established, the statement follows by the well-known regularity results of Morrey [4]. In order to derive (14), we exploit some ideas in [7]. For every $x \in \Omega$ and for every $0 < \rho < d_x$, we set

$$g_x(\rho) = \int_{B_\rho(x)} \langle \nabla u, A\nabla u\rangle.$$

We denote by $P = (p_{ij})$ the matrix defined by

$$P(x+\rho e^{it}) = J^*(t)A(x+\rho e^{it})J(t).$$

Note that $p_{11}(x+\rho e^{it}) = \langle e^{it}, A(x+\rho e^{it})e^{it}\rangle$ and $\det P = \det A$. By the divergence theorem and (1), for a.e. ρ we have

$$g_x(\rho) = \int_{S_\rho(x)} (u-\mu)\langle n, A\nabla u\rangle = \int_{S_\rho(x)} (u-\mu)\langle \underline{e}_1, P\overline{\nabla} u\rangle,$$

where n is the outward normal to $S_\rho(x)$, $\underline{e}_1=(1,0)$ and μ is any constant. In view of Hölder's inequality, we may write

$$g_x(\rho)\le\left(\int_{S_\rho(x)}p_{11}(u-\mu)^2\right)^{1/2}\left(\int_{S_\rho(x)}\frac{\langle\underline{e}_1,P\overline{\nabla}u\rangle^2}{p_{11}}\right)^{1/2}.$$

By inequality (10) with $a(t)=p_{11}(x+\rho e^{it})$, $b(t)=(\det A/p_{11})(x+\rho e^{it})$ and

$$w(t)=u(x+\rho e^{it})-\mu,\qquad \mu=\frac{1}{2\pi}\int_0^{2\pi}p_{11}(x+\rho e^{it})u(x+\rho e^{it})\,\mathrm{d}t,$$

we derive that

$$\int_{S_\rho(x)}p_{11}(u-\mu)^2\le C_A(x,\rho)\int_{S_\rho(x)}\frac{\det A}{p_{11}}u_t^2.$$

Therefore,

$$g_x(\rho)\le C_A^{1/2}(x,\rho)\left(\int_{S_\rho(x)}\frac{\det A}{p_{11}}u_t^2\right)^{1/2}\left(\int_{S_\rho(x)}\frac{\langle\underline{e}_1,P\overline{\nabla}u\rangle^2}{p_{11}}\right)^{1/2}.$$

At this point, we observe that any 2×2 symmetric matrix $B=(b_{ij})$ such that $b_{11}\neq0$ satisfies the following identity:

$$\langle\xi,B\xi\rangle=\frac{\langle\xi,B\underline{e}_1\rangle^2}{b_{11}}+\frac{\det B}{b_{11}}\langle\xi,\underline{e}_2\rangle^2, \tag{15}$$

for any $\xi\in\mathbb{R}^2$, where $\underline{e}_2=(0,1)$. Recalling that $u_\theta/\rho=(\overline{\nabla}u)_{22}$, in view of the elementary inequality $\sqrt{ab}\le(a+b)/2$ and of identity (15) with $B=P(x+\rho e^{i\theta})$ and $\xi=\overline{\nabla}u$, we obtain:

$$\begin{aligned}
g_x(\rho)\le&\rho\,C_A^{1/2}(x,\rho)\left(\int_{S_\rho(x)}\frac{\det A}{p_{11}}\left(\frac{u_t}{\rho}\right)^2\right)^{1/2}\left(\int_{S_\rho(x)}\frac{\langle\underline{e}_1,P\overline{\nabla}u\rangle^2}{p_{11}}\right)^{1/2}\\
=&\rho\,C_A^{1/2}(x,\rho)\left(\int_{S_\rho(x)}\frac{\det A}{p_{11}}\left(\overline{\nabla}u\right)_{22}^2\right)^{1/2}\left(\int_{S_\rho(x)}\frac{\langle\underline{e}_1,P\overline{\nabla}u\rangle^2}{p_{11}}\right)^{1/2}\\
\le&\frac{\rho}{2}\,C_A^{1/2}(x,\rho)\int_{S_\rho(x)}\left(\frac{\det P}{p_{11}}\left(\overline{\nabla}u\right)_{22}^2+\frac{\langle e_1,P\overline{\nabla}u\rangle^2}{p_{11}}\right)\\
=&\frac{\rho}{2}\,C_A^{1/2}(x,\rho)\int_{S_\rho(x)}\langle\overline{\nabla}u,P\overline{\nabla}u\rangle=\frac{\rho}{2}\,C_A^{1/2}(x,\rho)\int_{S_\rho(x)}\langle\nabla u,A\nabla u\rangle.
\end{aligned}$$

