ON DRAZIN INVERTIBILITY

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Abstract. The left Drazin spectrum and the Drazin spectrum coincide with the upper semi-B-Browder spectrum and the B-Browder spectrum, respectively. We also prove that some spectra coincide whenever \( T \) or \( T^* \) satisfies the single-valued extension property.

1. Introduction and preliminaries

Throughout this note \( L(X) \) will denote the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space \( X \). The operator \( T \in L(X) \) is said to be upper semi-Fredholm if its kernel \( \ker T \) is finite-dimensional and the range \( T(X) \) is closed, while \( T \in L(X) \) is said to be lower semi-Fredholm if \( T(X) \) is finite-codimensional. If either \( T \) is upper or lower semi-Fredholm, then \( T \) is said to be a semi-Fredholm operator, while \( T \) is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If \( T \in L(X) \) is semi-Fredholm, the classical index of \( T \) is defined by \( \text{ind}(T) := \dim \ker T - \text{codim} T(X) \).

The concept of semi-Fredholm operators has been generalized by Berkani ([9], [13] and [11]) in the following way: for every \( T \in L(X) \) and a nonnegative integer \( n \) let us denote by \( T \mid_n \) the restriction of \( T \) to \( T^n(X) \) viewed as a map from the space \( T^n(X) \) into itself (we set \( T \mid_0 = T \)). \( T \in L(X) \) is said to be semi-B-Fredholm, (resp. B-Fredholm, upper semi-B-Fredholm, lower semi-B-Fredholm,) if for some integer \( n \geq 0 \) the range \( T^n(X) \) is closed and \( T \mid_n \) is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case \( T \mid_m \) is a semi-Fredholm operator for all \( m \geq n \) ([13]). This enables one to define the index of a semi-B-Fredholm operator as \( \text{ind}(T) = \text{ind}(T \mid_n) \).

A bounded operator \( T \in L(X) \) is said to be a Weyl operator if \( T \) is a Fredholm operator having index 0. A bounded operator \( T \in L(X) \) is said to be B-Weyl if for some integer \( n \geq 0 \) the range \( T^n(X) \) is closed and \( T \mid_n \) is Weyl. The Weyl spectrum and the B-Weyl spectrum are defined, respectively, by

\[
\sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl} \}
\]

and

\[
\sigma_{bw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl} \}.
\]

Recall that the ascent of an operator \( T \in L(X) \) is defined as the smallest nonnegative integer \( p := p(T) \) such that \( \ker T^p = \ker T^{p+1} \). If such an integer does not
exist, we put \( p(T) = \infty \). Analogously, the descent of \( T \) is defined as the smallest nonnegative integer \( q := q(T) \) such that \( T^q(X) = T^{q+1}(X) \), and if such an integer does not exist, we put \( q(T) = \infty \). It is well known that if \( p(T) \) and \( q(T) \) are both finite, then \( p(T) = q(T) \); see [1] Theorem 3.3. Moreover, if \( \lambda \in \mathbb{C} \), the condition \( 0 < p(\lambda I - T) = q(\lambda I - T) < \infty \) is equivalent to saying that \( \lambda \) is a pole of the resolvent. In this case \( \lambda \) is an eigenvalue of \( T \) and an isolated point of the spectrum \( \sigma(T) \); see [17] Prop. 50.2.

The concept of Drazin invertibility [14] has been introduced in a more abstract setting than operator theory [14]. In the case of the Banach algebra \( L(X), T \in L(X) \) is said to be Drazin invertible (with a finite index) precisely when \( p(T) = q(T) < \infty \) and this is equivalent to saying that \( T = T_0 \oplus T_1 \), where \( T_0 \) is invertible and \( T_1 \) is nilpotent; see [19] Corollary 2.2 and [18] Prop. A]. Every \( B \)-Fredholm operator \( T \) admits the representation \( T = T_0 \oplus T_1 \), where \( T_0 \) is Fredholm and \( T_1 \) is nilpotent [11], so every Drazin invertible operator is \( B \)-Fredholm.

The concept of Drazin invertibility for bounded operators may be extended as follows.

