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A NOTE ON RESOLUTION OF RATIONAL AND HYPERSURFACE SINGULARITIES

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ABSTRACT. It is well known that the exceptional set in a resolution of a rational surface singularity is a tree of rational curves. We generalize the combinatoric part of this statement to higher dimensions and show that the highest cohomologies of the dual complex associated to a resolution of an isolated rational singularity vanish. We also prove that the dual complex associated to a resolution of an isolated hypersurface singularity is simply connected. As a consequence, we show that the dual complex associated to a resolution of a 3-dimensional Gorenstein terminal singularity has the homotopy type of a point.

1. INTRODUCTION

Let $o \in X$ be an isolated singularity of an algebraic variety (or an analytic space) X defined over a field of characteristic 0, $\dim X \geq 2$. Consider a good resolution $f\colon Y \to X$ (this means that the exceptional locus $Z \subset Y$ of f is a divisor with simple normal crossings). Let $Z = \sum Z_i$, where Z_i are irreducible. To the divisor Z we can associate the dual complex $\Gamma(Z)$. It is a CW-complex whose cells are standard simplexes $\Delta^j_{i_0\dots i_k}$ corresponding to the irreducible components $Z^j_{i_0\dots i_k}$ of the intersections $Z_{i_0} \cap \dots \cap Z_{i_k} = \bigcup_j Z^j_{i_0\dots i_k}$. The $k-1$-simplex $\Delta^{j'}_{i_0\dots\widehat{i_s}\dots i_k}$ is a face of the k-simplex $\Delta^j_{i_0\dots i_k}$ iff $Z^j_{i_0\dots i_k} \cap Z^{j'}_{i_0\dots\widehat{i_s}\dots i_k} \neq \varnothing$. If X and Y are surfaces, then $\Gamma(Z)$ is the usual resolution graph of f. Note that $\Gamma(Z)$ is a simplicial complex iff all the intersections $Z_{i_0} \cap \dots \cap Z_{i_k}$ are irreducible. This can be obtained for a suitable resolution. Also note that if $\dim X = n$, then $\dim(\Gamma(Z)) \leq n-1$.

The complex $\Gamma(Z)$ was first studied by G. L. Gordon in connection to the monodromy in families (see [7]). We say that $\Gamma(Z)$ is *the dual complex associated to the resolution* f. The main reason motivating the study of the dual complex is the fact that the homotopy type of $\Gamma(Z)$ depends only on the singularity $o \in X$

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but not on the choice of a resolution f. This is a consequence of the Abramovich-Karu-Matsuki-Włodarczyk Weak Factorization Theorem in the Logarithmic Category (see [1]). Indeed, this theorem reduces the problem to the case of a blowup $\sigma\colon (X', Z') \to (X, Z)$, where X and X' are smooth varieties with divisors Z and Z' with simple normal crossings, and the center of the blowup is admissible in some sense. It can be explicitly verified that $\Gamma(Z)$ is homotopy equivalent to $\Gamma(Z')$ (see [16]).

For example (it is taken from [7]), consider the singularity

$$\{x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} + x_1 x_2 \dots x_n = 0\} \subset \mathbb{C}^n .$$

A good resolution can be obtained just by blowing up the origin. The reader can easily prove that the exceptional divisor Z consists of n hyperplanes in a general position in $\mathbb{P}^{n-1}$. We see that the complex $\Gamma(Z)$ is the border of a standard $n-1$-dimensional simplex, and thus it has the homotopy type of the sphere S^{n-2}.

If $F\colon (Y, Z) \to (X, o)$ is a resolution of an isolated toric singularity (X, o), then the complex $\Gamma(Z)$ has the homotopy type of a point ([16]).

