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EQUIVARIANT EMBEDDING OF METRIZABLE G-SPACES IN LINEAR G-SPACES

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ABSTRACT. Given a Lie group G we study the class $\mathcal{M}_G$ of proper metrizable G-spaces with metrizable orbit spaces, and show that any G-space $X \in \mathcal{M}_G$ admits a closed G-embedding into a convex G-subset C of some locally convex linear G-space, such that X has some G-neighborhood in C which belongs to the class $\mathcal{M}_G$. As a corollary we see that any G-ANR for $\mathcal{M}_G$ is a G-ANE for $\mathcal{M}_G$.

1. INTRODUCTION

In this paper we study spaces in the class $\mathcal{M}_G$ of proper metrizable G-spaces with metrizable orbit spaces, where G is a Lie group. In the classical theory of retracts the Wojdysławski embedding theorem (see [Hu, Chapter III, Theorem 2.1]) ensures that any metrizable space can be embedded as a closed subset of a convex subspace of some Banach space.

In the equivariant case, S. Antonyan has proved that for any topological group G, any G-space X with a G-invariant metric admits a G-embedding as a closed G-subset of a convex G-subspace C of some Banach G-space B with a G-invariant norm [An1, Proposition 8 and Theorem 1]. However, the motivation for our study lies in the theory of extensors and retracts for the class $\mathcal{M}_G$, and hence we would like to know that B belongs to $\mathcal{M}_G$, or at least that a neighborhood of the image of X in C does.

It was shown by E. Elfving [E1, Theorem 3.11] that for a linear Lie group G, any Palais proper metrizable G-space X which is locally compact, separable and finite-dimensional, and which has finitely many orbit types, admits a closed G-embedding in a linear G-space such that G acts Palais properly on a G-neighborhood of the image. In [E2] the linearity assumption on the group G is dropped and the assumptions on the space X weakened; see Theorem 5.4.

Here we will prove the following theorem:

Theorem 5.1. *Let G be a Lie group and let $X \in \mathcal{M}_G$. Then there exists a G-embedding $e\colon X \to L$ where L is a locally convex linear G-space such that $e(X)$ is a closed subset of some G-invariant convex subset C of L and $e(X)$ has some*

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G-neighborhood V in C such that $V \in \mathcal{M}_G$.

Using Theorem 5.1 we show that the classes $\mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G$ and G-ANR-$\mathcal{M}_G$ are the same.

2. Preliminaries

Throughout the paper G will denote an arbitrary Lie group unless otherwise stated, where Lie groups are defined to be Hausdorff and second countable.

A G-space is a Hausdorff topological space X with a continuous action of the group G, namely a continuous map $\Phi\colon G \times X \to X$ such that $\Phi(e, x) = x$ and $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$, where $e \in G$ is the unit element. We usually denote $\Phi(g, x)$ by gx.

Let X be a G-space. Given any set $S \subset X$ and any subgroup H of G, denote $HS = \{gs \mid g \in H, s \in S\}$. A subset $S \subset X$ is said to be a G-subset if $GS = S$, and a neighborhood which is a G-subset is called a G-neighborhood. Given $x \in X$ we define the isotropy subgroup of G at x as the subgroup $G_x = \{g \in G \mid gx = x\}$.

Suppose X and Y are two G-spaces. A continuous map $f\colon X \to Y$ is a G-map if $f(gx) = gf(x)$ for every $g \in G$ and every $x \in X$. G-maps which are homeomorphisms, embeddings, retractions, etc., are called G-homeomorphisms, G-embeddings, G-retractions and so on.

A completely regular G-space X is said to be *Cartan* if every point has a neighborhood V such that the closure of the set $\{g \in G : gV \cap V \neq \emptyset\}$ is compact. The action of G on a completely regular G-space X is *proper* if for any pair of points $x, y \in X$ there exist neighborhoods V_x, V_y of x and y such that the closure of the set $\{g \in G\colon gV_x \cap V_y \neq \emptyset\}$ is compact. We say that X is a *proper G-space.* The action of G on a completely regular G-space X is *Palais proper* if for any point $x \in X$ there exists a neighborhood V_x such that any point $y \in X$ has a neighborhood V_y for which the closure of $\{g \in G : gV_x \cap V_y \neq \emptyset\}$ is compact. Then we say that X is a *Palais proper G-space.*

Clearly a Palais proper G-space is proper, and any proper G-space must be a Cartan G-space.

We will denote by $\mathcal{M}_G$ the class of proper metrizable G-spaces X which have a metrizable orbit space X/G. By [An-Ne, Theorem B], for a proper metrizable G-space X, the metrizability of X/G is equivalent to X/G being paracompact, or X admitting a G-invariant metric, where a metric d on a G-space X is said to be G-invariant if $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and all $g \in G$.

Let H be a closed subgroup of G. A subset S of a G-space X is an *H-slice* if GS is open in X and there exists a G-map $f\colon GS \to G/H$ such that $S = f^{-1}(eH)$. The set S is a *slice at the point* $x \in X$ if $x \in S$ and S is a G_x-slice.