**Definition 1.1.** \( T \in L(X) \) is said to be left Drazin invertible if \( p := p(T) < \infty \) and \( T^{p+1}(X) \) is closed; while \( T \in L(X) \) is said to be right Drazin invertible if \( q := q(T) < \infty \) and \( T^q(X) \) is closed.

It should be noted that the condition \( q = q(T) < \infty \) does not entail that \( T^q(X) \) is closed; see Example 5 of [21]. Clearly, \( T \in L(X) \) is both right and left Drazin invertible if and only if \( T \) is Drazin invertible. In fact, if \( 0 < p := p(T) = q(T) \), then \( T^p(X) = T^{p+1}(X) \) is the kernel of the spectral projection associated with the spectral set \( \{0\} \); see [17] Prop. 50.2.

The left Drazin spectrum is then defined as

\[
\sigma_{ld}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible} \},
\]

the right Drazin spectrum is defined as

\[
\sigma_{rd}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible} \},
\]

and the Drazin spectrum is defined as

\[
\sigma_{d}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible} \}.
\]

Obviously, \( \sigma_{d}(T) = \sigma_{ld}(T) \cup \sigma_{rd}(T) \).

A bounded operator \( T \in L(X) \) is said to be Browder (resp. upper semi-Browder, lower semi-Browder) if \( T \) is Fredholm and \( p(T) = q(T) < \infty \) (resp. \( T \) is upper semi-Fredholm and \( p(T) < \infty \), \( T \) is lower semi-Fredholm and \( q(T) < \infty \)). Every Browder operator is Weyl and hence, if

\[
\sigma_{b}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder} \}
\]

denotes the Browder spectrum of \( T \), then \( \sigma_{w}(T) \subseteq \sigma_{b}(T) \). In the sequel by \( \sigma_{ub}(T) \) we shall denote the upper semi-Browder spectrum of \( T \) defined by

\[
\sigma_{ub}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder} \}.
\]

Clearly, every bounded below operator \( T \in L(X) \) (\( T \) injective with closed range) is upper semi-Browder, while every surjective operator is lower semi-Browder. The classical approximate point spectrum of \( T \) will be denoted by \( \sigma_{a}(T) \) while by \( \sigma_{s}(T) \) we shall denote the surjectivity spectrum of \( T \).
It is natural to extend the concept of semi-Browder operators as follows: A bounded operator \( T \in L(X) \) is said to be \( B \)-Browder (resp. upper semi-\( B \)-Browder, lower semi-\( B \)-Browder) if for some integer \( n \geq 0 \) the range \( T^n(X) \) is closed and \( T^n \) is Browder (resp. upper semi-Browder, lower semi-Browder). The respective \( B \)-Browder spectra are denoted by \( \sigma_{bb}(T) \), \( \sigma_{usbb}(T) \) and \( \sigma_{lsbb}(T) \).

The main result of this paper establishes that \( T \in L(X) \) is \( B \)-Browder (respectively, upper semi-\( B \)-Browder, lower semi-Browder) if and only if \( T \) is Drazin invertible (respectively, left Drazin invertible, right Drazin invertible); consequently \( \sigma_{bb}(T) = \sigma_d(T) \), \( \sigma_{usbb}(T) = \sigma_{ld}(T) \) and \( \sigma_{lsbb}(T) = \sigma_{rd}(T) \). We also prove that many of the spectra before introduced coincide whenever \( T \), or its dual \( T^* \), satisfies the single-valued extension property.

2. SVEP and semi-\( B \)-Browder spectra

A useful tool in the Fredholm theory is given by the localized single-valued extension property. This property has an important role in local spectral theory; see the recent monographs by Laursen and Neumann [20] and Aiena [1].

**Definition 2.1.** Let \( X \) be a complex Banach space and \( T \in L(X) \). The operator \( T \) is said to have the single-valued extension property at \( \lambda_0 \in \mathbb{C} \) (abbreviated SVEP at \( \lambda_0 \)) if for every open disc \( D \) of \( \lambda_0 \), the only analytic function \( f: U \to X \) that satisfies the equation \( (\lambda I - T)f(\lambda) = 0 \) for all \( \lambda \in D \) is the function \( f \equiv 0 \). An operator \( T \in L(X) \) is said to have SVEP if \( T \) has SVEP at every point \( \lambda \in \mathbb{C} \).

