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ON DRAZIN INVERTIBILITY

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Abstract. The left Drazin spectrum and the Drazin spectrum coincide with the upper semi-B-Browder spectrum and the B-Browder spectrum, respectively. We also prove that some spectra coincide whenever T or T^* satisfies the single-valued extension property.

1. Introduction and preliminaries

Throughout this note $L(X)$ will denote the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. The operator $T \in L(X)$ is said to be *upper semi-Fredholm* if its kernel ker T is finite-dimensional and the range $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm* if $T(X)$ is finite-codimensional. If either T is upper or lower semi-Fredholm, then T is said to be a *semi-Fredholm operator*, while T is said to be a *Fredholm operator* if it is both upper and lower semi-Fredholm. If $T \in L(X)$ is semi-Fredholm, the classical *index* of T is defined by ind $(T) := \dim \ker T - \operatorname{codim} T(X)$.

The concept of semi-Fredholm operators has been generalized by Berkani ([9], [13] and [11]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi-B-Fredholm*, (resp. *B-Fredholm, upper semi-B-Fredholm*, *lower semi-B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([13]). This enables one to define the index of a semi-B-Fredholm operator as ind $T =$ ind $T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be a *Weyl operator* if T is a Fredholm operator having index 0. A bounded operator $T \in L(X)$ is said to be *B-Weyl* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *Weyl spectrum* and the *B-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}$$

and

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not } B\text{-Weyl}\}.$$

Recall that the *ascent* of an operator $T \in L(X)$ is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not

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exist, we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest nonnegative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$; see [1, Theorem 3.3]. Moreover, if $\lambda \in \mathbb{C}$, the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ is equivalent to saying that λ is a pole of the resolvent. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$; see [17, Prop. 50.2].

The concept of Drazin invertibility [14] has been introduced in a more abstract setting than operator theory [14]. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) precisely when $p(T) = q(T) < \infty$ and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent; see [19, Corollary 2.2] and [18, Prop. A]. Every B-Fredholm operator T admits the representation $T = T_0 \oplus T_1$, where T_0 is Fredholm and T_1 is nilpotent [11], so every Drazin invertible operator is B-Fredholm.

The concept of Drazin invertibility for bounded operators may be extended as follows.

Definition 1.1. $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed; while $T \in L(X)$ is said to be *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed.

It should be noted that the condition $q = q(T) < \infty$ does not entails that $T^q(X)$ is closed; see Example 5 of [21]. Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$, then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$; see [17, Prop. 50.2].

The *left Drazin spectrum* is then defined as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the *right Drazin spectrum* is defined as

$$\sigma_{\rm rd}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

and the *Drazin spectrum* is defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$.

A bounded operator $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp. T is upper semi-Fredholm and $p(T) < \infty$, T is lower semi-Fredholm and $q(T) < \infty$). Every Browder operator is Weyl and hence, if

$$\sigma_{\rm b}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}$$

denotes the Browder spectrum of T, then $\sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T)$. In the sequel by $\sigma_{\rm usb}(T)$ we shall denote the *upper semi-Browder spectrum* of T defined by

$$\sigma_{\rm usb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\}.$$

Clearly, every bounded below operator $T \in L(X)$ (T injective with closed range) is upper semi-Browder, while every surjective operator is lower semi-Browder. The classical *approximate point spectrum* of T will be denoted by $\sigma_{\rm a}(T)$ while by $\sigma_{\rm s}(T)$ we shall denote the *surjectivity spectrum* of T.

It is natural to extend the concept of semi-Browder operators as follows: A bounded operator $T \in L(X)$ is said to be *B-Browder* (resp. *upper semi-B-Browder*, *lower semi-B-Browder*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder). The respective B-Browder spectra are denoted by $\sigma_{\mathrm{bb}}(T)$, $\sigma_{\mathrm{usbb}}(T)$ and $\sigma_{\mathrm{lsbb}}(T)$.