Recalling the definition of g_x, we derive from the above inequality that

$$g_x(\rho)\le\frac{\rho}{2}\,C_A^{1/2}(x,\rho)\,g_x'(\rho),$$

for almost every $0<\rho<d_x$. In particular, for every $x\in\Omega$ and $\rho\in(0,d_x)$ we have

$$g_x(\rho)\le\frac{\rho}{2}\sup_{x\in\Omega,0<\rho<d_x}C_A^{1/2}(x,\rho)\,g_x'(\rho)=\frac{\rho}{2\beta_0(A)}\,g_x'(\rho)\qquad\text{in }(0,d_x).$$

The above implies that the function $\rho^{-2\beta_0(A)}g_x(\rho)$ is non-decreasing, and therefore bounded, in $(0,d_x)$. ∎

Proof of Theorem 1. In view of Lemma 1 with

$$a(t) = p_{11}(x + \rho e^{it}), \qquad b(t) = \frac{\det A}{p_{11}}(x + \rho e^{it}),$$

we have

$$C_A(x,\rho) \le \left(\frac{\frac{1}{2\pi}\int_0^{2\pi} \frac{p_{11}}{\sqrt{\det A}}(x+\rho e^{it})\mathrm{d}t}{\frac{4}{\pi}\arctan\left(\frac{\inf_{t\in(0,2\pi)}\det A(x+\rho e^{it})}{\sup_{t\in(0,2\pi)}\det A(x+\rho e^{it})}\right)^{1/4}} \right)^2.$$

Now the asserted estimate follows by Lemma 2. □

In order to prove Theorem 3, we need the following refinement of Lemma 1.

Lemma 3. *In the assumptions of Lemma* 1*, we have*

$$C(a,b) \le \inf_{\varphi,\psi\in\mathcal{B}} \frac{\max\varphi}{\min\psi}\left(\frac{\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{a\psi}{b\varphi}}}{\frac{4}{\pi}\arctan\sqrt{\frac{\min\sqrt{ab/\varphi\psi}}{\max\sqrt{ab/\varphi\psi}}}} \right)^2. \tag{16}$$

Proof. Fix $\varphi \in \mathcal{B}$ and $\psi \in \mathcal{B}$, and let $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ be a 2π-periodic function such that $\int_0^{2\pi} aw = 0$. Let

$$k = \frac{\int_0^{2\pi} aw/\varphi}{\int_0^{2\pi} a/\varphi}.$$

Then,

$$\int aw^2 \le \int a(w-k)^2. \tag{17}$$

Indeed, setting

$$f(s) := \int_0^{2\pi} a(w-s)^2, \qquad s \in \mathbb{R},$$

we find that

$$\min_{s\in\mathbb{R}} f(s) = f\left(\int_0^{2\pi} \frac{a}{\int_0^{2\pi} a} w\right) = f(0),$$

which establishes (17). Therefore, recalling the definition of k, we have

$$\begin{aligned}\int aw^2 &\le \max\varphi \int_0^{2\pi} \frac{a}{\varphi}(w-k)^2 \le \max\varphi\; C\left(\frac{a}{\varphi},\frac{b}{\psi}\right)\int_0^{2\pi}\frac{b}{\psi}w'^2\\ &\le \frac{\max\varphi}{\min\psi} C\left(\frac{a}{\varphi},\frac{b}{\psi}\right)\int_0^{2\pi} bw'^2.\end{aligned}$$