By [Pa2, Theorem 2.3.3] we know that in a Cartan G-space, there exists a slice at every point and the isotropy subgroup of G at any point is compact.

The following lemma describes how a slice at $x \in X$ induces an H-slice for any subgroup H of G which is conjugate to G_x.

Lemma 2.1. *Suppose that X is a Cartan G-space and let S be a slice at $x \in X$. Let H be a subgroup of G which is conjugate to G_x by an element $\bar{g} \in G$; that is, $H = \bar{g} G_x \bar{g}^{-1} = G_{\bar{g}x}$. Then $\bar{g}S$ is a slice at $\bar{g}x$, and in particular $\bar{g}S$ is an H-slice.*

Proof. By [Pa2, Proposition 1.1.5] the maps $G/G_x \to Gx$ and $G/G_{\bar{g}x} \to Gx$ given by $gG_x \mapsto gx$ and $gG_{\bar{g}x} \mapsto g\bar{g}x$, respectively, are G-homeomorphisms. Thus there is a G-homeomorphism $h\colon G/G_x \to G/G_{\bar{g}x}$ given by $h\colon G/G_x \approx_G Gx = G(\bar{g}x) \approx_G G/G_{\bar{g}x} = G/H$, where $\bar{g}G_x \mapsto \bar{g}x \mapsto eG_{\bar{g}x} = eH$ and so $h \circ f\colon G(\bar{g}S) = GS \to G/H$ is the mapping associated with $\bar{g}S$ as a $G_{\bar{g}x}$-slice since $(h \circ f)^{-1}(eG_{\bar{g}x}) = f^{-1}(\bar{g}G_x) = \bar{g}S$. □

The following characterization of slices is extremely useful.

Theorem 2.2 ([Pa2, Theorem 2.1.4])**.** *Suppose X is a Cartan G-space and let H be a compact subgroup of G. A subset $S \subset X$ is an H-slice if and only if the following conditions hold:*

i) *S is closed in GS,*
ii) *$S = HS$,*
iii) *$gS \cap S \neq \emptyset$ implies $g \in H$,*
iv) *GS is open in X,*
v) *S has a neighbohood V in GS such that the closure of $\{g \in G : gV \cap V \neq \emptyset\}$ is compact.*

We say that the open set GS is a *tubular neighborhood* (of x) if S is an H-slice (a slice at x).

Definition 2.3 (Tubular covering)**.** A *tubular covering* of a G-space X is a covering of X by tubular neighborhoods.

Lemma 2.4. *An open G-subset of a tubular neighborhood is a tubular neighborhood. Hence an open refinement by G-sets of a tubular covering is a tubular covering.*

Proof. Suppose H is a closed subgroup of G, let S be an H-slice and let $f\colon GS \to G/H$ be a corresponding G-map. Let U be an open G-invariant subset of GS. Now $S' = S \cap U$ is an H-slice since $GS' = U$ is open and $f' = f|U\colon U \to G/H$ is a G-map where $(f')^{-1}(eH) = f^{-1}(eH) \cap U = S \cap U = S'$. □

Lemma 2.5. *If $\{GS_k\}_{k\in K}$ is a family of pairwise disjoint tubular neighborhoods in X, where S_k is an H-slice for all $k \in K$, then $S = \bigcup_{k\in K} S_k$ is an H-slice.*

Proof. Here a corresponding map $f\colon GS = \bigcup_{k\in K} GS_k \to G/H$ is defined by $f|GS_k = f_k$ where $f_k\colon GS_k \to G/H$ is a map corresponding to the slice S_k. □

Given a closed subgroup H of G and an H-space S, there is an action of H on the product $G \times S$ given by $h(g, s) = (gh^{-1}, hs)$. We denote by $G \times_H S$ the quotient space $(G \times S)/H$, which is called the *twisted product* of G and S with respect to H. There is an action of G on $G \times_H S$ defined by the formula $\bar{g}[g, s] = [\bar{g}g, s]$.

Proposition 2.6 ([E1, Proposition 1.18])**.** *Let H be a closed subgroup of G and let S be an H-slice in a G-space X. Then*

$$G \times_H S \approx_G GS.$$

3. Countability of tubular coverings

The following lemma makes use of a technique originating with J. Milnor; see [Pa1, Theorem 1.8.2].

Lemma 3.1. *Let X be a Cartan G-space and suppose that X/G is paracompact.*[1] *Then X admits a countable locally finite tubular covering.*

[1] We assume that paracompact spaces are Hausdorff.