The main result of this paper establishes that $T \in L(X)$ is B-Browder (respectively, upper semi-B-Browder, lower semi-Browder) if and only if T is Drazin invertible (respectively, left Drazin invertible, right Drazin invertible); consequently $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{d}}(T)$, $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ld}}(T)$ and $\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{rd}}(T)$. We also prove that many of the spectra before introduced coincide whenever T, or its dual T^*, satisfies the single-valued extension property.

2. SVEP and semi-B-Browder spectra

A useful tool in the Fredholm theory is given by the localized single-valued extension property. This property has an important role in local spectral theory; see the recent monographs by Laursen and Neumann [20] and Aiena [1].

Definition 2.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ of λ_0, the only analytic function $f : U \to X$ that satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that the localized SVEP is inherited by the restriction to closed invariant subspaces; i.e., if T has SVEP at λ_0 and M is a closed T-invariant subspace of X, then $T|M$ has SVEP at λ_0. Moreover, the set $\Sigma(T)$ of all points $\lambda \in \mathbb{C}$ such that T does not have SVEP at λ is an open set contained in the interior of the spectrum of T. Consequently, if T has SVEP at each point λ of an open punctured disc $\mathbb{D} \setminus \{\lambda_0\}$ centered at λ_0, then T also has SVEP at λ_0.

We have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually,

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda; \tag{2}$$

see [1, Theorem 3.8]. Furthermore, from the definition of localized SVEP it is easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

Remark 2.2. The implications (1), (2), (3) and (4) are actually equivalences if T is a semi-Fredholm operator; see [5] or [1, Chapter 3].

Lemma 2.3. *If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:*
(i) *there exists $n \geq p+1$ such that $T^n(X)$ is closed;*
(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. Define $c_i'(T) := \dim(\ker T^i/\ker T^{i+1})$. Clearly, $p = p(T) < \infty$ entails that $c_i'(T) = 0$ for all $i \geq p$, so $k_i(T) := c_i'(T) - c_{i+1}'(T) = 0$ for all $i \geq p$. The equivalence then easily follows from [21, Lemma 12]. □

Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.4. $T \in L(X)$ is said to be *quasi-Fredholm of degree d* if there exists $d \in \mathbb{N}$ such that:
(a) $\mathrm{dis}(T) = d$,
(b) $T^n(X)$ is a closed subspace of X for each $n \geq d$,
(c) $T(X) + \ker T^d$ is a closed subspace of X.

It should be noted that by Proposition 2.5 of [13] every semi-B-Fredholm operator is quasi-Fredholm.

Theorem 2.5. *For every $T \in L(X)$ the following statements are equivalent:*
(i) *T is left Drazin invertible;*
(ii) *There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below;*
(iii) *T is semi-B-Fredholm and T has SVEP at 0.*
Dually, if $T \in L(X)$ the following statements are equivalent:
(iv) *T is right Drazin invertible;*
(v) *there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is onto;*
(vi) *T is semi-B-Fredholm and T^* has SVEP at 0.*

Proof. (i) $\Leftrightarrow$ (ii) Suppose that T is left Drazin invertible. Then $p = p(T) < \infty$ and $T^{p+1}(X)$ is closed. From Lemma 2.3 it follows that $T^p(X)$ is closed. By [1, Lemma 3.2] we have $\ker T \cap T^p(X) = \ker T_{[p]} = \{0\}$, so $T_{[p]}$ is injective. The range of $T_{[p]}$ is closed, since it coincides with $T^{p+1}(X)$; hence $T_{[p]}$ is bounded below, so the condition (ii) is satisfied.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below. Let us consider an element $x \in \ker T^{n+1}$. Clearly, $T(T^n x) = 0$ so $T^n x \in \ker T$. Since $T^n x \in T^n(X)$ it then follows that $T^n x \in \ker T \cap T^n(X) = \ker T_{[n]} = \{0\}$; thus $x \in \ker T^n$. Therefore, $\ker T^{n+1} = \ker T^n$, so T has finite ascent $p := p(T) \leq n$. The range of $T_{[n]}$ is the closed subspace $T^{n+1}(X)$, with $p+1 \leq n+1$. Therefore $T^{p+1}(X)$ is closed; thus T is left Drazin invertible.