Evidently, \( T \in L(X) \) has SVEP at every point of the resolvent \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). Moreover, from the identity theorem for analytic functions it is easily seen that \( T \) has SVEP at every point of the boundary \( \partial \sigma(T) \) of the spectrum. In particular, \( T \) has SVEP at every isolated point of the spectrum. Note that the localized SVEP is inherited by the restriction to closed invariant subspaces; i.e., if \( T \) has SVEP at \( \lambda_0 \) and \( M \) is a closed \( T \)-invariant subspace of \( X \), then \( T|_M \) has SVEP at \( \lambda_0 \). Moreover, the set \( \Sigma(T) \) of all points \( \lambda \in \mathbb{C} \) such that \( T \) does not have SVEP at \( \lambda \) is an open set contained in the interior of the spectrum of \( T \). Consequently, if \( T \) has SVEP at each point \( \lambda \) of an open punctured disc \( \mathbb{D} \setminus \{\lambda_0\} \) centered at \( \lambda_0 \), then \( T \) also has SVEP at \( \lambda_0 \).

We have

\[(1) \quad p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda,\]

and dually,

\[(2) \quad q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda;\]

see [1] Theorem 3.8]. Furthermore, from the definition of localized SVEP it is easily seen that

\[(3) \quad \sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda,\]

and dually,

\[(4) \quad \sigma_s(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.\]

**Remark 2.2.** The implications (1), (2), (3) and (4) are actually equivalences if \( T \) is a semi-Fredholm operator; see [5] or [1, Chapter 3].
Lemma 2.3. If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:

(i) there exists $n \geq p + 1$ such that $T^n(X)$ is closed;

(ii) $T^n(X)$ is closed for all $n \geq p$.

Proof. Define $c'_i(T) := \dim(\ker T^i/\ker T^{i+1})$. Clearly, $p = p(T) < \infty$ entails that $c'_i(T) = 0$ for all $i \geq p$, so $k_i(T) := c'_i(T) - c'_{i+1}(T) = 0$ for all $i \geq p$. The equivalence then easily follows from [21, Lemma 12].

Define

\[ \Delta(T) := \{ n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T \}. \]

The degree of stable iteration is defined as $\text{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\text{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.4. $T \in L(X)$ is said to be quasi-Fredholm of degree $d$ if there exists $d \in \mathbb{N}$ such that:

(a) $\text{dis}(T) = d$,

(b) $T^n(X)$ is a closed subspace of $X$ for each $n \geq d$,

(c) $T(X) + \ker T^d$ is a closed subspace of $X$.

It should be noted that by Proposition 2.5 of [13] every semi-B-Fredholm operator is quasi-Fredholm.

Theorem 2.5. For every $T \in L(X)$ the following statements are equivalent:

(i) $T$ is left Drazin invertible;

(ii) There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below;

(iii) $T$ is semi-B-Fredholm and $T$ has SVEP at 0.

Dually, if $T \in L(X)$ the following statements are equivalent:

(iv) $T$ is right Drazin invertible;

(v) there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is onto;

(vi) $T$ is semi-B-Fredholm and $T^*$ has SVEP at 0.

Proof. (i) $\iff$ (ii) Suppose that $T$ is left Drazin invertible. Then $p = p(T) < \infty$ and $T^{p+1}(X)$ is closed. From Lemma 2.3 it follows that $T^p(X)$ is closed. By [11, Lemma 3.2] we have $\ker T \cap T^p(X) = \ker T_{[p]} = \{0\}$, so $T_{[p]}$ is injective. The range of $T_{[p]}$ is closed, since it coincides with $T^{p+1}(X)$; hence $T_{[p]}$ is bounded below, so the condition (ii) is satisfied.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below. Let us consider an element $x \in \ker T^{n+1}$. Clearly, $T(T^n x) = 0$ so $T^n x \in \ker T$. Since $T^n x \in T^n(X)$ it then follows that $T^n x \in \ker T \cap T^n(X) = \ker T_{[n]} = \{0\}$; thus $x \in \ker T^n$. Therefore, $\ker T^{n+1} = \ker T^n$, so $T$ has finite ascent $p := p(T) \leq n$. The range of $T_{[n]}$ is the closed subspace $T^{n+1}(X)$, with $p + 1 \leq n + 1$. Therefore $T^{p+1}(X)$ is closed; thus $T$ is left Drazin invertible.