Proof. By [An4, Proposition 3.6] we know that G has at most countably many compact conjugacy types, represented by compact subgroups H_n of G where $n \in \mathbb{N}$. Suppose that $\{GS_i\}_{i\in I}$ is a tubular covering of X such that for each $i \in I$, S_i is a slice at some point $x_i \in X$, where G_{x_i} is a compact subgroup of G. Then $gG_{x_i}g^{-1} = H_{n_i}$ for some $g \in G$ and some $n_i \in \mathbb{N}$, and by Lemma 2.1, gS_i is a slice at gx_i and $G_{gx_i} = gG_{x_i}g^{-1} = H_{n_i}$. Thus we may assume from the beginning that each S_i is an H_{n_i}-slice for some $n_i \in \mathbb{N}$. Let $\{\varphi_i\colon X \to [0,1]\}_{i\in I}$ be a G-invariant partition of unity subordinate to $\{GS_i\}_{i\in I}$. Such a partition of unity exists because X/G is paracompact.

For each finite $T \subset I$, denote

$$W(T) = \{x \in X : \varphi_i(x) > \varphi_j(x) \text{ for all } i \in T \text{ and for all } j \in I \setminus T\}.$$

Denote by $u_T\colon X \to [0,1]$ the continuous G-invariant map

$$u_T(x) = \max\{0, \min_{i\in T, j\in I\setminus T}(\varphi_i(x) - \varphi_j(x))\}.$$

Then $W(T) = u_T^{-1}(0,1]$ is open and G-invariant in X.

If $x \in W(T)$, then $\varphi_i(x) > \varphi_j(x) \geq 0$ for all $i \in T, j \in I \setminus T$, so in particular $x \in \varphi_i^{-1}(0,1]$ for all $i \in T$. Thus $W(T)$ is an open G-invariant subset of $\varphi_i^{-1}(0,1] \subset GS_i$ for each $i \in T$; hence $W(T)$ is a tubular neighborhood by Lemma 2.4.

If Card $T =$ Card T' and $T \neq T'$, then $W(T) \cap W(T') = \emptyset$, because if $i \in T \setminus T'$, $j \in T' \setminus T$ and $x \in W(T) \cap W(T')$, then simultaneously $\varphi_i(x) > \varphi_j(x)$ and $\varphi_j(x) < \varphi_i(x)$, which is impossible.

Define

$$W_n^m = \bigcup\{W(T) : \text{Card } T = n \text{ and } W(T) \text{ is an } H_m\text{-tubular neighborhood}\}$$

for all $m, n \in \mathbb{N}$. Now W_n^m is an H_m-tubular neighborhood by Lemma 2.5, because it is a disjoint union of H_m-tubular neighborhoods. It follows that $\{W_n^m\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ is a countable tubular covering of X.

Denote by $\pi_X\colon X \to X/G$ the canonical projection. Since X/G is paracompact, the open covering $\{\pi_X(W_n^m)\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of X/G admits a precise locally finite refinement by [Du, Chapter VIII, Theorem 1.4], and in particular this refinement is countable. Denote it by $\{V_n\}_{n\in\mathbb{N}}$. Now $\{\pi_X^{-1}(V_n)\}$ is a countable locally finite refinement of $\{W_n^m\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ by G-neighborhoods; hence it is a countable locally finite tubular covering of X by Lemma 2.4, and we are done. □

4. Homeomorphism properties of isovariant G-maps

The main result in this section is an important homeomorphism property of isovariant G-maps between Cartan G-spaces, which we will use in proving our main theorem. Recall that a map $f\colon X \to Y$ between G-spaces is isovariant if $G_x = G_{f(x)}$ for all $x \in X$. Our lemma builds on the following result for compact transformation groups:

Lemma 4.1 ([Br, Exercise 10 of Chapter I])**.** *Let H be a compact Hausdorff topological group, and let $f\colon X \to Y$ be an isovariant H-map between H-spaces. Then f is an open map if and only if the induced map $\bar{f}\colon X/H \to Y/H$ is an open map.*

Lemma 4.2. *Suppose that X and Y are Cartan G-spaces and that $f\colon X \to Y$ is an isovariant G-map. Then f is a G-homeomorphism if and only if the induced map $\bar{f}\colon X/G \to Y/G$ is a homeomorphism.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow[\bar{f}]{} & Y/G \end{array}$$

where π_X and π_Y denote the canonical projections. Since π_X is continuous and π_Y is open, it is clear that if f is open, then so is $\bar{f}$. Hence if f is a homeomorphism, so is $\bar{f}$.