This implies (16), using (9) and the fact that φ, ψ are arbitrary. □

Proof of Theorem 3. The proof follows by the same arguments used to prove Theorem 1, replacing (10) by (16). □

3. Proof of Theorem 2

We define

$$\mu = \frac{4}{\pi} \arctan M^{-(1-\tau)/2}, \qquad c = \frac{2}{1+M^{-\tau}}.$$

We prove

Proposition 1. *For every $\tau \in [0,1]$, let A_τ be the coefficient matrix defined in Theorem 2. There exists $m_0 > 1$ such that there holds*

$$\beta(A_\tau) = \frac{\mu}{c} = \frac{2}{\pi}(1+M^{-\tau}) \arctan M^{-(1-\tau)/2}$$

for all $M \in (1, m_0^{1/\tau})$ if $\tau > 0$, and with no restriction on M if $\tau = 0$. Furthermore, let $u_\tau \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ be defined in polar coordinates by

$$u_\tau(\rho,\theta) = \rho^{\mu/c} w_\tau(\theta),$$

where

$$w_\tau(\theta) = \begin{cases} \sin[\mu(c^{-1}\theta - \pi/4)], & \text{if } \theta \in [0, c\pi/2), \\ M^{-(1-\tau)/2} \cos[\mu\left(c^{-1}M^\tau(\theta - c\pi/2) - \pi/4\right)], & \text{if } \theta \in [c\pi/2, \pi), \\ -\sin[\mu(c^{-1}(\theta-\pi) - \pi/4)], & \text{if } \theta \in [\pi, \pi + c\pi/2), \\ -M^{-(1-\tau)/2} \cos[\mu(c^{-1}M^\tau(\theta - \pi - c\pi/2) - \pi/4)], & \text{if } \theta \in [\pi + c\pi/2, 2\pi). \end{cases}$$

Then, u_τ is a weak solution to the elliptic equation (1) with $A = A_\tau$, and its Hölder exponent is μ/c. In particular, $\alpha(A_\tau) = \beta(A_\tau)$.

In order to prove Proposition 1, we begin by proving some lemmas.

Lemma 4. *For every $x \neq 0$ let $\theta = \arg x$ and let*

$$A(x) = A(\theta) = J(\theta)K(\theta)J^*(\theta),$$

where

$$K(\theta) = \begin{pmatrix} k_1(\theta) & 0 \\ 0 & k_2(\theta) \end{pmatrix}$$

for some positive and bounded, 2π-periodic functions k_1, k_2. Then, in polar coordinates, equation (1) takes the form:

$$\begin{cases} (\rho k_1 u_\rho)_\rho + \left(\frac{k_2}{\rho} u_\theta\right)_\theta = 0 & \text{in } (0,+\infty) \times \mathbb{R}, \\ u \ 2\pi\text{−periodic in } \theta. \end{cases} \tag{18}$$

If $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, $u \not\equiv 0$, is of the form $u(\rho,\theta) = R(\rho)\Theta(\theta)$, then u satisfies (18) if and only if $R(\rho) = \rho^\gamma$ for some constant $\gamma > 0$ and Θ is a 2π-periodic weak solution to the equation

$$-(k_2\Theta')' = \gamma^2 k_1 \Theta \qquad \text{in } \mathbb{R}. \tag{19}$$

Proof. By definition, u satisfies

$$\int_{\mathbb{R}^2} \langle \nabla u, A\nabla v \rangle = 0 \qquad \forall v \in C_c^\infty(\mathbb{R}^2).$$