In this paper, we study the dual complex associated to a resolution in the case when X is a rational or a hypersurface singularity defined over the field $\mathbb{C}$ of complex numbers. We show that if $f\colon Y \to X$ is a good resolution of an isolated rational singularity $o \in X$, $\dim X = n$, then $H^{n-1}(\Gamma(Z), \mathbb{C}) = 0$ (see Theorem 2.2). The proof is a generalization of M. Artin's argument from [3] to the n-dimensional case. The main new ingredient is the lemma on the degeneracy of a spectral sequence associated to a divisor with simple normal crossings on a Kähler manifold (Lemma 2.4). If X is an isolated hypersurface singularity, $\dim X \geq 3$, then $\pi(\Gamma(Z)) = 0$ (see Theorem 3.1). This result is based on the well known fact that the link of an isolated hypersurface singularity of dimension $n \geq 3$ is simply connected ([14]). These results allow us to prove that the homotopy type of the dual complex associated to a resolution of an isolated rational hypersurface 3-dimensional singularity is trivial (Corollary 3.3). As an application, we show that the dual complex associated to a resolution of a 3-dimensional Gorenstein terminal singularity has the homotopy type of a point (Corollary 3.4).

We prove our theorems for algebraic varieties, but everything also holds for analytic spaces (with obvious changes).

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2. Rational singularities

Recall

Definition 2.1. An algebraic variety (or an analytic space) X has *rational singularities* if X is normal and for any resolution $f\colon Y \to X$ all the sheaves $R^i f_* \mathcal{O}_Y$ vanish, $i > 0$.

In the sequel, when we say that f is a good resolution we additionally assume that all the intersections $Z_{i_0} \cap \cdots \cap Z_{i_k}$ of prime components of the exceptional divisor $Z = \sum Z_i$ of f are irreducible, thus $\Gamma(Z)$ is a simplicial complex.

The following theorem can be considered as a generalization of the classical fact that the exceptional locus in a resolution of a rational surface singularity is a tree of rational curves ([3]).

Theorem 2.2. *Let $o \in X$ be an isolated rational singularity of a variety (or an analytic space) X of dimension $n \geq 2$, and let $f\colon Y \to X$ be a good resolution with the exceptional divisor Z. Then the highest (complex) cohomologies of the complex $\Gamma(Z)$ vanish:*

$$H^{n-1}(\Gamma(Z), \mathbb{C}) = 0\,.$$

Proof. Let $Z = \sum_{i=1}^N Z_i$ be the decomposition of the divisor Z to its prime components Z_i. We can assume that X is projective (since the given singularity is isolated) and f is obtained by a sequence of smooth blowups (Hironaka's resolution [12]). Thus all Z_i and Y are Kähler manifolds.

The sheaves $R^i f_* \mathcal{O}_Y$ are concentrated at the point o. Via Grothendieck's theorem on formal functions (see [11], (4.2.1), and [9], Ch. 4, Theorem 4.5 for the analytic case) the completion of the stalk of the sheaf $R^i f_* \mathcal{O}_Y$ at the point o is

$$\varprojlim_{(r)\to(\infty)} H^i(Z, \mathcal{O}_{Z_{(r)}})\,, \tag{1}$$

where $(r) = (r_1, \ldots, r_N)$ and $Z_{(r)} = \sum_{i=1}^N r_i Z_i$. If $(r) \geq (s)$, i. e., $r_i \geq s_i$ $\forall i$, there is a natural surjective map g of sheaves on Z:

$$g\colon \mathcal{O}_{Z_{(r)}} \to \mathcal{O}_{Z_{(s)}}\,.$$

Since dimension of Z is $n-1$, the map g induces a surjective map of cohomologies

$$H^{n-1}(Z, \mathcal{O}_{Z_{(r)}}) \to H^{n-1}(Z, \mathcal{O}_{Z_{(s)}})\,.$$

Recall that the given singularity $o \in X$ is rational, and thus the projective limit (1) is 0. Therefore the cohomology group $H^{n-1}(Z, \mathcal{O}_Z)$ vanishes, too (because the projective system in (1) is surjective). Now it follows from Lemma 2.3 below that $H^{n-1}(\Gamma(Z), \mathbb{C}) = 0$. □