Now suppose $\bar{f}$ is a homeomorphism; it then easily follows from the bijectivity of $\bar{f}$ and the fact that any isovariant G-map restricts to a bijection on the orbits that f is a bijection. It remains to show that f is open. Being Cartan, Y has a covering $\{GS_i^Y\}_{i\in I}$ where each S_i^Y is a slice at a point $y_i \in Y$ such that the isotropy subgroups G_{y_i} are compact for all $i \in I$. Let $i \in I$. Then $S_i^X = f^{-1}(S_i^Y)$ is a slice at $x_i = f^{-1}(y_i)$. Note that by isovariance $G_{y_i} = G_{x_i}$. We consider now only the restrictions

$$\begin{array}{ccc} GS_i^X & \xrightarrow{\ f\ } & Gf(S_i^X) = GS_i^Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ GS_i^X/G & \xrightarrow[\bar{f}]{} & Gf(S_i^X)/G = GS_i^Y/G. \end{array}$$

Since $GS_i^X/G \approx S_i^X/G_{x_i}$ and $GS_i^Y/G \approx S_i^Y/G_{x_i}$ by [E1, Lemma 1.23], we see that the restriction $f|S_i^X\colon S_i^X \to S_i^Y$ induces a map $\tilde{f}\colon S_i^X/G_{x_i} \to S_i^Y/G_{x_i}$ given by $\tilde{f}(G_{x_i}s) = G_{x_i}(f(s))$, which makes the following diagram commutative:

$$\begin{array}{ccc} S_i^X & \xrightarrow{\ f\ } & S_i^Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ S_i^X/G_{x_i} & \xrightarrow[\tilde{f}]{} & S_i^Y/G_{x_i} \\ {\scriptstyle \approx}\downarrow & & \downarrow{\scriptstyle \approx} \\ GS_i^X/G & \xrightarrow[\bar{f}]{} & GS_i^Y/G. \end{array}$$

Hence since $\bar{f}$ is open, so is $\tilde{f}$. The isotropy subgroup G_{x_i} is compact by [Pa2, Theorem 2.3.3] since the G-action on X is Cartan and thus by Lemma 4.1 the restriction $f|S_i^X$ is open.

The map $\mathrm{id}_G \times f|S_i^X$ induces a map $G \times_{G_{x_i}} f|S_i^X : G \times_{G_{x_i}} S_i^X \to G \times_{G_{x_i}} S_i^Y$ as in [Br, Chapter II, Proposition 2.1] which makes the following diagram commutative:

$$\begin{array}{ccc}
G \times S_i^X & \xrightarrow{\mathrm{id}_G \times f|S_i^X} & G \times S_i^Y \\
\downarrow & & \downarrow \\
G \times_{G_{x_i}} S_i^X & \xrightarrow{G \times_{G_{x_i}} f|S_i^X} & G \times_{G_{x_i}} S_i^Y \\
{\scriptstyle [g,s]\mapsto gs,\ \approx_G}\downarrow & & \downarrow{\scriptstyle [g,y]\mapsto g_y,\ \approx_G} \\
GS_i^X & \xrightarrow[f]{} & GS_i^Y
\end{array}$$

and since $f|S_i^X$ is open, $G \times_{G_{x_i}} f|S_i^X$ is open by [Br, Proposition II 2.1]. Hence $f|GS_i^X$ is open onto its image; i.e., it is a homeomorphism onto its image.

Since the S_i^Y are slices, the set GS_i^Y is open in Y for each $i \in I$. Hence f is open; i.e., it is a homeomorphism. □

5. Embedding theorem

Now we are able to prove the following theorem, which is the main result of this note:

Theorem 5.1. *Let G be a Lie group and let $X \in \mathcal{M}_G$. Then there exists a G-embedding $e \colon X \to L$ where L is a locally convex linear G-space such that $e(X)$ is a closed subset of some G-invariant convex subset C of L and $e(X)$ has some G-neighborhood V in C such that $V \in \mathcal{M}_G$.*

We will prove Theorem 5.1 by finding an isovariant G-map $e \colon X \to L$ which induces an embedding $X/G \to L/G$ under suitable conditions, and by applying Lemma 4.2. The same method has been used by J. Jaworowski [Ja] and by G. Bredon [Br, Chapter II.10] for compact Lie group actions.

Definition 5.2 (LCL G-space)**.** A locally convex linear G-space (for short, an LCL G-space) is a G-space L which is a locally convex topological vector space where each element $g \in G$ represents a linear map $L \to L$.

Lemma 5.3. *A product of LCL G-spaces with diagonal action is an LCL G-space.*

Proof. Let X_i be a family of LCL G-spaces where the indices i run through some set I. Set $X = \prod_{i \in I} X_i$; now it is clear that X is a topological vector space and a G-space where each $g \in G$ represents a linear map $X \to X$. Furthermore X is locally convex.

Indeed, suppose U is an open neighborhood of $x \in X$. Then there is some basic open set $V = \prod_{i \in I} V_i$ such that $x \in V \subset U$. For V to be an element in the basis, we know that each V_i is open in X_i and $V_i = X_i$ for all but finitely many $i \in I$. Now for each $V_i \neq X_i$ there exists a convex neighborhood W_i of $\mathrm{pr}_i(x)$ such that $W_i \subset V_i$. Whenever $V_i = X_i$, set $W_i = X_i$. Define $W = \prod_{i \in I} W_i$. Now W is a convex neighborhood of x and $W \subset V \subset U$. □

We are going to use the following result, which is obtained in [E2, Section 3], although there it is not explicitly stated as a theorem:

Theorem 5.4. *Let G be a Lie group and assume that a G-space $X \in \mathcal{M}_G$ admits a metric d which satisfies*

$$(*) \quad \forall\, r > 0\ \forall\, x \in X\colon \bar{B}_r^d(x) \text{ is compact,}$$

where $\bar{B}_r^d(x)$ denotes the closed ball of radius r about x with respect to the metric d.