In polar coordinates centered at 0, recalling that $\overline{\nabla}u = (u_\rho, u_\theta/\rho) = J^*\nabla u$, we have

$$\begin{aligned} 0 = \int_{\mathbb{R}^2} \langle \nabla u, A\nabla v\rangle &= \int_{(0,+\infty)\times(0,2\pi)} \langle \overline{\nabla}u, K(\theta)\overline{\nabla}v\rangle \rho \mathrm{d}\rho\mathrm{d}\theta \\ &= \int_{(0,+\infty)\times(0,2\pi)} \left(\rho k_1 u_\rho v_\rho + \frac{k_2}{\rho} u_\theta v_\theta\right) \mathrm{d}\rho\mathrm{d}\theta, \end{aligned}$$

for every $v \in C_c^\infty(\mathbb{R}^2)$. Integration by parts yields (18). Now suppose that $u(\rho,\theta) = R(\rho)\Theta(\theta)$. In view of Nikodym's theorem, R and Θ are absolutely continuous on $(0,+\infty)$ and $\mathbb{R}$, respectively. Choosing v of the form $v(\rho,\theta) = \varphi(\rho)\psi(\theta)$ with $\varphi \in C_c^\infty(0,+\infty)$ and $\psi \in C_c^\infty(0,2\pi)$, we derive from the above that

$$\int_0^{+\infty} \rho R'\varphi' \mathrm{d}\rho \int_0^{2\pi} k_1 \Theta\psi \mathrm{d}\theta + \int_0^{+\infty} \frac{R}{\rho}\varphi \mathrm{d}\rho \int_0^{2\pi} k_2 \Theta'\psi' \mathrm{d}\theta = 0.$$

Since φ, ψ are arbitrary, we conclude that

$$\frac{\int_0^{+\infty} (\rho R')'\varphi \mathrm{d}\rho}{\int_0^{+\infty} R\rho^{-1}\varphi \mathrm{d}\rho} = -\frac{\int_0^{2\pi} (k_2\Theta')'\psi \mathrm{d}\theta}{\int_0^{2\pi} k_1\Theta\psi \mathrm{d}\theta} = \tau,$$

for some constant $\tau \in \mathbb{R}$. It follows that

$$\int_0^{+\infty} (\rho R')'\varphi \mathrm{d}\rho = \tau \int_0^{+\infty} \frac{R}{\rho}\varphi \mathrm{d}\rho \qquad \forall \varphi \in C_c^\infty(0,+\infty)$$

and

$$\int_0^{2\pi} (k_2\Theta')'\psi \mathrm{d}\theta = -\tau \int_0^{2\pi} k_1\Theta\psi \mathrm{d}\theta \qquad \forall \psi \in C_c^\infty(0,2\pi).$$

By regularity, R is smooth in $(0,+\infty)$ and satisfies $(\rho R')' = \tau R\rho^{-1}$ in $(0,+\infty)$. Recalling that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, we derive $R(\rho) = \rho^\gamma$ with $\gamma > 0$, and $\tau = \gamma^2 > 0$. Furthermore, (19) is also established. □

Lemma 5. *Suppose A satisfies the assumptions of Lemma 4. Then, for all $x \in \mathbb{R}^2$, $\rho > 0$ and $t \in \mathbb{R}$ such that $x + \rho e^{it} \neq 0$, we have*

$$\text{(20)} \quad \frac{\langle e^{it}, A(x+\rho e^{it})e^{it}\rangle}{\sqrt{\det A(x+\rho e^{it})}} = \sqrt{\frac{k_1(\theta(t))}{k_2(\theta(t))}} \cos^2(\theta(t) - t) + \sqrt{\frac{k_2(\theta(t))}{k_1(\theta(t))}} \sin^2(\theta(t) - t),$$

where

$$\theta(t) = \arg(x + \rho e^{it}).$$

Proof. Using the fact that $J^*(\theta)e^{it} = e^{i(t-\theta)}$ for all $t, \theta \in \mathbb{R}$, we have

$$\begin{aligned} \langle e^{it}, A(x+\rho e^{it})e^{it}\rangle &= \langle J^*(\theta(t))e^{it}, K(\theta(t))J^*(\theta(t))e^{it}\rangle \\ &= k_1(\theta(t))\cos^2(t-\theta(t)) + k_2(\theta(t))\sin^2(t-\theta(t)). \end{aligned}$$

Now (20) follows easily. □

We shall need the following property from Euclidean geometry. As we have not found a proof in the literature, we include one here.