Lemma 2.3. *Let $Z = \sum Z_i$ be a reduced divisor with simple normal crossings on a compact Kähler manifold Y, $\dim Y = n$, and assume that $H^k(Z, \mathcal{O}_Z) = 0$ for some k, $1 \leq k \leq n-1$. Then the k-th cohomologies with coefficients in $\mathbb{C}$ of the complex $\Gamma(Z)$ vanish, too:*

$$H^k(\Gamma(Z), \mathbb{C}) = 0\,.$$

Proof. Let us introduce some notation. Let $Z^0 = \bigsqcup_i Z_i$ be the disjoint union of the irreducible components Z_i and $Z^p = \bigsqcup_{i_0<i_1<\cdots<i_p} Z_{i_0 i_1 \ldots i_p}$ where $Z_{i_0 i_1 \ldots i_p} = Z_{i_0} \cap Z_{i_1} \cdots \cap Z_{i_p}$. By $\varphi_p\colon Z^p \to Z$ denote the natural map. Consider the structure sheaves $\mathcal{O}_{Z^p} = \bigoplus_{i_0<\cdots<i_p} \mathcal{O}_{Z_{i_0 \ldots i_p}}$, the sheaves $\mathcal{A}^{p,q} = \bigoplus_{i_0<\cdots<i_p} \mathcal{A}^{p,q}_{Z_{i_0\ldots i_p}}$ of differential forms of bidegree $(0,q)$ on Z^p and their direct images $\mathcal{K}^p = \varphi_{p*}\mathcal{O}_{Z^p}$ and $\mathcal{K}^{p,q} = \varphi_{p*}\mathcal{A}^{p,q}$ on Z. The sequence of sheaves $\{\mathcal{K}^p\}_{p\geq 1}$ forms a complex via the combinatoric differentials $\delta^p\colon \mathcal{K}^p \to \mathcal{K}^{p+1}$, where if

$$a = \bigoplus a_{i_0\ldots i_p} \in \mathcal{K}^p(U) = \bigoplus_{i_0<\cdots<i_p} \varphi_{p*}\mathcal{O}_{Z_{i_0\ldots i_p}}(U)$$

is a section of the sheaf $\mathcal{K}^p$ over an open set $U \subseteq Z$, then

$$(\delta(a))_{i_0\ldots i_p i_{p+1}}(U) = \sum_{j=0}^{p+1} (-1)^j (a_{i_0\ldots\widehat{i_j}\ldots i_{p+1}})|_{Z_{i_0\cdots i_{p+1}}\cap U}\,.$$

Note that there is also a natural injection of $\mathcal{O}_Z$ into $\mathcal{K}^0$. The sequence of sheaves

$$0 \to \mathcal{O}_Z \to \mathcal{K}^0 \xrightarrow{\delta^0} \mathcal{K}^1 \xrightarrow{\delta^1} \dots$$

is exact. This is easy to check by considering the stalks; in particular, the exactness at $\mathcal{K}^0$ is a consequence of the following fact which holds locally in a sufficiently small neighborhood of every point of Z: if $\{f_i\}$ is a collection of regular functions on Z_i such that their restrictions onto intersections $Z_i \cap Z_j$ coincide, then there exists a regular function f on Z such that $f|_{Z_i} = f_i$ for all i (it is important here that the divisor Z has normal crossings). Therefore the complexes

$$\mathcal{O}^* : \quad \mathcal{O}_Z \to 0 \to 0 \to \dots$$

and

$$\mathcal{K}^* : \quad \mathcal{K}^0 \xrightarrow{\delta^0} \mathcal{K}^1 \xrightarrow{\delta^1} \mathcal{K}^2 \xrightarrow{\delta^2} \dots$$

are quasi-isomorphic. It is clear that the hypercohomologies of the first complex are isomorphic to the cohomologies of Z with coefficients in the structure sheaf: $\mathbf{H}^p(\mathcal{O}^*) \simeq H^p(Z, \mathcal{O}_Z)$. Now let us calculate the hypercohomologies of the complex $\mathcal{K}^*$ by using the acyclic resolutions

$$\mathcal{K}^p \to \mathcal{K}^{p,0} \xrightarrow{\bar\partial} \mathcal{K}^{p,1} \xrightarrow{\bar\partial} \dots ,$$

where $\bar\partial$ is the Dolbeaux differential.