Then there exists a Banach G-space B and a closed G-embedding $i\colon X \to B$ such that $i(X) \subset B \setminus \{\bar{0}\}$ and G acts properly on $B \setminus \{\bar{0}\}$.

Remark 5.5.

- i) Note that the metric d satisfying $(*)$ can be any metric on X, independent of the G-action.
- ii) A normed vector space is locally convex.
- iii) From the construction of the space B one easily sees that the norm in B is G-invariant, inducing a G-invariant metric on B and, by restriction, on $B \setminus \{\bar{0}\}$. Hence, $B \setminus \{\bar{0}\} \in \mathcal{M}_G$.

Suppose that the G-space X has a global H-slice S with H a compact subgroup of G, i.e. $X = GS$. Then there exists an isovariant H-map

$$\varphi\colon S \to \prod_{n\in\mathbb{N}} E_n$$

where each E_n is a Euclidean representation space for H (see [An2, Lemma 5]). The twisted product $G \times_H E_n$ is a G-space, and there is an H-embedding $i_n\colon E_n \to G \times_H E_n$ defined by $i_n(x) = [e, x]$. This defines an isovariant H-map

$$\tilde{\varphi}\colon S \xrightarrow{\varphi} \prod_{n\in\mathbb{N}} E_n \xrightarrow{\prod i_n} \prod_{n\in\mathbb{N}} G \times_H E_n.$$

Using this we obtain a G-map $\psi\colon X = GS \to \prod_{n\in\mathbb{N}} G \times_H E_n$ by setting $\psi(gx) = g\tilde{\varphi}(x)$.

Lemma 5.6. *The map ψ is isovariant.*

Proof. By equivariance $G_{gs} \subset G_{\psi(gs)}$ for all $gs \in GS$. Thus we only need to show that $G_{gs} \supset G_{\psi(gs)}$.

Assume that $g\tilde{\varphi}(S) \cap \tilde{\varphi}(S) \neq \emptyset$ for some $g \in G$. Then $[g, \varphi_n(s_1)] = [e, \varphi_n(s_2)]$ for all $n \in \mathbb{N}$ and for some $s_1, s_2 \in S$, where $\varphi_n = \mathrm{pr}_n \circ \varphi$. But then $g \in H$.

If we now assume that $\bar{g} \in G_{\psi(gs)}$ for some $gs \in GS$, then $\bar{g}\psi(gs) = \psi(gs)$, giving $\bar{g}g\tilde{\varphi}(s) = g\tilde{\varphi}(s)$; hence $g^{-1}\bar{g}g\tilde{\varphi}(s) = \tilde{\varphi}(s)$ and thus by the previous argument $g^{-1}\bar{g}g \in H_{\tilde{\varphi}(s)} = H_s$.

But then $g^{-1}\bar{g}gs = s$; hence $\bar{g}gs = gs$ and thus $\bar{g} \in G_{gs}$. It follows that $G_{\psi(gs)} = G_{gs}$ for all $gs \in GS$ and thus ψ is isovariant. □

Now $G \times_H E_n$ is a proper G-space by [E1, Proposition 1.3], it is second countable and it is a manifold because $G \times_H E_n \to G/H$ is a vector bundle by [Kaw, Theorem 2.26]. It has a metric with the property $(*)$ because any second countable manifold can be embedded as a closed subset of some Euclidean space (see [H-W, Theorem V.3] for the embedding theorem, and see [Br, Chapter III, Corollary 10.2] for the closedness). Note that this embedding does not need to be a G-embedding, as the condition $(*)$ on the metric is independent of the action of G.

Furthermore, $(G \times_H E_n)/G$ is metrizable by [Pa2, Theorem 4.3.4], giving $G \times_H E_n \in \mathcal{M}_G$. Hence, by Theorem 5.4 we obtain a G-embedding $G \times_H E_n \to B_n$, where B_n is a Banach G-space and G acts properly on $B_n \setminus \{\bar{0}\}$. This gives an isovariant G-map

$$\tilde{\psi}\colon X = GS \xrightarrow{\psi} \prod_{n\in\mathbb{N}} G \times_H E_n \to \prod_{n\in\mathbb{N}} B_n =: \tilde{Z},$$

where $\tilde{\psi}(GS) \subset \tilde{Z} \setminus \{\bar{0}\}$.