(ii) $\iff$ (iii) Assume (i) or equivalently (ii). Then $T$ has SVEP at 0, since $p(T) < \infty$ and $T_{[n]}$ is upper semi-Fredholm, so $T$ is upper semi-B-Fredholm.

Conversely, suppose that $T$ is semi-B-Fredholm and $T$ has SVEP at 0. By Proposition 3.2 of [10] if $T$ quasi-Fredholm, in particular if $T$ is semi-B-Fredholm, then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is semi-regular (i.e., it has closed range and its kernel is contained in the range of each iterate of $T_{[n]}$). Since the restriction $T_{[n]}$ has SVEP at 0, from Theorem 2.49 of [11] it then follows that $T_{[n]}$ is bounded below.
Theorem 2.7. [2] More generally for quasi-Fredholm operators, may be characterized as follows:

\[ T\text{is upper semi-Browder for some } n \iff \text{upper semi-Fredholm, then } \hat{T} \text{ is onto. Moreover, } T^n(X) \text{ is closed by assumption. Conversely, if (v) holds, then } T^{n+1}(X) = T^n(X) \text{ so } q := q(T) \leq n. \text{ Obviously, } T^q(X) = T^n(X) \text{ is closed.} \]

(v) \iff (vi). Assume (v), or equivalently (iv). Since \( q := q(T) < \infty \), then \( T^* \) has SVEP at 0 and, clearly, \( T_n \) is lower semi-Fredholm, so (vi) holds. The opposite implication has been proved in [2] Theorem 2.7. \( \square \)

Corollary 2.6. \( T \in L(X) \) is Drazin invertible if and only if \( T \) is semi-B-Fredholm and both \( T \) and \( T^* \) have SVEP at 0.

The condition that \( T \), or \( T^* \), has SVEP at 0 for semi-B-Fredholm operators, more generally for quasi-Fredholm operators, may be characterized as follows:

Theorem 2.7. [2] Suppose that \( T \in L(X) \) is quasi-Fredholm. Then the following statements are equivalent:

(i) \( T \) has SVEP at 0;
(ii) \( \sigma(T) \) does not cluster at 0.

Dually, if \( T \in L(X) \) is quasi-Fredholm, then the following statements are equivalent:

(iii) \( T^* \) has SVEP at 0;
(iv) \( \sigma(T) \) does not cluster at 0.

Given \( n \in \mathbb{N} \) let us denote by \( \hat{T}_n : X/\ker T^n \to X/\ker T^n \) the quotient map defined canonically by \( \hat{T}_n \hat{x} := \hat{T}x \) for each \( \hat{x} \in \hat{X} := X/\ker T^n \), where \( x \in \hat{x} \).

Lemma 2.8. Suppose that \( T \in L(X) \) and \( T^n(X) \) is closed for some \( n \in \mathbb{N} \). If \( T_n \) is upper semi-Fredholm, then \( \hat{T}_n \) is upper semi-Fredholm and \( \text{ind} \hat{T}_n = \text{ind} T_n \). Analogous statements hold if \( T_n \) is assumed to be lower semi-Fredholm, Weyl, upper or lower semi-Browder, respectively.

Proof. The operator \([T^n] : X/\ker T^n \to T^n(X)\) defined by \([T^n]\hat{x} = T^n x, \text{ where } x \in \hat{x},\) is a bijection, and it easy to check that \([T^n]\hat{T}_n = T_n[T^n],\) from which the statements follow. \( \square \)

Theorem 2.9. Suppose that \( T \in L(X) \). Then the following equivalences hold:

(i) \( T \) is upper semi-B-Browder if and only if \( T \) is left Drazin invertible.
(ii) \( T \) is lower semi-B-Browder if and only if \( T \) is right Drazin invertible.
(iii) \( T \) is B-Browder if and only if \( T \) is Drazin invertible.

Proof. (i) Trivially, every bounded below operator is upper semi-Browder. By Theorem 2.6 if \( T \) is left Drazin invertible, then \( T \) is upper semi-B-Browder.