Lemma 6. *Let $\mathcal{C}$ be a (two-sided) cone with vertex at the origin and let $x \in \mathbb{R}^2$ be such that $|x| < 1$. Then*

$$\text{(21)} \qquad |\mathcal{C} \cap S_1(x)| = |\mathcal{C} \cap S_1(0)|.$$

Proof. We denote by A, B, C, D the intersection points of $\mathcal{C}$ with $S_1(x)$ taken in, say, counterclockwise order. We have to show that $\angle AxB+\angle CxD = \angle AOB+\angle COD = 2\angle AOB$. We set $\alpha = \angle AxB$, $\beta = \angle CxD$, $\varepsilon = \angle xAC = \angle xCA$, $\delta = \angle xBD = \angle xDB$, $\eta = \angle ABx = \angle BAx$, $\theta = \angle CDx = \angle DCx$, $\varphi = \angle AOB = \angle COD$. Then, summing the angles of the triangles AxB, CxD, AOB, COD, respectively, we obtain

$$\begin{aligned} &\alpha + 2\eta = \pi, && \eta - \delta + \eta + \varepsilon + \varphi = \pi, \\ &\beta + 2\theta = \pi, && \theta - \varepsilon + \theta + \delta + \varphi = \pi. \end{aligned}$$

Summation of these equations yields $\alpha+\beta = 2\pi - 2(\eta+\theta)$ and $2(\eta+\theta) = 2\pi - 2\varphi$, from which we derive the desired equality $\alpha + \beta = 2\varphi$. □

For every $x \in \mathbb{R}^2$ and for every $\rho > 0$ we define

$$f(x,\rho) = \frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \frac{\langle n, A_\tau\, n\rangle}{\sqrt{\det A_\tau}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle e^{it}, A_\tau(x+\rho e^{it})e^{it}\rangle}{\sqrt{\det A_\tau(x+\rho e^{it})}}\mathrm{d}t,$$

where A_τ is the matrix defined in Theorem 2. We note that

$$f(x,\rho) = f\left(\frac{x}{\rho}, 1\right). \tag{22}$$

We prove the following.

Lemma 7. *There exists $m_0 > 1$ such that for all $x \in \mathbb{R}^2$ and for all $\rho > 0$ there holds*

$$f(x,\rho) \le f(0,1) = c = \frac{2}{1+M^{-\tau}}$$

for all $M \in (1, m_0^{1/\tau})$ if $\tau > 0$, and with no restriction on M if $\tau = 0$.

Proof. Throughout this proof, we let

$$m := M^\tau$$

and

$$\mathcal{C} := \left\{x \in \mathbb{R}^2 \setminus \{0\} : \ \arg x \in [\frac{\pi}{2}c, \pi) \cup [\pi + \frac{\pi}{2}c, 2\pi)\right\}.$$

Then

$$K_\tau(x) = \begin{cases} \mathrm{diag}(M, M^{1-2\tau}), & \text{if } x \in \mathcal{C}, \\ \mathrm{Id}_{\mathbb{R}^2}, & \text{otherwise.} \end{cases}$$

In view of Lemma 5, it follows that

$$\begin{aligned} &\frac{\langle e^{it}, A_\tau(x+\rho e^{it})e^{it}\rangle}{\sqrt{\det A_\tau(x+\rho e^{it})}} \\ &\qquad = \begin{cases} m\cos^2(\theta(t)-t) + m^{-1}\sin^2(\theta(t)-t)\,, & \text{if } x+\rho e^{it} \in \mathcal{C}, \\ 1, & \text{otherwise,} \end{cases} \end{aligned}$$

and in view of (22), we may assume $\rho = 1$.