Lemma 5.7. *There is a G-invariant metric d on $\tilde{Z}$, which induces a pseudometric $\bar{d}$ on $\tilde{Z}/G$.*

Proof. We have $\tilde{Z} = \Pi_{n\in\mathbb{N}} B_n$ where each B_n is a Banach G-space with a G-invariant metric d_n induced by the norm as noted in Remark 5.5. In each B_n we define a new metric e_n by setting $e_n(x,y) = \min\{d_n(x,y), \frac{1}{n}\}$. The metric e_n is equivalent to d_n by [Du, Chapter IX, Theorem 3.3] and e_n is G-invariant because d_n is so.

Denote by $\pi_m\colon \tilde{Z} = \Pi_{n\in\mathbb{N}} B_n \to B_m$ the m^{th} projection; we define a metric $d\colon \tilde{Z} \times \tilde{Z} \to \mathbb{R}$ by setting

$$d(z,z') = \sup_{m\in\mathbb{N}} e_m(\pi_m(z), \pi_m(z')).$$

The map d metrizes the product topology on $\tilde{Z}$ by [Du, Chapter IX, Theorem 7.2] since $e_n(B_n) \to 0$ as $n \to \infty$, and d is G-invariant because each e_n is so.

The metric d induces a pseudometric $\bar{d}$ on $\tilde{Z}/G$ by $\bar{d}(\tilde{x},\tilde{y}) = d(Gx, Gy)$, where $\tilde{x} = \pi(x)$, $\tilde{y} = \pi(y)$, and $\pi\colon \tilde{Z} \to \tilde{Z}/G$ is the canonical projection. The pseudometric $\bar{d}$ induces the quotient topology on $\tilde{Z}/G$ since the arguments

$$y \in B_d(x,r) \Rightarrow \bar{d}(\tilde{x},\tilde{y}) \le d(x,y) < r \Rightarrow \pi(y) = \tilde{y} \in B_{\bar{d}}(\tilde{x},r)$$

and

$$\begin{aligned} \tilde{y} \in B_{\bar{d}}(\tilde{x},r) \quad &\Rightarrow \inf\nolimits_{g\in G} d(x,gy) = \bar{d}(\tilde{x},\tilde{y}) < r \Rightarrow gy \in B_d(x,r) \text{ for some } g \in G \\ &\Rightarrow \tilde{y} = \pi(gy) \in \pi B_d(x,r) \end{aligned}$$

imply that $\pi B_d(x,r) = B_{\bar{d}}(\tilde{x},r)$. □

The pseudometric $\bar{d}$ restricted to the set $(\tilde{Z} \setminus \{\bar{0}\})/G$ is the same as the pseudometric induced by the metric d restricted to $\tilde{Z} \setminus \{\bar{0}\}$; hence the restriction of $\bar{d}$ is a pseudometric inducing the quotient topology on $(\tilde{Z} \setminus \{\bar{0}\})/G$, denoted by $\bar{d}|$.

Lemma 5.8. *G acts properly on $\tilde{Z} \setminus \{\bar{0}\}$.*

Proof. G acts properly on $B_n \setminus \{\bar{0}\}$ for each $n \in \mathbb{N}$ and hence the space $\tilde{Z}_n = \left(\prod_{i=1}^{n-1} B_i\right) \times (B_n \setminus \{\bar{0}\}) \times \left(\prod_{i=n+1}^{\infty} B_i\right)$ is a proper G-space for all $n \in \mathbb{N}$. Furthermore $\tilde{Z} \setminus \{\bar{0}\} = \bigcup_{n\in\mathbb{N}} \tilde{Z}_n$. Now $\tilde{Z} \setminus \{\bar{0}\}$ is a Cartan G-space since any completely regular G-space which is the union of open Cartan G-subspaces is a Cartan G-space. Since $(\tilde{Z} \setminus \{\bar{0}\})/G$ admits a pseudometric $\bar{d}|$, it is regular, and it follows that the action of G on $\tilde{Z} \setminus \{\bar{0}\}$ is Palais proper by [Pa2, Proposition 1.2.5], so it is certainly proper. □

Lemma 5.9. *The pseudometric* $\bar{d}|$ *on* $(\tilde{Z} \setminus \{\bar{0}\})/G$ *is a metric.*

Proof. Since G acts properly on $\tilde{Z}\setminus\{\bar{0}\}$ by Lemma 5.8, the quotient space $(\tilde{Z}\setminus\{\bar{0}\})/G$ is Hausdorff by [Pa2, Proposition 1.1.4], and we have seen that $\bar{d}|$ induces this topology. But then $\bar{d}|$ must be a metric. □

Next we pass from the global slice situation above to the more complicated situation with arbitrary spaces from $\mathcal{M}_G$.

Lemma 5.10. *Let* X *be as in Theorem* 5.1. *There exists an LCL* G*-space* Z *and an isovariant* G*-map* $\tilde{f}\colon X \to Z$ *such that* $\tilde{f}(X) \subset Z \setminus \{\bar{0}\}$ *and* $Z \setminus \{\bar{0}\} \in \mathcal{M}_G$.