Conversely, suppose that \( T \) is upper semi-B-Browder. By Lemma 2.8 then \( \hat{T}_n \) is upper semi-Browder for some \( n \in \mathbb{N} \) and hence by Remark 2.2 the condition \( p(\hat{T}_n) < \infty \) is equivalent to saying that \( \sigma_n(\hat{T}_n) \) does not cluster at 0. Let \( D(0, \varepsilon) \) be an open ball centered at 0 such that \( D(0, \varepsilon) \setminus \{0\} \) does not contain points of \( \sigma_n(\hat{T}_n) \), so

\[ \ker (\lambda I - \hat{T}_n) = \{0\} \text{ for all } 0 < |\lambda| < \varepsilon. \]

Since the restriction \( T|\ker T^n \) is nilpotent we also have that \( D(0, \varepsilon) \setminus \{0\} \subseteq \rho(T|\ker T^n), \rho(T|\ker T^n) \) the resolvent of \( T|\ker T^n \), so

\[ (\lambda I - T)(\ker T^n) = \ker T^n \text{ for all } 0 < |\lambda| < \varepsilon. \]
Since for all $0 < |\lambda| < \varepsilon$ we also have $\ker (\lambda I - T) \ker T^n = \{0\}$, it then easily follows that $\ker (\lambda I - T) = \{0\}$, i.e. $\lambda I - T$ is injective for all $0 < |\lambda| < \varepsilon$.

We show now that $(\lambda I - T)(X)$ is closed for all $0 < |\lambda| < \varepsilon$.

Set $\bar{X} := X/\ker T^n$ and let $w \in (\lambda I - T)(X)$ be arbitrary. Then there exists $x \in X$ such that $w = (\lambda I - T)x$ and hence $\hat{w} = (\lambda I - \hat{T}_n)\hat{x} \in (\lambda I - \hat{T}_n)(\bar{X})$. Since $\lambda \notin \sigma_\alpha(\hat{T}_n)$, then $(\lambda I - \hat{T}_n)(\bar{X})$ is closed, and hence there exists a sequence $(w_n) \subset X$ such that $(\lambda I - \hat{T}_n)w_n \to \hat{w}$ as $n \to +\infty$; thus

$$(\lambda I - T)w_n - w \to z_n \in \ker T^n.$$  

From (i) we know that there exists $y_n \in \ker T^n$ such that $z_n = (\lambda I - T)y_n$, and hence

$$(\lambda I - T)w_n - (\lambda I - T)y_n = (\lambda I - T)(w_n - y_n) \to w,$$

so that $(\lambda I - T)(X)$ is closed. We have shown that $\lambda I - T$ is bounded below for all $0 < |\lambda| < \varepsilon$ and, consequently, $0$ is an isolated point of $\sigma_\alpha(T)$. This implies that $T$ has SVEP at $0$ and since by assumption $T$ is upper semi-B-Browder from Theorem 2.8, we then conclude that $T$ is left Drazin invertible.

(ii) By Theorem 2.5, if $T$ is right Drazin invertible, then there exists $n \in \mathbb{N}$ such that $T_n$ is onto and hence lower semi-B-Browder.

Conversely, suppose that $T$ is lower semi-B-Browder and let $n \in \mathbb{N}$ such that $T_n$ is lower semi-Browder. By Lemma 2.8, then $\hat{T}_n$ is lower semi-Browder and hence the condition $q(\hat{T}_n) < \infty$ is equivalent to saying that $\sigma_\alpha(\hat{T}_n)$ does not cluster at $0$. Let $\mathbb{D}(0, \varepsilon)$ be an open ball centered at $0$ such that $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ does not contain points of $\sigma_\alpha(\hat{T}_n)$. As in the proof of part (i) we have $(\lambda I - T)(\ker T^n) = \ker T^n$ for all $0 < |\lambda| < \varepsilon$. We show that $(\lambda I - T)(X) = X$ for all $0 < |\lambda| < \varepsilon$. Since $\lambda I - \hat{T}_n$ is onto, for each $x \in X$ there exists $\hat{y} \in X$ such that $(\lambda I - \hat{T}_n)\hat{y} = \hat{x}$ and hence

$$x - (\lambda I - T)y \in \ker T^n = (\lambda I - T)(\ker T^n).$$

Consequently, there exists $z \in \ker T^n$ such that $x - (\lambda I - T)y = (\lambda I - T)z$, from which it follows that

$$x = (\lambda I - T)(z + y) \in (\lambda I - T)(X).$$

We have proved that $\lambda I - T$ is onto for all $0 < |\lambda| < \varepsilon$; thus $\sigma_\alpha(T)$ does not cluster at $0$ and consequently $T^*$ has SVEP at $0$. By Theorem 2.5 we then conclude that $T$ is right Drazin invertible.