Consider the bigraded sequence of groups

$$K^{p,q} = H^0(\mathcal{K}^{p,q}, Z) \simeq \bigoplus_{i_0 < \dots < i_p} H^0(\varphi_{p*}\mathcal{A}^{p,q})$$

endowed with the differentials $\bar\partial \colon K^{p,q} \to K^{p,q+1}$ and $\delta \colon K^{p,q} \to K^{p+1,q}$, where $\bar\partial$ is the Dolbeaux differential and δ is the combinatoric differential defined as follows: if $\alpha = \bigoplus \alpha_{i_0 \dots i_p} \in K^{p,q}$, then

$$(\delta(\alpha))_{i_0 \dots i_p i_{p+1}} = \sum_{j=0}^{p+1} (-1)^{q+j} (\alpha_{i_0 \dots \widehat{i_j} \dots i_{p+1}})|_{Z_{i_0 \cdots i_{p+1}}} .$$

These differentials satisfy the equality $\bar\partial\delta + \delta\bar\partial = 0$, thus $(K^{*,*}, \delta, \bar\partial)$ is a bicomplex. Let (K^*, d), $K^n = \bigoplus_{p+q=n} K^{p,q}$, $d = \delta + \bar\partial$ be the associated complex. Now we can write that $\mathbf{H}^p(\mathcal{K}^*) = H^p(\mathcal{K}^*, d)$.

There is a filtration

$$F^p K^n = \bigoplus_{\substack{p'+q=n \\ p' \geq p}} K^{p',q}$$

on the complex K^*. It is known (see [10], Ch. 3, section 5) that the spectral sequence E_r associated to the filtration F^pK^n converges to the cohomologies $H^*(K^*, d)$ and

$$E_0^{p,q} \simeq K^{p,q};$$
$$E_1^{p,q} \simeq H^q_{\bar\partial}(K^{p,*});$$
$$E_2^{p,q} \simeq H^p_\delta(H^q_{\bar\partial}(K^{*,*})).$$

In particular,

$$E_1^{p,0} \simeq H^0_{\bar\partial}\Big(\bigoplus_{i_0<\dots<i_p} H^0(\varphi_{p*}\mathcal{A}^{p,*})\Big)$$
$$\simeq \bigoplus_{i_0<\dots<i_p} \mathbb{C}.$$

Therefore the cochain complex

$$0 \to E_1^{0,0} \xrightarrow{\delta} E_1^{1,0} \xrightarrow{\delta} \dots$$

is isomorphic to the cochain complex that one uses to calculate cohomologies of $\Gamma(Z)$ (here we denote by the same letter δ the map between cohomologies induced by the combinatoric differential). It follows that

$$E_2^{p,0} \simeq H^p(\Gamma(Z), \mathbb{C})\,.$$

We shall show that the spectral sequence E_r degenerates in E_2. The method of the proof is based on the standard technique of the theory of the mixed Hodge structures. We learned this from [13], Chapter 4, §2. Also compare [8]. Since the result about E_r can be of a particular interest, we state it as a separate lemma.

Lemma 2.4. *Let $Z = \sum Z_i$ be a reduced divisor with simple normal crossings on a compact Kähler manifold Y, and let E_r be the associated spectral sequence as described above. Then $d_r = 0$ for all $r \geq 2$, i.e., this spectral sequence degenerates in E_2.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & & \\
\cdots & \xrightarrow{\delta} & K^{p,q} & \xrightarrow{\delta} & K^{p+1,q} & \xrightarrow{\delta} & K^{p+2,q} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & & \\
\cdots & \xrightarrow{\delta} & K^{p,q-1} & \xrightarrow{\delta} & K^{p+1,q-1} & \xrightarrow{\delta} & K^{p+2,q-1} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & & \uparrow{\scriptstyle\bar\partial} & &
\end{array}$$

where $\bar\partial\delta + \delta\bar\partial = 0$. First we take cohomologies in the vertical rows and obtain the sequences

$$\begin{array}{c}
\dots\dots \\
\dots \to H^q(K^{p,*}) \xrightarrow{\delta} H^q(K^{p+1,*}) \xrightarrow{\delta} H^q(K^{p+2,*}) \to \dots \\
\dots \to H^{q-1}(K^{p,*}) \xrightarrow{\delta} H^{q-1}(K^{p+1,*}) \xrightarrow{\delta} H^{q-1}(K^{p+2,*}) \to \dots \\
\dots\dots
\end{array}$$