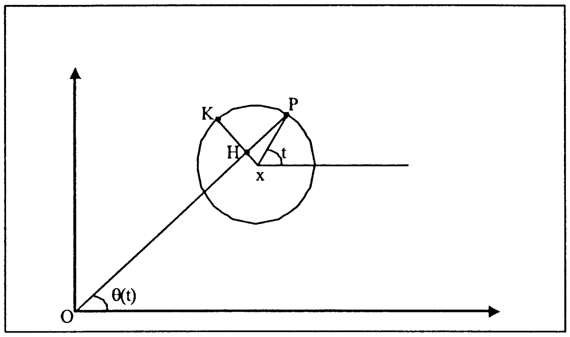

FIGURE 1. Geometrical interpretation of (23)

Case (i): $|x|<1$. We estimate:

$$\begin{aligned}2\pi f(x,1) =&2\pi-|\mathcal{C}\cap S_1(x)|\\&+\int_{\mathcal{C}\cap S_1(x)}\left\{m\cos^2\left(\theta(t)-t\right)+\frac{1}{m}\sin^2\left(\theta(t)-t\right)\right\}\mathrm{d}t\\=&2\pi-|\mathcal{C}\cap S_1(x)|+\int_{\mathcal{C}\cap S_1(x)}\left\{\left(m-\frac{1}{m}\right)\cos^2\left(\theta(t)-t\right)+\frac{1}{m}\right\}\mathrm{d}t\\\leq&2\pi-|\mathcal{C}\cap S_1(x)|+m|\mathcal{C}\cap S_1(x)|=2\pi+(m-1)|\mathcal{C}\cap S_1(x)|.\end{aligned}$$

From Lemma 6 we derive $|\mathcal{C}\cap S_1(x)|=|\mathcal{C}\cap S_1(0)|$, and therefore we obtain the estimate

$$2\pi f(x,1)\leq 2\pi+(m-1)|\mathcal{C}\cap S_1(0)|.$$

On the other hand, when $x=0$ we have $\theta(t)-t\equiv 0$, and consequently

$$\frac{\langle e^{it},A_\tau(e^{it})e^{it}\rangle}{\sqrt{\det A_\tau(e^{it})}}=\begin{cases}m, & \text{if } e^{it}\in\mathcal{C},\\1 & \text{otherwise.}\end{cases}$$

It follows that $2\pi f(0,1)=|S_1(0)\setminus\mathcal{C}|+m|\mathcal{C}\cap S_1(0)|=2\pi+(m-1)|\mathcal{C}\cap S_1(0)|$, and recalling the definition of $\mathcal{C}$ and c, we obtain $f(0,1)=c=2/(1+m^{-1})$. Hence, the desired estimate $f(x,1)\leq f(0,1)$ follows in the case $|x|<1$ with no restriction on M.

Case (ii): $|x|\geq 1$. We set $h(t)=\cos^2(\theta(t)-t)$. By elementary geometrical arguments, for every $0\leq k\leq 1$ we have

$$|\{h(t)\leq k\}|=4\arcsin\sqrt{k}. \tag{23}$$

Indeed, in Figure 1 we have $|\theta(t)-t|=\angle xPO$. Taking $Kx\perp OP$ we have $h(t)=|PH|^2$, $|\overset{\frown}{PK}|=\arcsin|PH|$. Now (23) follows by symmetry. Since $mh(t)+m^{-1}(1-h(t))\geq 1$ if and only if $h(t)\geq(m+1)^{-1}$, we estimate

$$2\pi f(x,1)\leq\left|\left\{h(t)\leq\frac{1}{m+1}\right\}\right|+\int_{\{h(t)\geq(m+1)^{-1}\}}\left\{mh(t)+\frac{1}{m}(1-h(t))\right\}\mathrm{d}t.$$

By virtue of (23), we derive

$$2\pi f(x,1)\leq 4\arcsin\sqrt{\frac{1}{m+1}}+\int_{\{h(t)\geq(m+1)^{-1}\}}\left\{mh(t)+\frac{1}{m}(1-h(t))\right\}\mathrm{d}t. \tag{24}$$

