THE PROJECTIVE $\pi$-CHARACTER BOUNDS THE ORDER OF A $\pi$-BASE

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Abstract. All spaces below are Tychonov. We define the projective $\pi$-character $p\pi\chi(X)$ of a space $X$ as the supremum of the values $\pi\chi(Y)$ where $Y$ ranges over all (Tychonov) continuous images of $X$. Our main result says that every space $X$ has a $\pi$-base whose order is $\leq p\pi\chi(X)$; that is, every point in $X$ is contained in at most $p\pi\chi(X)$-many members of the $\pi$-base. Since $p\pi\chi(X) \leq t(X)$ for compact $X$, this is a significant generalization of a celebrated result of Shapirovskii.

Let us start by recalling a few definitions and basic facts. A $\pi$-base $B$ of a space $X$ (resp. a local $\pi$-base at a point $x \in X$) is a family of non-empty open sets such that every non-empty open set (resp. every neighbourhood of $x$) includes some member of $B$. The $\pi$-weight $\pi(X)$ of $X$ is the smallest infinite cardinal such that $X$ has a $\pi$-base of at most that cardinality. The $\pi$-character $\pi\chi(x, X)$ of $x$ in $X$ is the smallest cardinality of a local $\pi$-base at $x \in X$, and

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$$

is the $\pi$-character of the space $X$. Finally, the local tightness at $x \in X$ is the smallest cardinal $\kappa$ such that if $x$ belongs to the closure $\overline{A}$ of a set $A$, then there is $B \subset A$ with $|B| \leq \kappa$ and $x \in \overline{B}$; moreover

$$t(X) = \sup\{t(x, X) : x \in X\}$$

is the tightness of the space $X$.

Shapirovskii proved the following two important results concerning these cardinal functions for compacta: If $X$ is compact, then $\pi\chi(X) \leq t(X)$. Moreover $X$ has a $\pi$-base $B$ of order $\leq t(X)$; i.e. every point of $X$ is contained in at most $t(X)$-many members of $B$. (A trivial consequence is that if $t(X)^+ \leq t(X)$—i.e. among $t(X)^+$-many open sets there are always $t(X)^+$-many with non-empty intersection—then $X$ has a $\pi$-base of cardinality at most $t(X)$.) The first result was proved in [5], and alternative proofs were given in [1] and [3]. The second result first appeared in [5] and then in [8]. A very short and elegant new proof (using a variant of Shapirovskii’s “algebraic” approach to free sequences) was presented in [9].

Arhangel’skii has recently introduced in [2] the concept of a space of countable projective $\pi$-character and noticed that any compact space of countable tightness...
has countable projective $\pi$-character. Then he showed that a compact space of countable projective $\pi$-character that has $\omega_1$ as a caliber is separable (or equivalently, has a countable $\pi$-base), thereby strengthening the above consequence of Shapirovskii’s result for countably tight compacta.

In this paper we introduce the general concept of projective $\pi$-character and give the following significant generalization of Shapirovskii’s full result: Any Tychonov space has a $\pi$-base of order at most the projective $\pi$-character of the space. Not only is this result stronger for compacta, because it replaces tightness with projective $\pi$-character that is smaller, but somewhat surprisingly it extends to all Tychonov spaces.

Let $\varphi$ be any cardinal function defined on a class $\mathcal{C}$ of topological spaces. We define the projective version $p\varphi$ of $\varphi$ on $\mathcal{C}$ as follows. For any $X \in \mathcal{C}$ we let $p\varphi(X)$ be the supremum of the values $\varphi(Y)$ where $Y$ ranges over all continuous images of $X$ belonging to $\mathcal{C}$. In particular, we shall consider the case in which $\varphi = \pi X$, the $\pi$-character defined on the class of Tychonov spaces. It is easy to show then that a Tychonov space $X$ has countable projective $\pi$-character in the sense of [2] if $p\pi(X) \leq \omega$.