(ii) $\Leftrightarrow$ (iii) Assume (i) or equivalently (ii). Then T has SVEP at 0, since $p(T) < \infty$ and $T_{[n]}$ is upper semi-Fredholm, so T is upper semi-B-Fredholm.

Conversely, suppose that T is semi-B-Fredholm and T has SVEP at 0. By Proposition 3.2 of [10] if T quasi-Fredholm, in particular if T is semi-B-Fredholm, then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is semi-regular (i.e., it has closed range and its kernel is contained in the range of each iterate of $T_{[n]}$). Since the restriction $T_{[n]}$ has SVEP at 0, from Theorem 2.49 of [1] it then follows that $T_{[n]}$ is bounded below.

(iv) $\Leftrightarrow$ (v) If $q := q(T) < \infty$, then $T(T^q(X)) = T^{q+1}(X) = T^q(X)$, so $T_{[q]}$ is onto. Moreover, $T^q(X)$ is closed by assumption. Conversely, if (v) holds, then $T^{n+1}(X) = T^n(X)$ so $q := q(T) \leq n$. Obviously, $T^q(X) = T^n(X)$ is closed.

(v) $\Leftrightarrow$ (vi). Assume (v), or equivalently (iv). Since $q := q(T) < \infty$, then T^* has SVEP at 0 and, clearly, $T_{[n]}$ is lower semi-Fredholm, so (vi) holds. The opposite implication has been proved in [2, Theorem 2.7]. □

Corollary 2.6. *$T \in L(X)$ is Drazin invertible if and only if T is semi-B-Fredholm and both T and T^* have SVEP at 0.*

The condition that T, or T^*, has SVEP at 0 for semi-B-Fredholm operators, more generally for quasi-Fredholm operators, may be characterized as follows:

Theorem 2.7. [2] *Suppose that $T \in L(X)$ is quasi-Fredholm. Then the following statements are equivalent:*

(i) *T has SVEP at 0;*

(ii) *$\sigma_{\mathrm{a}}(T)$ does not cluster at 0.*

Dually, if $T \in L(X)$ is quasi-Fredholm, then the following statements are equivalent:

(iii) *T^* has SVEP at 0;*

(iv) *$\sigma_{\mathrm{s}}(T)$ does not cluster at 0.*

Given $n \in \mathbb{N}$ let us denote by $\widehat{T}_n : X/\ker T^n \to X/\ker T^n$ the quotient map defined canonically by $\widehat{T}_n\, \hat{x} := \widehat{Tx}$ for each $\hat{x} \in \widehat{X} := X/\ker T^n$, where $x \in \hat{x}$.

Lemma 2.8. *Suppose that $T \in L(X)$ and $T^n(X)$ is closed for some $n \in \mathbb{N}$. If $T_{[n]}$ is upper semi-Fredholm, then $\widehat{T}_n$ is upper semi-Fredholm and ind $\widehat{T}_n$ = ind $T_{[n]}$. Analogous statements hold if $T_{[n]}$ is assumed to be lower semi-Fredholm, Weyl, upper or lower semi-Browder, respectively.*

Proof. The operator $[T^n] : X/\ker T^n \to T^n(X)$ defined by

$$[T^n]\hat{x} = T^n x, \quad \text{where } x \in \hat{x},$$

is a bijection, and it easy to check that $[T^n]\widehat{T}_n = T_{[n]}[T^n]$, from which the statements follow. □

Theorem 2.9. *Suppose that $T \in L(X)$. Then the following equivalences hold:*

(i) *T is upper semi-B-Browder if and only if T is left Drazin invertible.*

(ii) *T is lower semi-B-Browder if and only if T is right Drazin invertible.*

(iii) *T is B-Browder if and only if T is Drazin invertible.*

Proof. (i) Trivially, every bounded below operator is upper semi-Browder. By Theorem 2.5 if T is left Drazin invertible, then T is upper semi-B-Browder.