Similarly, let $0 < \varepsilon \le m$ and note that $1 + \varepsilon(m-1)/m \le m$. We have that $mh(t) + m^{-1}(1 - h(t)) \ge 1 + \varepsilon(m-1)/m$ if and only if $h \ge (1+\varepsilon)/(m+1)$. Therefore, we estimate in turn that

$$\begin{aligned}
&\int_{\{h(t)\ge(m+1)^{-1}\}} \left\{ mh(t) + \frac{1}{m}(1-h(t)) \right\} \mathrm{d}t \\
&\le \left(1+\varepsilon\frac{m-1}{m}\right)\left|\left\{\frac{1}{m+1} \le h \le \frac{1+\varepsilon}{m+1}\right\}\right| + m\left(2\pi - \left|\left\{h \le \frac{1+\varepsilon}{m+1}\right\}\right|\right) \\
&= 4\left(1+\varepsilon\frac{m-1}{m}\right)\left(\arcsin\sqrt{\frac{1+\varepsilon}{m+1}} - \arcsin\sqrt{\frac{1}{m+1}}\right) \\
&\qquad + m\left(2\pi - 4\arcsin\sqrt{\frac{1+\varepsilon}{m+1}}\right) \\
&= 2\pi m - 4\left(1+\varepsilon\frac{m-1}{m}\right)\arcsin\sqrt{\frac{1}{m+1}} \\
&\qquad - 4(m-1)\left(1-\frac{\varepsilon}{m}\right)\arcsin\sqrt{\frac{1+\varepsilon}{m+1}}.
\end{aligned}$$

Hence,

$$f(x,1) \le 2\pi m - 4\varepsilon\frac{m-1}{m}\arcsin\sqrt{\frac{1}{m+1}} - 4(m-1)\left(1-\frac{\varepsilon}{m}\right)\arcsin\sqrt{\frac{1+\varepsilon}{m+1}},$$

and it suffices to check that there exist $\varepsilon > 0$ and $m_0 > 1$ such that

$$\begin{aligned}
&\frac{1}{2\pi}\left(2\pi m - 4\varepsilon\frac{m-1}{m}\arcsin\sqrt{\frac{1}{m+1}} - 4(m-1)\left(1-\frac{\varepsilon}{m}\right)\arcsin\sqrt{\frac{1+\varepsilon}{m+1}}\right) \\
&\qquad\qquad \le \frac{2}{1+m^{-1}},
\end{aligned}$$

for all $1 < m \le m_0$. Upon factorization, the above is equivalent to

$$\frac{m(m-1)}{m+1} \le (m-1)\left[\frac{2\varepsilon}{\pi m}\arcsin\sqrt{\frac{1}{m+1}} + \left(1-\frac{\varepsilon}{m}\right)\frac{2}{\pi}\arcsin\sqrt{\frac{1+\varepsilon}{m+1}}\right]. \tag{25}$$

Therefore, if $\tau = 0$, we have $m = 1$, and (25) holds with no restriction on M. If $\tau > 0$, we have $m - 1 > 0$, and (25) is verified if and only if

$$\frac{m}{m+1} \le \frac{2}{\pi}\arcsin\sqrt{\frac{1+\varepsilon}{m+1}} - \frac{2\varepsilon}{\pi m}\left(\arcsin\sqrt{\frac{1+\varepsilon}{m+1}} - \arcsin\sqrt{\frac{1}{m+1}}\right). \tag{26}$$

Let $\delta = m - 1$ and consider the function ζ defined by

$$\begin{aligned}
&\zeta(\varepsilon,\delta) \\
&= \frac{2}{\pi}\left[\arcsin\sqrt{\frac{1+\varepsilon}{2+\delta}} - \frac{\varepsilon}{1+\delta}\left(\arcsin\sqrt{\frac{1+\varepsilon}{2+\delta}} - \arcsin\sqrt{\frac{1}{2+\delta}}\right)\right] - \frac{1+\delta}{2+\delta}.
\end{aligned}$$