(iii) Clear.

\begin{corollary}
 For every $T \in L(X)$ we have

$$\sigma_{usbb}(T) = \sigma_{ld}(T), \quad \sigma_{lsbb}(T) = \sigma_{rd}(T), \quad \sigma_{bb}(T) = \sigma_d(T).$$

\end{corollary}

3. Browder type theorems

Let us denote by $USBF^-(X)$ the class of all upper semi-B-Fredholm operators with index less than or equal to $0$, while by $LSBF^+(X)$ we denote the class of all lower semi-B-Fredholm operators with index greater than or equal to $0$. Set

$$\sigma_{usbf^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin USBF^-(X)\}$$

and

$$\sigma_{lsbf^+}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin LSBF^+(X)\}.$$
Theorem 3.1. If $T \in L(X)$, then the following equalities hold:

(i) $\sigma_{usbf}(T) = \sigma_{usbf-}(T) \cup \text{acc} \sigma_s(T)$.
(ii) $\sigma_{lsbf}(T) = \sigma_{lsbf+}(T) \cup \text{acc} \sigma_s(T)$.
(iii) $\sigma_{lb}(T) = \sigma_{lbw}(T) \cup \text{acc} \sigma(T)$.

Proof. The proof of the equalities (i), (iii) may be found in [6] and [7]. To show the equality (ii), we observe first that

$$\sigma_{lb+}(T) \subseteq \sigma_{rd}(T).$$

Indeed, if $\lambda \notin \sigma_{rd}(T)$, then, by Theorem 2.10 of [10] $\lambda I - T_n$ is onto some $n \in \mathbb{N}$, hence lower semi-Fredholm and

$$\text{ind}(\lambda I - T) = \text{ind}(\lambda I - T_n) = \alpha(\lambda I - T_n) \geq 0;$$

thus $\lambda \notin \sigma_{lb+}(T)$.

By Corollary 2.10 in order to show the inclusion $\sigma_{lsbf}(T) \subseteq \sigma_{lsbf+}(T) \cup \text{acc} \sigma_s(T)$ we need only to prove that $\text{acc} \sigma_s(T) \subseteq \sigma_{lsbf}(T)$. If $\lambda \notin \sigma_{lsbf}(T) = \sigma_{rd}(T)$, then $\lambda I - T$ is right Drazin invertible, and hence by Theorem 2.5 $\lambda I - T$ is $\sigma$-semi-B-Fredholm with $\varphi(\lambda I - T) < \infty$. By Corollary 4.8 of [16] it then follows that $\lambda I - T$ is onto in a punctured disc centered at $\lambda$; thus $\lambda \notin \text{acc} \sigma_s(T)$.

To show the opposite inclusion $\sigma_{lsbf}(T) \subseteq \sigma_{lsbf+}(T) \cup \text{acc} \sigma_s(T)$, suppose that $\lambda \notin \sigma_{lsbf+}(T) \cup \text{acc} \sigma_s(T)$. Since $\lambda \notin \text{acc} \sigma_s(T)$, then $T^*$ has SVEP at $\lambda$. Since $\lambda I - T$ is lower semi-B-Fredholm by Theorem 2.5 then $\lambda I - T$ is right Drazin invertible. By Corollary 2.10 then $\lambda \notin \sigma_{rd}(T) = \sigma_{lsbf}(T)$, so the equality (ii) is proved.

A bounded operator $T \in L(X)$ is said to satisfy Browder’s theorem if $\sigma_w(T) = \sigma_b(T)$. Denote by $\sigma_{w^{-}}(T)$ the essential approximate point spectrum of $T$, defined as the complement in $\mathbb{C}$ of the set of all $\lambda$ such that $\lambda I - T$ is upper semi-Fredholm with $\text{ind} T \leq 0$. The operator $T \in L(X)$ is said to satisfy a-Browder’s theorem if $\sigma_{w^{-}}(T) = \sigma_{ab}(T)$; see for instance [4].