Conversely, suppose that T is upper semi-B-Browder. By Lemma 2.8, then $\widehat{T}_n$ is upper semi-Browder for some $n \in \mathbb{N}$ and hence by Remark 2.2 the condition $p(\widehat{T}_n) < \infty$ is equivalent to saying that $\sigma_{\mathrm{a}}(\widehat{T}_n)$ does not cluster at 0. Let $\mathbb{D}(0,\varepsilon)$ be an open ball centered at 0 such that $\mathbb{D}(0,\varepsilon) \setminus \{0\}$ does not contain points of $\sigma_{\mathrm{a}}(\widehat{T}_n)$, so

$$\ker(\lambda I - \widehat{T}_n) = \{0\} \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{5}$$

Since the restriction $T|\ker T^n$ is nilpotent we also have that $\mathbb{D}(0,\varepsilon) \setminus \{0\} \subseteq \rho(T|\ker T^n)$, $\rho(T|\ker T^n)$ the resolvent of $T|\ker T^n$, so

$$(\lambda I - T)(\ker T^n) = \ker T^n \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{6}$$

Since for all $0 < |\lambda| < \varepsilon$ we also have $\ker(\lambda I - T|\ker T^n) = \{0\}$, it then easily follows that $\ker(\lambda I - T) = \{0\}$, i.e. $\lambda I - T$ is injective for all $0 < |\lambda| < \varepsilon$.

We show now that $(\lambda I - T)(X)$ is closed for all $0 < |\lambda| < \varepsilon$.

Set $\hat{X} := X/\ker T^n$ and let $w \in (\lambda I - T)(X)$ be arbitrary. Then there exists $x \in X$ such that $w = (\lambda I - T)x$ and hence $\hat{w} = (\lambda I - \widehat{T}_n)\hat{x} \in (\lambda I - \widehat{T}_n)(\hat{X})$. Since $\lambda \notin \sigma_{\rm a}(\widehat{T}_n)$, then $(\lambda I - \widehat{T}_n)(\hat{X})$ is closed, and hence there exists a sequence $(w_n) \subset X$ such that $(\lambda I - \widehat{T}_n)\hat{w_n} \to \hat{w}$ as $n \to +\infty$; thus

$$(\lambda I - T)w_n - w \to z_n \in \ker T^n.$$

From (6) we know that there exists $y_n \in \ker T^n$ such that $z_n = (\lambda I - T)y_n$, and hence

$$(\lambda I - T)w_n - (\lambda I - T)y_n = (\lambda I - T)(w_n - y_n) \to w,$$

so that $(\lambda I - T)(X)$ is closed. We have shown that $\lambda I - T$ is bounded below for all $0 < |\lambda| < \varepsilon$ and, consequently, 0 is an isolated point of $\sigma_{\rm a}(T)$. This implies that T has SVEP at 0 and since by assumption T is upper semi-B-Browder from Theorem 2.5, we then conclude that T is left Drazin invertible.

(ii) By Theorem 2.5, if T is right Drazin invertible, then there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is onto and hence lower semi-Browder.

Conversely, suppose that T is lower semi-B-Browder and let $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Browder. By Lemma 2.8, then $\widehat{T}_n$ is lower semi-Browder and hence the condition $q(\widehat{T}_n) < \infty$ is equivalent to saying that $\sigma_{\rm s}(\widehat{T}_n)$ does not cluster at 0. Let $\mathbb{D}(0, \varepsilon)$ be an open ball centered at 0 such that $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ does not contain points of $\sigma_{\rm s}(\widehat{T}_n)$. As in the proof of part (i) we have $(\lambda I - T)(\ker T^n) = \ker T^n$ for all $0 < |\lambda| < \varepsilon$. We show that $(\lambda I - T)(X) = X$ for all $0 < |\lambda| < \varepsilon$. Since $\lambda I - \widehat{T}_n$ is onto, for each $x \in X$ there exists $y \in X$ such that $(\lambda I - \widehat{T}_n)\hat{y} = \hat{x}$ and hence

$$x - (\lambda I - T)y \in \ker T^n = (\lambda I - T)(\ker T^n).$$

Consequently, there exists $z \in \ker T^n$ such that $x - (\lambda I - T)y = (\lambda I - T)z$, from which it follows that

$$x = (\lambda I - T)(z + y) \in (\lambda I - T)(X).$$

We have proved that $\lambda I - T$ is onto for all $0 < |\lambda| < \varepsilon$; thus $\sigma_{\rm s}(T)$ does not cluster at 0 and consequently T^* has SVEP at 0. By Theorem 2.5 we then conclude that T is right Drazin invertible.