Proof. By Lemma 3.1 we may assume that $\{GS_n\}_{n\in\mathbb{N}}$ is a locally finite tubular covering of X. By Lemma 2.4 and by the normality of X/G we may let $\{GR_n\}_{n\in\mathbb{N}}$ and $\{GT_n\}_{n\in\mathbb{N}}$ be similar coverings such that $\overline{GT}_n \subset GR_n \subset \overline{GR}_n \subset GS_n$ for each $n \in \mathbb{N}$. According to the discussion above each GS_n admits an isovariant G-map $\tilde{f}_n$ into an LCL G-space Z_n, where $\tilde{f}_n(GS_n) \subset Z_n \setminus \{\bar{0}\}$, and G acts properly on $Z_n \setminus \{\bar{0}\}$. For each $n \in \mathbb{N}$ we may construct a G-invariant map $\lambda_n\colon X \to I$ such that $\lambda_n(\overline{GT}_n) = \{1\}$ and $\lambda_n(X \setminus GR_n) = \{0\}$, since X/G is normal.

We obtain a continuous and isovariant G-map $\tilde{f}\colon X \to \prod_{n\in\mathbb{N}} Z_n =: Z$ by setting $\tilde{f}(x) = (\lambda_1(x)\tilde{f}_1(x), \ldots, \lambda_i(x)\tilde{f}_i(x), \ldots)$.

Clearly $\tilde{f}(X) \subset Z \setminus \{\bar{0}\}$, and Z, being a product of LCL G-spaces, is an LCL G-space by Lemma 5.3.

Furthermore, Z admits a G-invariant metric which induces a pseudometric on Z/G just as in Lemma 5.7, G acts properly on $Z \setminus \{\bar{0}\}$ as in Lemma 5.8 and the pseudometric on Z/G restricts to a metric on $(Z \setminus \{\bar{0}\})/G$ as in Lemma 5.9. In other words, $Z \setminus \{\bar{0}\} \in \mathcal{M}_G$. □

Lemma 5.11. *With* X *as in Theorem* 5.1, *there exists a* G*-embedding* $e\colon X \to L$ *where* L *is an LCL* G*-space,* $e(X) \subset C$ *for some convex* G*-subset* C *of* L *and* $e(X)$ *has a* G*-neighborhood* V *in* C *such that* $V \in \mathcal{M}_G$.

Proof. Let $h\colon X/G \to B$ be an embedding of the metrizable space X/G into a Banach space B such that $h(X/G)$ is a closed subset of C' where C' is a convex subset of B. Such a map exists by the Wojdysławski embedding theorem. With Z as in Lemma 5.10, define a map $e\colon X \to Z \times B =: L$ by setting $e = (\tilde{f}, h \circ \pi_X)$ where $\tilde{f}$ is the isovariant G-map obtained in Lemma 5.10 and $\pi_X\colon X \to X/G$ is the natural projection. Let G act trivially on B. Now we have

i) e is an isovariant G-map because $\tilde{f}$ is isovariant.
ii) e is injective since the induced map $\bar{e}\colon X/G \to Z/G \times B$ is injective and e is isovariant. The map $\bar{e}$ is injective because h is injective.
iii) The map $\bar{e}$ is a homeomorphism onto its image, whose inverse $\bar{e}^{-1}\colon \bar{e}(X/G) \to X/G$ is given by $h^{-1} \circ \mathrm{pr}_2$, where $\pi_Z\colon Z \to Z/G$ is the natural projection and $\mathrm{pr}_2\colon Z/G \times B \to B$ is the projection to the second coordinate.
iv) $e(X) \subset Z \times C' = C$, which is convex, and G acts properly on $Z\setminus\{\bar{0}\} \times C' = V$, which is a neighborhood of $e(X)$ in C.

Now $e(X) \subset V$, which is proper, and in particular $e(X)$ is Cartan. Thus by Lemma 4.2 the map e is a homeomorphism onto its image. The space $V/G = (Z \setminus \{\bar{0}\})/G \times C'$ is metrizable; thus $V \in \mathcal{M}_G$. □

Lemma 5.12. *The image $e(X)$ is a closed subset of $Z \times C'$.*

Proof. Assume that $y = (u, v) \in (Z \times C') \setminus e(X)$. In case $v \notin h(X/G)$, since h is a closed embedding, there exists a neighborhood U of v in C' such that $U \cap h(X/G) = \emptyset$. Thus $Z \times U$ is a neighborhood of (u, v) which does not intersect $e(X)$.

In case $v \in h(X/G)$ we have $v = h(\pi_X(x))$ for some $x \in X$ and then we must have $Gu \cap G\tilde{f}(x) = \emptyset$ (if not, then $u = g\tilde{f}(x) = \tilde{f}(gx)$ for some $g \in G$, giving $v = h(\pi_X(x)) = h(\pi_X(gx))$, which implies $y = (u, v) = e(gx) \in e(X)$).