Here δ is the induced map between cohomologies. Then we calculate δ-cohomologies and obtain the differential

$$d_2\colon H^p_\delta(H^q_{\bar\partial}(K^{*,*})) \to H^{p+2}_\delta(H^{q-1}_{\bar\partial}(K^{*,*}))$$

that acts as described below.

An element

$$\bar{\bar{a}} \in H^p_\delta(H^q_{\bar\partial}(K^{*,*}))$$

is a class of those $\bar{a} \in H^q_{\bar\partial}(K^{p,*})$ modulo $\delta(H^q_{\bar\partial}(K^{p-1,*}))$ that map to 0 under the action of δ: $\delta(\bar{a}) = 0$ in $H^q_{\bar\partial}(K^{p+1,*})$. But $\bar{a}$ is a class of $a \in K^{p,q} \mod \bar\partial K^{p,q-1}$ such that $\bar\partial a = 0 \in K^{p,q+1}$. Therefore we can choose a representative $a \in K^{p,q}$ for $\bar{\bar{a}}$ such that $\bar\partial a = 0$ and $\delta a = 0$ modulo $\bar\partial K^{p+1,q-1}$ in $K^{p+1,q}$. It follows that there exists an element $a' \in K^{p+1,q-1}$ such that $\bar\partial a' = \delta a$. Map this a' down to $K^{p+1,q-1}$: $\delta(a') \in K^{p+2,q-1}$. It can be verified by standard methods that $\delta(a')$ correctly determines a class $\overline{\overline{\delta(a')}}$ in $H^{p+2}_\delta(H^{q-1}_{\bar\partial}(K^{*,*}))$ and the differential d_2 is defined as follows:

$$d_2(\bar{\bar{a}}) = \overline{\overline{\delta(a')}}\,.$$

Differentials d_r, $r \geq 3$, can be obtained by iterating this construction. For example, d_3 is defined for $a \in K^{p,q}$ such that $\delta(a') = 0$ modulo $\bar\partial K^{p+2,q-1}$, thus there is an $a'' \in K^{p+2,q-2}$ such that $\bar\partial(a'') = \delta(a')$, and $d_3\colon E_3^{p,q} \to E_3^{p+3,q-2}$ is induced by correspondence $a \to \delta(a'')$ (see, e.g., [5], A.3.13.4).

Our aim is to show that $d_r = 0$, $r \geq 2$. The differential d_2 is trivial if the representative $a \in K^{p,q}$ can be chosen in such a way that $\delta(a)$ is exactly 0 but not only 0 modulo $\bar\partial K^{p+1,q-1}$. But this is true because there are harmonic differential forms in the class $\bar{a} \in H^q_{\bar\partial}(K^{p,*})$ and we can take a to be a harmonic form. The form δa is defined by means of restrictions onto subvarieties and linear operations. All varieties we consider are Kähler, hence δa is also harmonic. But it is $0 \mod (\bar\partial K^{p+1,q-1})$ and thus is exactly 0. Further, this $\delta a = 0$ can be lifted to $a' = 0$ in $K^{p+1,q-1}$, thus $\delta(a') = 0$ and so forth. This also shows that $d_r = 0$ for all $r \geq 3$. □

Now let us come back to the proof of Lemma 2.3. We have $E_2 = E_\infty$; therefore $H^p(\Gamma(Z), \mathbb{C}) \simeq E_2^{p,0}$ is a subgroup of

$$H^p(K^*, d) \simeq \mathbf{H}^p(K^*) \simeq \mathbf{H}^p(\mathcal{O}^*) \simeq H^p(Z, \mathcal{O}_Z)\,.$$

If $H^p(\Gamma(Z))$ is not trivial, then $H^p(Z, \mathcal{O}_Z)$ is also not trivial. □