Also, as was already mentioned, if $X$ is compact Hausdorff, then we have $p\pi(X) \leq t(X)$. In fact, this follows because $t(Y) \leq t(X)$ for any continuous image of $X$ and, by Shapirovskii’s first result above, $\pi(X) \leq t(Y)$ for every compact $Y$. But are $p\pi(X)$ and $t(X)$ really different? Arhangel’skii asked more specifically if there is a compactum of countable projective $\pi$-character that is not countably tight; see [2], problem 7. Our next example yields such a compactum.

**Example 1.** Let $X$ be a compactification of $\omega$ whose remainder is (homeomorphic to) the ordinal $\omega_1 + 1$. Then $p\pi(X) \leq \omega < t(X)$.

**Proof.** It is obvious that $t(\omega_1, X) = t(X) = \omega_1$. To see $p\pi(X) \leq \omega$, consider any continuous surjection $f : X \to Y$. If $f(\omega_1) = p$ is an isolated point in $Y$, then there is an $\alpha < \omega_1$ such that $f$ is constant on the interval $[\alpha, \omega_1]$; hence $Y$ is countable and compact and so, trivially, $\pi(Y) \leq w(Y) = \omega$.

If, however, $p$ is not isolated, then $Y$ has a countable dense subset $S$ with $p \notin S$. So there is a closed $G_\delta$ set $F$ such that $p \in F \subseteq Y \setminus S$, and again we can find an $\alpha < \omega_1$ such that $f[\alpha, \omega_1] \subseteq F$. But then $G = Y \setminus F$ is countable and dense open in $Y$; moreover $w(G) = \omega$ because every countable and locally compact space is second countable. So we have $\pi(Y) \leq \pi(Y) = w(Y) = \omega$. \qed

We recall from [3] that $\pi sw(X)$ denotes the $\pi$-separating weight of a space $X$, that is, the minimum order of a $\pi$-base of $X$; see p. 74 of [3].

With this we may now formulate our main result as follows.

**Theorem 2.** For any Tychonov space $X$ we have $\pi sw(X) \leq p\pi(X)$. In particular, any Tychonov space of countable projective $\pi$-character has a point-countable $\pi$-base.

Our proof of Theorem 2 will go along similar lines as Shapirovskii’s proof of the weaker result $\pi sw(X) \leq t(X)$ for compact spaces. The main idea of that was to show that the compactum $X$ admits an irreducible map onto a subspace of the $\Sigma(t(X))$-power of the unit interval. The role of irreducible maps in our proof will be played by a new, more general, type of maps that we shall call $\pi$-irreducible. So we shall first define and deal with these maps. (The referee has pointed out to us that [3] is an excellent source concerning Shapirovskii’s original method.)
**Definition 3.** Let $f$ be a continuous map of $X$ onto $Y$. We say that the map $f$ is $\pi$-irreducible if for every proper closed subset $F \subset X$ its image $f[F]$ is not dense in $Y$.

Clearly, an onto map $f$ is $\pi$-irreducible iff the $f$-image of a non-dense set is non-dense. Also, it is obvious that a closed map is $\pi$-irreducible iff it is irreducible; consequently the two concepts coincide for maps between compact Hausdorff spaces.

The following proposition will be used in the proof of Theorem 2 and explains our terminology.

**Proposition 4.** Let $f$ be a continuous map of $X$ onto $Y$. Then the following five statements are equivalent:

1. $f$ is $\pi$-irreducible;
2. for every $\pi$-base $\mathcal{B}$ of $X$ and for every $B \in \mathcal{B}$ the $f$-image of its complement, $f[X \setminus B]$, is not dense in $Y$;
3. there is a $\pi$-base $\mathcal{B}$ of $X$ such that for every $B \in \mathcal{B}$ the $f$-image $f[X \setminus B]$ is not dense in $Y$;
4. for every $\pi$-base $\mathcal{C}$ of $Y$ the family $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a $\pi$-base of $X$;
5. there is a $\pi$-base $\mathcal{C}$ of $Y$ such that $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a $\pi$-base of $X$.