If $u \neq \bar{0}$, then u and $\tilde{f}(x)$ are points in the open subset $Z \setminus \{\bar{0}\}$ of Z, where $(Z \setminus \{\bar{0}\})/G$ is metrizable. Thus there exist disjoint G-neighborhoods U and W of Gu and $G\tilde{f}(x)$ in $Z \setminus \{\bar{0}\}$ and hence in Z. If $u = \bar{0}$, then $\pi_Z(u)$ is closed in Z/G, which is pseudometrizable and hence regular, so there exist again disjoint G-neighborhoods U and W of Gu and $G\tilde{f}(x)$ in Z. Thus we see that in any case there exist disjoint G-neighborhoods U and W of Gu and $G\tilde{f}(x)$ in Z.

Since $\tilde{f}$ is continuous and equivariant there exists a G-neighborhood $\tilde{W}$ of Gx in X such that $\tilde{f}(\tilde{W}) \subset W$. Since π_X is an open map and h is an embedding there exists an open neighborhood M of $h(\pi_X(x))$ in C' such that $M \cap h(\pi_X(X)) = h(\pi_X(\tilde{W}))$.

Clearly $U \times M$ is an open neighborhood of y in $Z \times C'$; we show that it is disjoint from $e(X)$. If $(x, w) \in e(X) \cap (U \times M)$, then $x = \tilde{f}(z) \in U$ and $w = h(\pi_X(z)) \in M$ for some $z \in X$. But then $\tilde{f}(z) \notin W$ giving $z \notin \tilde{W}$ and since $\tilde{W}$ is G-invariant we have $\pi_X(z) \notin \pi_X(\tilde{W})$, which gives $h(\pi_X(z)) \notin h(\pi_X(\tilde{W}))$ since h is injective.

However, $z \in X$ and $h(\pi_X(z)) \in M$ implies that $h(\pi_X(z)) \in M \cap h(\pi_X(X)) = h(\pi_X(\tilde{W}))$, which gives a contradiction. Hence we must have $e(X) \cap (U \times M) = \emptyset$.

It follows that $e(X)$ is closed in $Z \times C'$. □

The rest of the proof of Theorem 5.1. If we now set $L = Z \times B$, then L is an LCL G-space. Set $C = Z \times C'$ and $V = (Z \setminus \{\bar{0}\}) \times C'$ as before. Then V is a G-neighborhood of $e(X)$ in C, C is a G-invariant convex subset of L, $e(X)$ is closed in V, and $V \in \mathcal{M}_G$. Hence the theorem is true. □

6. Application

We say that a G-space Y is a G-equivariant absolute neighborhood extensor (G-ANE) for $\mathcal{M}_G$, written $Y \in G$-ANE-$\mathcal{M}_G$, if for any G-space $X \in \mathcal{M}_G$ and any closed G-invariant subset A in X with a G-map $f: A \to Y$ there exists a G-extension $F: U \to Y$ of f over some G-neighborhood U of A in X.

We say that a G-space $X \in \mathcal{M}_G$ is a G-equivariant absolute neighborhood retract (G-ANR) for $\mathcal{M}_G$, written $X \in G$-ANR-$\mathcal{M}_G$, if, whenever there exists a closed G-embedding $i: X \to Y$ of X into some G-space $Y \in \mathcal{M}_G$, then there exists a G-neighborhood retraction $r: U \to i(X)$ where U is a G-neighborhood of $i(X)$ in Y.

It is easy to show that then $\mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G \subset G$-ANR-$\mathcal{M}_G$. Here we show that the two classes are the same, following the classical proof from the non-equivariant case (see, for instance, [Hu, Chapter III, Theorem 3.2]).

Corollary 6.1. *G-ANR-$\mathcal{M}_G$=$\mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G$.*

Proof. We should show that G-ANR-$\mathcal{M}_G \subset \mathcal{M}_G \cap G$-ANE-$\mathcal{M}_G$. Suppose that $Y \in G$-ANR-$\mathcal{M}_G$. Then $Y \in \mathcal{M}_G$ by definition. By Theorem 5.1 we may assume that Y is a closed G-subset of a convex G-subset C of some LCL G-space, where Y

has a G-neighborhood U in C such that $U \in \mathcal{M}_G$. Since $Y \in G$-ANR-$\mathcal{M}_G$, there exists a G-neighborhood V of Y in U (and hence in C) and a G-retraction $r\colon V \to Y$. Let $X \in \mathcal{M}_G$ and let A be a closed G-invariant subset of X with a G-map $f\colon A \to Y$. By the equivariant Dugundji extension theorem [An3, Corollary 1] we know that $C \in G$-ANE-$\mathcal{M}_G$; hence the map $i \circ f\colon A \to Y \hookrightarrow C$ admits a G-extension $h\colon W \to C$, where W is a G-neighborhood of A in X. Set $W' = W \cap h^{-1}(V)$. Now W' is a G-neighborhood of A in X and the map $r \circ h|W'\colon W' \to V \to Y$ is a neighborhood G-extension of f. It follows that $Y \in G$-ANE-$\mathcal{M}_G$. □

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