(iii) Clear. □

Corollary 2.10. *For every $T \in L(X)$ we have*

$$\sigma_{\rm usbb}(T) = \sigma_{\rm ld}(T), \quad \sigma_{\rm lsbb}(T) = \sigma_{\rm rd}(T), \quad \sigma_{\rm bb}(T) = \sigma_{\rm d}(T).$$

3. Browder type theorems

Let us denote by $USBF^-(X)$ the class of all upper semi-B-Fredholm operators with index less than or equal to 0, while by $LSBF^+(X)$ we denote the class of all lower semi-B-Fredholm operators with index greater than or equal to 0. Set

$$\sigma_{\rm usbf^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin USBF^-(X)\}$$

and

$$\sigma_{\rm lsbf^+}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin LSBF^+(X)\}.$$

Theorem 3.1. *If $T \in L(X)$, then the following equalities hold:*
(i) $\sigma_{\rm usbb}(T) = \sigma_{\rm usbf^-}(T) \cup acc\,\sigma_{\rm a}(T)$.
(ii) $\sigma_{\rm lsbb}(T) = \sigma_{\rm lsbf^+}(T) \cup acc\,\sigma_{\rm s}(T)$.
(iii) $\sigma_{\rm bb}(T) = \sigma_{\rm bw}(T) \cup acc\,\sigma(T)$.

Proof. The proof of the equalities (i), (iii) may be found in [6] and [7]. To show the equality (ii), we observe first that

$$\sigma_{\rm lsbf^+}(T) \subseteq \sigma_{\rm rd}(T). \tag{7}$$

Indeed, if $\lambda \notin \sigma_{\rm rd}(T)$, then, by Theorem 2.5, $\lambda I - T_{[n]}$ is onto some $n \in \mathbb{N}$, hence lower semi-Fredholm and

$$\mathrm{ind}(\lambda I - T) = \mathrm{ind}(\lambda I - T_{[n]}) = \alpha(\lambda I - T_{[n]}) \geq 0;$$

thus $\lambda \notin \sigma_{\rm lsbf^+}(T)$.

By Corollary 2.10, in order to show the inclusion $\sigma_{\rm lsbb}(T) \supseteq \sigma_{\rm lsbf^+}(T) \cup \mathrm{acc}\,\sigma_{\rm s}(T)$ we need only to prove that $\mathrm{acc}\,\sigma_{\rm s}(T) \subseteq \sigma_{\rm lsbb}(T)$. If $\lambda \notin \sigma_{\rm lsbb}(T) = \sigma_{\rm rd}(T)$, then $\lambda I - T$ is right Drazin invertible, and hence by Theorem 2.5, $\lambda I - T$ is T is semi-B-Fredholm with $q(\lambda I - T) < \infty$. By Corollary 4.8 of [16] it then follows that $\lambda I - T$ is onto in a punctured disc centered at λ; thus $\lambda \notin \mathrm{acc}\,\sigma_{\rm s}(T)$.

To show the opposite inclusion $\sigma_{\rm lsbb}(T) \subseteq \sigma_{\rm lsbf^+}(T) \cup \mathrm{acc}\,\sigma_{\rm s}(T)$, suppose that $\lambda \notin \sigma_{\rm lsbf^+}(T) \cup \mathrm{acc}\,\sigma_{\rm s}(T)$. Since $\lambda \notin \mathrm{acc}\,\sigma_{\rm s}(T)$, then T^* has SVEP at λ. Since $\lambda I - T$ is lower semi-B-Fredholm by Theorem 2.5, then $\lambda I - T$ is right Drazin invertible. By Corollary 2.10, then $\lambda \notin \sigma_{\rm rd}(T) = \sigma_{\rm lsbb}(T)$, so the equality (ii) is proved. □

A bounded operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$. Denote by $\sigma_{\rm usf^-}(T)$ the *essential approximate point spectrum* of T, defined as the complement in $\mathbb{C}$ of the set of all λ such that $\lambda I - T$ is upper semi-Fredholm with $\mathrm{ind}\,T \leq 0$. The operator $T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\sigma_{\rm usf^-}(T) = \sigma_{\rm ub}(T)$; see for instance [4].