According to [12], a bounded operator $T \in L(X)$ is said to satisfy the generalized Browder’s theorem if $\sigma(T) \setminus \sigma_{w}(T) = \sigma_s(T)$, while $T \in L(X)$ is said to satisfy the generalized a-Browder’s theorem if $\sigma_a(T) \setminus \sigma_{usbf-}(T) = \sigma_{ld}(T)$.

Note that in all the papers concerning generalized Browder’s theorems (see for instance [7], [15], [12], [8]), there is no trace of the role of $B$-Browder spectra. Our Corollary 2.10 shows that this is only apparent. In fact, by Corollary 2.10 we have:

generalized Browder’s theorem holds for $T \Leftrightarrow \sigma_{bw}(T) = \sigma_{hh}(T)$.

while

generalized a-Browder’s theorem holds for $T \Leftrightarrow \sigma_{usbf-}(T) = \sigma_{usbf}(T)$.

Browder’s theorem may be characterized by localized SVEP: Browder’s theorem (resp. generalized Browder’s theorem) holds for $T$ if and only if $T$ has SVEP at every $\lambda \notin \sigma_w(T)$ (resp. $T$ has SVEP at every $\lambda \notin \sigma_{bw}(T)$, see [7]), while a-Browder’s theorem (resp. generalized a-Browder’s theorem) holds for $T$ if and only if $T$ has SVEP at every $\lambda \notin \sigma_{usbf-}(T)$ (resp. $T$ has SVEP at every $\lambda \notin \sigma_{usbf-}(T)$, see [8]). The inclusions $\sigma_{bw}(T) \subseteq \sigma_w(T)$ and $\sigma_{usbf-}(T) \subseteq \sigma_{usbf}(T)$ immediately entail that the generalized Browder’s theorem implies Browder’s theorem, and, analogously, the generalized a-Browder’s theorem implies a-Browder’s theorem. The main result of a very recent paper [8] proves that Browder’s theorem and the generalized Browder’s theorem (respectively, a-Browder’s theorem and the
Theorem 3.2. For every $T \in L(X)$ the following equivalences hold:

(i) $\sigma_{w}(T) = \sigma_{b}(T) \Leftrightarrow \sigma_{bw}(T) = \sigma_{bb}(T)$.

(ii) $\sigma_{usb}(T) = \sigma_{ub}(T) \Leftrightarrow \sigma_{usbb}(T) = \sigma_{usbf}(T)$.

Proof. (i) We have only to show the implication $\Rightarrow$. Assume that $\sigma_{w}(T) = \sigma_{b}(T)$. Clearly, $\sigma_{bw}(T) \subseteq \sigma_{bb}(T)$ for all $T \in L(X)$. To show the opposite inclusion, assume that $\lambda_{0} \notin \sigma_{bw}(T)$, i.e. that $\lambda_{0}I - T$ is $B$-Weyl. By [13, Corollary 3.2], then there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is Weyl and hence Browder for all $\lambda \in \mathbb{D} \setminus \{\lambda_{0}\}$. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$, then both $T$ and $T^{*}$ have SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_{0}\}$, and hence both $T$ and $T^{*}$ have SVEP at $\lambda_{0}$. By Theorem 2.5, then $\lambda_{0}I - T$ is Drazin invertible, or equivalently $\lambda_{0} \notin \sigma_{bb}(T)$. Hence $\sigma_{bw}(T) = \sigma_{bb}(T)$.