**Proof.** We shall show (3)$\Rightarrow$(4) and (5)$\Rightarrow$(1) only because the other three implications of the cycle are trivial.

So, let $\mathcal{B}$ be as in (3) and $\mathcal{C}$ be any $\pi$-base of $Y$. For every non-empty open set $U$ in $X$ choose $B \in \mathcal{B}$ with $B \subset U$. Then there is a $C \in \mathcal{C}$ such that $C \cap f[X \setminus B] = \emptyset$, and hence $f^{-1}(C) \subset B \subset U$.

Now, let $\mathcal{C}$ be as in (5) and $F$ be a proper closed subset of $X$. Then there is a $C \in \mathcal{C}$ with $F \cap f^{-1}(C) = \emptyset$; consequently we have $f[F] \cap C = \emptyset$ and so $f[F]$ is not dense in $Y$. \hfill \square

**Corollary 5.** If $f : X \to Y$ is $\pi$-irreducible, then $\pi(X) = \pi(Y)$.

**Proof.** $\pi(X) \leq \pi(Y)$ is immediate from part (4) of Proposition 4. To see $\pi(X) \geq \pi(Y)$ first note that for any non-empty open $U \subset X$ the interior of $f[U]$ in $Y$ is non-empty. So for any $\pi$-base $\mathcal{B}$ of $X$ the family $\{\text{Int}_Y(f[B]) : B \in \mathcal{B}\}$ is a $\pi$-base of $Y$. Indeed, this is because if $V$ is non-empty open in $Y$ and $B \in \mathcal{B}$ with $B \subset f^{-1}(V)$, then $f[B] \subset V$. \hfill \square

We now consider another key ingredient of the proof of our main result: certain specially embedded subspaces of Tychonov cubes. As usual, we shall denote the unit interval $[0, 1]$ by $I$. The members of the Tychonov cube $I^\kappa$ will be construed as functions from $\kappa$ to $I$. So if $x \in I^\kappa$ and $\alpha < \kappa$, then $x \upharpoonright \alpha$ is the projection of $x$ to the subproduct $I^\alpha$.

**Definition 6.** We say that $Y \subset I^\kappa$ is 0-embedded in the Tychonov cube $I^\kappa$ if

$$\{y \upharpoonright \alpha : y \in Y \text{ and } y(\alpha) = 0\}$$

is dense in the projection $Y \upharpoonright \alpha = \{y \upharpoonright \alpha : y \in Y\}$ for every $\alpha < \kappa$.

We now present two results concerning 0-embedded subspaces of Tychonov cubes which will be crucial in the proof of our main theorem and are also interesting in themselves.
Theorem 7. Assume that $Y$ is 0-embedded in the Tychonov cube $I^\kappa$ where $\kappa$ is a regular cardinal and $y \in Y$ is such that $y(\alpha) > 0$ for all $\alpha < \kappa$. Then $\pi \chi(y, Y) = \kappa$.

Proof. Of course, only $\pi \chi(y, Y) \geq \kappa$ needs to be proven. To see this, let $U$ be any family of elementary open sets in $I^\kappa$ such that $|U| < \kappa$ and $U \cap Y \neq \emptyset$ for all $U \in U$. Every elementary open set $U \in U$ is supported by a finite subset of $\kappa$; hence the regularity of $\kappa$ implies the existence of an ordinal $\alpha < \kappa$ such that the support of each $U \in U$ is included in $\alpha$.

Since $Y$ is 0-embedded in $I^\kappa$, this implies that for every $U \in U$ we may pick a point $y_U \in U \cap Y$ such that $y_U(\alpha) = 0$. But then $y(\alpha) > 0$ clearly implies that the point $y$ is not in the closure of the set $\{y_U : U \in U\}$; consequently $U$ cannot be a local $\pi$-base at $y$ in $Y$, completing the proof. \(\square\)

From Theorem 7, we can immediately obtain the following useful corollary about the projective $\pi$-character of 0-embedded subspaces of Tychonov cubes.