According to [12], a bounded operator $T \in L(X)$ is said to satisfy the *generalized Browder's theorem* if $\sigma(T) \setminus \sigma_{\rm bw}(T) = \sigma_{\rm d}(T)$, while $T \in L(X)$ is said to satisfy the *generalized a-Browder's theorem* if $\sigma_{\rm a}(T) \setminus \sigma_{\rm usbf^-}(T) = \sigma_{\rm ld}(T)$.

Note that in all the papers concerning generalized Browder's theorems (see for instance [7], [15], [12], [8]), there is no trace of the role of B-Browder spectra. Our Corollary 2.10 shows that this is only apparent. In fact, by Corollary 2.10 we have:

$$\text{generalized Browder's theorem holds for } T \Leftrightarrow \sigma_{\rm bw}(T) = \sigma_{\rm bb}(T),$$

while

$$\text{generalized } a\text{-Browder's theorem holds for } T \Leftrightarrow \sigma_{\rm usbf^-}(T) = \sigma_{\rm usbb}(T).$$

Browder's theorem may be characterized by localized SVEP: Browder's theorem (resp. generalized Browder's theorem) holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{\rm w}(T)$ ([3]) (resp. T has SVEP at every $\lambda \notin \sigma_{\rm bw}(T)$, see [7]), while a-Browder's theorem (resp. generalized a-Browder's theorem) holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{\rm usf^-}(T)$ ([4]) (resp. T has SVEP at every $\lambda \notin \sigma_{\rm usbf^-}(T)$, see([6]). The inclusions $\sigma_{\rm bw}(T) \subseteq \sigma_{\rm w}(T)$ and $\sigma_{\rm usf^-}(T) \subseteq \sigma_{\rm usbf^-}(T)$ immediately entail that the generalized Browder's theorem implies Browder's theorem, and, analogously, the generalized a-Browder's theorem implies a-Browder's theorem. The main result of a very recent paper [8] proves that Browder's theorem and the generalized Browder's theorem (respectively, a-Browder's theorem and the generalized a-Browder's theorem) are equivalent. These results may be shown in a few lines as follows:

Theorem 3.2. *For every $T \in L(X)$ the following equivalences hold:*
(i) $\sigma_{\rm w}(T) = \sigma_{\rm b}(T) \Leftrightarrow \sigma_{\rm bw}(T) = \sigma_{\rm bb}(T)$.
(ii) $\sigma_{\rm usf^-}(T) = \sigma_{\rm ub}(T) \Leftrightarrow \sigma_{\rm usbf^-}(T) = \sigma_{\rm usbb}(T)$.

Proof. (i) We have only to show the implication $\Rightarrow$. Assume that $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$. Clearly, $\sigma_{\rm bw}(T) \subseteq \sigma_{\rm bb}(T)$ for all $T \in L(X)$. To show the opposite inclusion, assume that $\lambda_0 \notin \sigma_{\rm bw}(T)$, i.e. that $\lambda_0 I - T$ is B-Weyl. By [13, Corollary 3.2], then there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is Weyl and hence Browder for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$, then both T and T^* have SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$, and hence both T and T^* have SVEP at λ_0. By Theorem 2.5, then $\lambda_0 I - T$ is Drazin invertible, or equivalently $\lambda_0 \notin \sigma_{\rm bb}(T)$. Hence $\sigma_{\rm bw}(T) = \sigma_{\rm bb}(T)$.