3. Hypersurface singularities

Theorem 3.1. *Let $o \in X$ be an isolated hypersurface singularity of an algebraic variety (or an analytic space) X of dimension at least 3 defined over the field $\mathbb{C}$ of complex numbers. If $f\colon Y \to X$ is a good resolution of $o \in X$ and Z its exceptional divisor, then the fundamental group of $\Gamma(Z)$ is trivial:*

$$\pi(\Gamma(Z)) = 0\,.$$

Proof. Let n be the dimension of X. We can assume that X is a hypersurface in $\mathbb{C}^{n+1}$ and the singular point o coincides with the origin. Consider the link M of singularity $o \in X$, i.e., the intersection of X with a sphere S^{2n+1} of sufficiently small radius around the origin. The link M is an $(n-2)$-connected smooth manifold ([14], Corollary 2.9, Theorem 5.2); in particular, M is simply connected.

We can also consider M as the border of a tubular neighborhood of the exceptional divisor $Z \subset Y$. It is known (see [2]) that there is a surjective map $\varphi\colon M \to Z$ whose fibers are tori. It follows that the induced map $\varphi^*\colon \pi(M) \to \pi(Z)$ is also surjective and hence $\pi(Z) = 0$.

It remains to show that $\pi(Z)=0$ implies $\pi(\Gamma(Z))=0$. It is enough to construct a surjective map $\psi\colon Z\to\Gamma(Z)$ with connected fibers. The following lemma is, essentially, a partial case of the general construction of a map from a topological space $Z=\bigcup Z_i$ to the nerve $\Gamma(Z)$ corresponding to the covering $\{Z_i\}$ (see [4], p. 355). As in section 2, we assume that the intersections $Z_{i_0\dots i_p}$ are irreducible so that $\Gamma(Z)$ is a simplicial complex.

Lemma 3.2. *Let Z be a divisor with simple normal crossings on an algebraic veriety or an analytic space X, and let $\Gamma(Z)$ be the corresponding dual complex. Then there exists a map $\psi\colon Z\to\Gamma(Z)$ which is (i) simplicial in some triangulations of Z and $\Gamma(Z)$, (ii) surjective, and (iii) has connected fibers.*

Proof. First, let us take a triangulation Σ' of Z such that all the intersections $Z_{i_0\dots i_p}$ are subcomplexes. Next we make the barycentric subdivision Σ of Σ' and the barycentric subdivision of the complex $\Gamma(Z)$. Now let v be a vertex of Σ belonging to the subcomplex $Z_{i_0\dots i_p}$ but not to any smaller subcomplex $Z_{i_0\dots i_p i_{p+1}}$:

$$v\in Z_{i_0\dots i_p}\,,\quad v\notin Z_{i_0\dots i_p i_{p+1}}\;\forall\, i_{p+1}\,.$$

Then let

$$\psi(v)=\text{ the center of the simplex }\Delta_{i_0\dots i_p}\,.$$

This determines the map ψ completely as a simplicial map (depending on the triangulation Σ'). It is clear from our construction that ψ is surjective.

We claim that the map ψ has connected fibers. Indeed, first observe that ψ can be represented as a composition of topological contractions of connected subcomplexes

$$Z_{i_0\dots i_p}\setminus(\text{the open neighborhoods of intersections of } Z_{i_0\dots i_p} \text{ with } Z_{i_{p+1}}\;\forall i_{p+1}\neq i_0,\dots,i_p)\,. \tag{2}$$

By an open neighborhood of a subcomplex K we mean the union of interior points of all simplicial stars of Σ with centers on K. All complexes in (2) are connected because the codimension of $Z_{i_0\dots i_p}\cap Z_{i_{p+1}}$ in $Z_{i_0\dots i_p}$ is 2.

Further, the contraction of a subcomplex K from (2) can be factored into one-by-one contraction of maximal simplexes of K. The preimage of every connected set under such a contraction is connected since the preimage of every simplex is a simplex. Therefore the map ψ has all the needed properties. □

□

Some important types of singularities are rational hypersurface. Combining Theorems 2.2 and 3.1, we can obtain some precise results in the 3-dimensional case.