Corollary 8. If $Y$ is 0-embedded in the Tychonov cube $I^\kappa$, then for every non-isolated point $y \in Y$ we have

$$p \pi \chi(Y) \geq \left| \{\alpha : y(\alpha) > 0\} \right|,$$

and if $y \in Y$ is isolated, then $\{\alpha : y(\alpha) > 0\}$ is finite.

Our next result shows that every Tychonov space admits a $\pi$-irreducible map onto a suitable 0-embedded subspace of a Tychonov cube.

Theorem 9. Let $X$ be any Tychonov space of $\pi$-weight $\pi(X) = \kappa$. Then there is a $\pi$-irreducible map $f$ of $X$ onto a 0-embedded subspace $Y$ of the Tychonov cube $I^\kappa$.

Proof. To begin with, let us choose a $\pi$-base $\mathcal{B}$ of $X$ with $|\mathcal{B}| = \kappa$ and fix a well-ordering $\prec$ of $\mathcal{B}$ of order-type $\kappa$.

We shall define by transfinite induction on $\alpha < \kappa$ the co-ordinate maps $g_\alpha = p_\alpha \circ f : X \to I$, where $p_\alpha(y) = y(\alpha)$ is the $\alpha$th co-ordinate projection, and sets $B_\alpha \in \mathcal{B}$. So assume that $\alpha < \kappa$ and for all $\beta < \alpha$ the maps $g_\beta : X \to I$ and the sets $B_\beta \in \mathcal{B}$ have been defined.

Let $f_\alpha : X \to I^\kappa$ be the map whose $\beta$th co-ordinate map is $g_\beta$ for all $\beta < \alpha$ and set $Y_\alpha = f_\alpha[X]$. Then, in view of Corollary 5, the map $f_\alpha : X \to Y_\alpha$ cannot be $\pi$-irreducible because $\pi(Y_\alpha) < \kappa = \pi(X)$; hence using part (2) of Proposition 4 there is a member $B \in \mathcal{B}$ for which $f_\alpha[X \setminus B]$ is dense in $Y_\alpha$. Let $B_\alpha$ be the $\prec$-first such member of $\mathcal{B}$. We then define $g_\alpha : X \to I$ as any continuous function that is identically 0 on $X \setminus B_\alpha$ and takes the value 1 at some point in $B_\alpha$. As was intended, with the induction completed we let $f : X \to Y^\kappa$ be the unique map having the $g_\alpha$ for $\alpha < \kappa$ as its co-ordinate functions and we also set $Y = f[X]$.

Note first that if $\beta < \alpha$, then $B_\beta \prec B_\alpha$. Indeed, since we have $Y_\beta = Y_\alpha \upharpoonright \beta$, the density of $f_\alpha[X \setminus B_\alpha]$ in $Y_\alpha$ implies that $f_\beta[X \setminus B_\alpha]$ is dense in $Y_\beta$; hence $B_\alpha \prec B_\beta$ would contradict the choice of $B_\beta$. Moreover, by our construction, $f_{\beta+1}[X \setminus B_\beta]$ is not dense in $Y_{\beta+1}$ and consequently $f_\alpha[X \setminus B_\beta]$ is not dense in $Y_\alpha$, which implies $B_\alpha \neq B_\beta$.

Since $\mathcal{B}$ is of order type $\kappa$ under $\prec$, it follows from this that for every $B \in \mathcal{B}$ there is an $\alpha < \kappa$ with $B \prec B_\alpha$. But then, by the choice of $B_\alpha$, we have that $f_\alpha[X \setminus B]$ is not dense in $Y_\alpha = Y \upharpoonright \alpha$ and hence $f[X \setminus B]$ cannot be dense in $Y$. Using part (3) of Proposition 4 implies that $f$ is indeed a $\pi$-irreducible map of $X$ onto $Y$. 