(ii) Also here it suffices to prove the implication $\Rightarrow$. Assume that $\sigma_{usb}(T) = \sigma_{ub}(T)$. Clearly, $\sigma_{usbb}(T) \subseteq \sigma_{usf}(T)$ for all $T \in L(X)$. Suppose that $\lambda_{0} \notin \sigma_{usf}(T)$. Then $\lambda_{0}I - T \in USBF^{-}(X)$ and by [13, Corollary 3.2] there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is upper semi-Fredholm with index less than or equal to 0 for all $\lambda \in \mathbb{D} \setminus \{\lambda_{0}\}$. From assumption then $\lambda I - T$ is upper semi-Browder; hence $p(\lambda I - T) < \infty$. Thus, $T$ has SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_{0}\}$ and hence $T$ also has SVEP at $\lambda_{0}$. By Theorem 2.5 we then conclude that $\lambda_{0} \notin \sigma_{ub}(T) = \sigma_{usbb}(T)$, so the equality $\sigma_{usf}(T) = \sigma_{usbb}(T)$ is proved.

The following result shows that many of the spectra considered before coincide whenever $T$ or $T^{*}$ has SVEP.

Theorem 3.3. Suppose that $T \in L(X)$. Then the following statements hold:

(i) If $T$ has SVEP, then

$$\sigma_{lbf}(T) = \sigma_{lbb}(T) = \sigma_{d}(T) = \sigma_{bw}(T).$$

(ii) If $T^{*}$ has SVEP, then

$$\sigma_{usf}(T) = \sigma_{usbb}(T) = \sigma_{bw}(T) = \sigma_{d}(T).$$

(iii) If both $T$ and $T^{*}$ have SVEP, then

$$\sigma_{usf}(T) = \sigma_{lbf}(T) = \sigma_{bw}(T) = \sigma_{d}(T).$$

Proof. (i) By Theorem 3.1 and Corollary 2.10 we have

$$\sigma_{lbf}(T) \subseteq \sigma_{lbb}(T) = \sigma_{rd}(T) \subseteq \sigma_{d}(T).$$

We show now that $\sigma_{d}(T) \subseteq \sigma_{lbf}(T)$. Assume that $\lambda \notin \sigma_{lbf}(T)$. We may assume $\lambda = 0$. Since $T$ is lower semi-$B$-Fredholm and since $T^{*}$ has SVEP, in particular $T^{*}$ has SVEP at 0, by Theorem 2.5 then $T$ is right Drazin invertible or, equivalently, lower semi-$B$-Browder. Therefore there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Fredholm and $q(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\text{ind } T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{lbf}(T)$, we also have $\text{ind } T_{[n]} \geq 0$ from which we obtain $\text{ind } T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $p(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence $T$ is $B$-Browder. By part (iii) of Theorem 2.10 then $T$ is Drazin invertible, so $0 \notin \sigma_{d}(T)$, as desired. Finally, since $T$ has SVEP by which the $T$ satisfies the generalized Browder’s theorem, we have $\sigma_{bw}(T) = \sigma_{d}(T)$ and the equalities (8) are proved.
(ii) The inclusion $\sigma_{\text{sbf}}(T) \subseteq \sigma_{\text{ubhf}}(T) = \sigma_{\text{id}}(T) \subseteq \sigma_d(T)$ holds for every $T \in L(X)$ by Theorem 3.1 and Corollary 2.10.

We show that $\sigma_d(T) \subseteq \sigma_{\text{ubhf}}(T)$. Suppose that $\lambda \notin \sigma_{\text{ubhf}}(T)$ and assume that $\lambda = 0$. Since $T$ is upper semi-$B$-Fredholm, then there exists $n \in \mathbb{N}$ such that $T_n$ is upper semi-Fredholm. The restriction $T_n := T|T^n(X)$ has SVEP, in particular has SVEP at 0 and hence, see Remark 2.2, $p(T_n) < \infty$. By Theorem 3.4 of [1] it then follows that $\text{ind } T_n \leq 0$. On the other hand, since $\lambda \notin \sigma_{\text{sbf}}(T)$, we also have $\text{ind } T_n \geq 0$ from which we obtain $\text{ind } T_n = 0$. This implies, again by Theorem 3.4 of [1], that also $q(T_n) < \infty$, so that $T_n$ is Browder and hence $T$ is $B$-Browder. By part (iii) of Theorem 2.9 then $T$ is Drazin invertible, so $0 \notin \sigma_d(T)$, as desired. Finally, since $T$ has SVEP, then $T$ satisfies the generalized Browder's theorem, so $\sigma_{\text{bw}}(T) = \sigma_d(T)$.

(iii) Clear from parts (i), (ii).

\[ \Box \]

References


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