(ii) Also here it suffices to prove the implication $\Rightarrow$. Assume that $\sigma_{\rm usf^-}(T) = \sigma_{\rm ub}(T)$. Clearly, $\sigma_{\rm usbf^-}(T) \subseteq \sigma_{\rm usf}(T)$ for all $T \in L(X)$. Suppose that $\lambda_0 \notin \sigma_{\rm usbf^-}(T)$. Then $\lambda_0 I - T \in USBF^-(X)$ and by [13, Corollary 3.2] there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is upper semi-Fredholm with index less than or equal to 0 for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. From assumption then $\lambda I - T$ is upper semi-Browder; hence $p(\lambda I - T) < \infty$. Thus, T has SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$ and hence T also has SVEP at λ_0. By Theorem 2.5 we then conclude that $\lambda_0 \notin \sigma_{\rm ld}(T) = \sigma_{\rm usbb}(T)$, so the equality $\sigma_{\rm usbf^-}(T) = \sigma_{\rm usbb}(T)$ is proved. □

The following result shows that many of the spectra considered before coincide whenever T or T^* has SVEP.

Theorem 3.3. *Suppose that $T \in L(X)$. Then the following statements hold:*
(i) *If T has SVEP, then*

$$\sigma_{\rm lsbf^+}(T) = \sigma_{\rm lsbb}(T) = \sigma_{\rm d}(T) = \sigma_{\rm bw}(T). \tag{8}$$

(ii) *If T^* has SVEP, then*

$$\sigma_{\rm usbf^-}(T) = \sigma_{\rm usbb}(T) = \sigma_{\rm bw}(T) = \sigma_{\rm d}(T). \tag{9}$$

(iii) *If both T and T^* have SVEP, then*

$$\sigma_{\rm usbf^-}(T) = \sigma_{\rm lsbf^+}(T) = \sigma_{\rm bw}(T) = \sigma_{\rm d}(T). \tag{10}$$

Proof. (i) By Theorem 3.1 and Corollary 2.10 we have

$$\sigma_{\rm lsbf^+}(T) \subseteq \sigma_{\rm lsbb}(T) = \sigma_{\rm rd}(T) \subseteq \sigma_{\rm d}(T).$$

We show now that $\sigma_{\rm d}(T) \subseteq \sigma_{\rm lsbf^+}(T)$. Assume that $\lambda \notin \sigma_{\rm lsbf^+}(T)$. We may assume $\lambda = 0$. Since T is lower semi-B-Fredholm and since T^* has SVEP, in particular T^* has SVEP at 0, by Theorem 2.5 then T is right Drazin invertible or, equivalently, lower semi-B-Browder. Therefore there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Fredholm and $q(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\operatorname{ind} T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{\rm lsbf^+}(T)$, we also have $\operatorname{ind} T_{[n]} \geq 0$ from which we obtain $\operatorname{ind} T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $p(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence T is B-Browder. By part (iii) of Theorem 2.9 then T is Drazin invertible, so $0 \notin \sigma_{\rm d}(T)$, as desired. Finally, since T has SVEP by which the T satisfies the generalized Browder's theorem, we have $\sigma_{\rm bw}(T) = \sigma_{\rm d}(T)$ and the equalities (8) are proved.

(ii) The inclusion $\sigma_{\mathrm{lsbf}^-}(T) \subseteq \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{d}}(T)$ holds for every $T \in L(X)$ by Theorem 3.1 and Corollary 2.10.

We show that $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$. Suppose that $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$ and assume that $\lambda = 0$. Since T is upper semi-B-Fredholm, then there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is upper semi-Fredholm. The restriction $T_{[n]} := T|T^n(X)$ has SVEP, in particular has SVEP at 0 and hence, see Remark 2.2, $p(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\operatorname{ind} T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T)$, we also have $\operatorname{ind} T_{[n]} \geq 0$ from which we obtain $\operatorname{ind} T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $q(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence T is B-Browder. By part (iii) of Theorem 2.9, then T is Drazin invertible, so $0 \notin \sigma_{\mathrm{d}}(T)$, as desired. Finally, since T has SVEP, then T satisfies the generalized Browder's theorem, so $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$.

(iii) Clear from parts (i), (ii). □

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