Then (26) is equivalent to $\zeta(\varepsilon,\delta) \ge 0$. We note that $\zeta(0,0) = \frac{2}{\pi}\arcsin\sqrt{\frac{1}{2}} - \frac{1}{2} = 0$. By Taylor's expansion, there exists $\varepsilon_0 > 0$ such that the strict inequality $\zeta(\varepsilon_0, 0) > 0$ is satisfied. Hence, by continuity, there exists $\delta_0 > 0$ such that $\zeta(\varepsilon_0,\delta) > 0$ for all $\delta \in (0,\delta_0)$. Setting $m_0 = 1 + \delta_0$, we conclude that (26) is satisfied for $\varepsilon = \varepsilon_0$

and for all $\delta \in (0, \delta_0)$. It follows that the statement of the lemma holds with $M_0 = m_0^{1/\tau}$. $\square$

Proof of Proposition 1. We note that for all $x \in \mathbb{R}^2$, $\rho > 0$ we have

$$\frac{\inf_{S_\rho(x)} \det A}{\sup_{S_\rho(x)} \det A} \geq M^{-2(1-\tau)} = \frac{\inf_{S_1(0)} \det A}{\sup_{S_1(0)} \det A}.$$

Therefore, in view of Lemma 7 we have

$$\begin{aligned}\beta(A_\tau) &= \left(\sup_{x \in \mathbb{R}^2, \rho > 0} \frac{f(x,\rho)}{\frac{4}{\pi} \arctan \left(\frac{\inf_{S_\rho(x)} \det A}{\sup_{S_\rho(x)} \det A} \right)^{1/4}} \right)^{-1} \\ &= \left(\frac{f(0,1)}{4\pi^{-1} \arctan M^{-(1-\tau)/2}} \right)^{-1} \\ &= \frac{2}{\pi}(1 + M^{-\tau}) \arctan M^{-(1-\tau)/2} = \frac{\mu}{c}.\end{aligned}$$

On the other hand, by a direct check we see that w_τ is a 2π-periodic weak solution to the equation

$$-(k_{\tau,2} w_\tau')' = \frac{\mu}{c} k_{\tau,1} w_\tau \qquad \text{in } \mathbb{R},$$

where $k_{\tau,1}$, $k_{\tau,2}$ denote the diagonal entries of K_τ. It follows by Lemma 4 that u_τ satisfies (1) with $A = A_\tau$. Since w_τ is Lipschitz continuous, u_τ is Hölder continuous with exponent $\beta(A_\tau)$. $\square$

Proof of Theorem 2. The proof is a direct consequence of Proposition 1. $\square$

Acknowledgements

I am grateful to an anonymous referee for useful comments leading to a clearer exposition of the results.

References

[1] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Acc. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR0093649 (20:172)

[2] R. Giova, A weighted Wirtinger inequality, Ricerche Mat. **54**, No. 1 (2005), 293–302 (2006). MR2290221 (2007k:26047)

[3] T. Iwaniec and C. Sbordone, Quasiharmonic fields, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** No. 5 (2001), 519–572. MR1849688 (2002i:30021)

[4] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Am. Math. Soc. **43**, No. 1 (1938), 126–166. MR1501936

[5] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math. **13** (1960), 457–468. MR0170091 (30:332)

[6] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. **80**, No. 5 (1958), 931–954. MR0100158 (20:6592)

[7] L.C. Piccinini and S. Spagnolo, On the Hölder continuity of solutions of second order elliptic equations in two variables, Ann. Scuola Norm. Sup. Pisa **26**, No. 2 (1972), 391–402. MR0361422 (50:13867)

[8] T. Ricciardi, A sharp Hölder estimate for elliptic equations in two variables, Proc. Roy. Soc. Edinburgh **135A** (2005), 165–173. MR2119847 (2006f:35019)
[9] T. Ricciardi, A sharp weighted Wirtinger inequality, Boll. U.M.I. (8) **8-B** (2005), 259–267. MR2122985 (2005m:26039)

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy
E-mail address: tonia.ricciardi@unina.it