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Finally, by our construction, for every $\alpha < \kappa$ the image $f_\alpha[X \setminus B_\alpha]$ is dense in $Y_\alpha = Y \setminus \{\alpha\}$; moreover we have
\[ f_\alpha[X \setminus B_\alpha] \subset \{y \mid \alpha \in Y \text{ and } y(\alpha) = 0\}, \]
and consequently $Y$ is indeed 0-embedded in $I^\kappa$. \hfill \square

Let us now recall that the $k$th $\Sigma_\lambda$-power of $I$, denoted by $\Sigma_\lambda(I, \kappa)$, is the subspace of $I^\kappa$ consisting of all points whose support is of size at most $\lambda$. The support of a point $y \in I^\kappa$ is the set $\{\alpha < \kappa : y(\alpha) > 0\}$. Thus, from Theorem 9 and from Corollary 8—moreover from the trivial fact that $p\pi\chi(Y) \leq p\pi\chi(X)$ if $Y$ is any continuous image of $X$—we immediately obtain the following result.

**Corollary 10.** If $X$ is a Tychonov space such that $\pi(X) = \kappa$ and $p\pi\chi(X) = \lambda$, then some $\pi$-irreducible image $Y$ of $X$ embeds into $\Sigma_\lambda(I, \kappa)$.

This corollary is clearly a strengthening of the following result of Shapirovskii from [6] (see also 3.22 of [3]): If $X$ is compact Hausdorff, then some irreducible image of $X$ embeds into $\Sigma_{\pi\chi(X)}$-power of $I$.

The proof of our main theorem, Theorem 2, can now be easily established by recalling the following result of Shapirovskii from [6] (see also 3.24).

**Theorem (Shapirovskii).** If the space $Y$ embeds into a $\Sigma_\lambda$-power of $I$, then $\pi_{sw}(Y) \leq \lambda$.

**Proof of Theorem 2.** Now, to prove Theorem 2 consider any non-discrete Tychonov space $X$. By Corollary 10 then $X$ has a $\pi$-irreducible image $Y$ that embeds into a $\Sigma_\lambda$-power of $I$, where $\lambda = p\pi\chi(X)$. By the previous theorem of Shapirovskii the space $Y$ has a $\pi$-base $C$ of order at most $\lambda$. But by part (4) of Proposition 4 $\{f^{-1}(C) : C \in C\}$ is a $\pi$-base of $X$ that clearly has the same order as $C$.

The following result is then an immediate consequence of Theorem 2.

**Corollary 11.** Let $X$ be any Tychonov space and $\kappa > p\pi\chi(X)$ be a cardinal such that $\kappa$ is a caliber of $X$. Then $\pi(X) < \kappa$.

Since $t(X) \geq p\pi\chi(X)$ for a compact Hausdorff space $X$, this corollary implies Shapirovskii’s theorem saying that if $t(X)^+$ is a caliber of such a space $X$, then $\pi(X) \leq t(X)$. Moreover, it also extends from compacta to all Tychonov spaces Arhangel’skii’s result from [2] saying that spaces of countable projective $\pi$-character and having $\omega_1$ as a caliber are separable.

Let us conclude by pointing out that neither Theorem 2 nor Corollary 11 remain valid if the projective $\pi$-character $p\pi\chi$ is replaced by the simple $\pi$-character $\pi\chi$ in them. In fact, it has recently been shown in [3] that there are even first countable spaces whose $\pi$-separating weight is as large as you wish. Moreover, in the same paper it was also shown that it is consistent to have first countable spaces with caliber $\omega_1$ which have uncountable $\pi$-weight (or equivalently, density). However, since first countability implies countable tightness, none of these examples are (or could be) compact, so the following intriguing questions remain open.

**Problem 12.** Let $X$ be a compact Hausdorff space of countable $\pi$-character. Does $X$ have a point-countable $\pi$-base? If, in addition, $\omega_1$ is a caliber of $X$, is $X$ then separable?
References


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