Corollary 3.3. *Let $o\in X$ be an isolated rational hypersurface singularity of dimension 3. If $f\colon Y\to X$ is a good resolution with the exceptional divisor Z, then the dual complex $\Gamma(Z)$ associated to the resolution f has the homotopy type of a point.*

Proof. We know from Theorems 2.2 and 3.1 that $\Gamma(Z)$ is simply connected and $H^2(\Gamma(Z),\mathbb{C})=0$. Since $\dim X=3$, we have $\dim(\Gamma(Z))\le 2$. Thus $H_2(\Gamma(Z),\mathbb{Z})=0$. Now Corollary 3.3 follows from the Inverse Hurevich and Whitehead Theorems. □

For instance, 3-dimensional Gorenstein terminal singularities are exactly isolated compound Du Val points (up to an analytic equivalence; for details see [15]). Here it is sufficient to us that compound Du Val points are hypersurface singularities. On the other hand, terminal singularities (and, moreover, canonical) are rational (see [6]). Combining these results with Corollary 3.3, we get

Corollary 3.4. *Let $o \in X$ be a 3-dimensional Gorenstein terminal singularity and let $f: Y \to X$ be a good resolution with the exceptional divisor Z. Then the dual complex $\Gamma(Z)$ of f has the homotopy type of a point.*

References

[1] Abramovich, D., Karu, K., Matsuki, K., Włodarczyk, J. *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), 531–572. MR1896232 (2003c:14016)

[2] A'Campo, N. *La fonction zeta d'une monodromie*, Comment. Math. Helvetici **50** (1975), 233–248. MR0371889 (51:8106)

[3] Artin, M. *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136. MR0199191 (33:7340)

[4] Dold, A. *Lectures on Algebraic Topology*, Springer-Verlag, 1972. MR0415602 (54:3685)

[5] Eisenbud, D. *Commutative Algebra with a View toward Algebraic Geometry*, Springer-Verlag, 1995. MR1322960 (97a:13001)

[6] Elkik, R. *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), 1–6. MR621766 (83a:14003)

[7] Gordon, G. L. *On a simplicial complex associated to the monodromy*, Transactions of the AMS **261** (1980), 93–101. MR576865 (81j:32017)

[8] Gordon, G. L. *On the degeneracy of a spectral sequence associated to normal crossings*, Pacific J. Math. **90**(2) (1980), 389–396. MR600638 (83f:32011)

[9] Grauert, H., et al. *Several complex variables VII. Sheaf-theoretical methods in complex analysis*, Encyclopedia of Mathematical Sciences, 74, Springer-Verlag, Berlin, 1994. MR1326617 (96k:32001)

[10] Griffiths, P., Harris, J. *Principles of Algebraic Geometry*, v. 1, John Wiley and Sons, New York, 1978. MR507725 (80b:14001)

[11] Grothendieck, A. *Éléments de géométrie algébrique*, Ch. III, Institut des Hautes Études Scientifiques, Publications Mathématiques, No. 11 (1961). MR0217085 (36:177c)

[12] Hironaka, H. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Annals of Math. **79** (1964), 109–326. MR0199184 (33:7333)

[13] Kulikov, Vik. S., Kurchanov, P. F. *Complex algebraic varieties: periods of integrals and Hodge structures*, Algebraic Geometry, III, 1–217. Encyclopaedia Math. Sci., 36, Springer, Berlin, 1998. MR1602375

[14] Milnor, J. *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, 61, Princeton University Press, Princeton, NJ, 1968. MR0239612 (39:969)

[15] Reid, M. *Minimal Models of Canonical 3-folds*, Algebraic Varieties and Analytic Varieties, Adv. Studies in Pure Math., Kinokuniya and North-Holland (1983), V. 1, 131–180. MR715649 (86a:14010)

[16] Stepanov, D. A. *A note on the dual complex associated to a resolution of singularities* (Russian), Uspekhi Matem. Nauk **61**(1) (2006), 185–186. MR2239783 (2007c:14010)

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