COMPUTATION OF THE MORDELL-TORNHEIM ZETA VALUES

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Abstract. In this paper the authors present several algorithmic formulas which are potentially useful in computing the following Mordell-Tornheim zeta values:
\[ \zeta_{MT,r}(s_1, \ldots, s_r; s) := \sum_{m_1, \ldots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} \]
for the special cases
\[ \zeta_{MT,r}(1, \ldots, 1; s) \quad \text{and} \quad \zeta_{MT,r}(0, \ldots, 0; s). \]
Some interesting (known or new) consequences and illustrative examples are also considered.

1. Introduction

The following double series:
\[ \sum_{m_1, m_2=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} (m_1 + m_2)^s} \quad (s_1, s_2, s \in \mathbb{C}) \]
was defined and studied by Tornheim (see [14]). Subsequently, Mordell (see [10]) considered a multiple series in the form:
\[ \sum_{m_1, \ldots, m_r=1}^{\infty} \frac{1}{m_1 \cdots m_r \prod_{j=0}^{k \in \mathbb{N}_0} (m_1 + \cdots + m_r + a + j)} \]
\[ (r \in \mathbb{N} := \{1, 2, 3, \ldots\}; k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; a > -r). \]

In particular, he deduced that (see [10])
\[ \sum_{m_1, m_2=1}^{\infty} \frac{1}{m_1 m_2 (m_1 + m_2)} = 2 \zeta(3), \]
where $\zeta(z)$ denotes the Riemann zeta function defined by (see, for details, [13, Section 2.3])

\begin{equation}
\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{k^z} \quad (\Re(z) > 1).
\end{equation}

For

$$r \in \mathbb{N} \quad \text{and} \quad s_1, \cdots, s_r, s \in \mathbb{C},$$

Matsumoto (see [8] and [9]) defined the Mordell-Tornheim $r$-fold zeta function by

\begin{equation}
\zeta_{MT,r}(s_1, \cdots, s_r; s) := \sum_{m_1, \cdots, m_r = 1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s}.
\end{equation}

He showed that the function $\zeta_{MT,r}(s_1, \cdots, s_r; s)$ can be continued meromorphically to the whole $(r + 1)$-dimensional complex space and the multiple series in (1.5) is absolutely convergent when $\Re(s_j) > 1$ ($j = 1, \cdots, r$) and $\Re(s) > 0$.

By using the following result of Mordell (see [10]):

$$\sum_{m_1, \cdots, m_r = 1}^{\infty} \frac{1}{m_1 \cdots m_r (m_1 + \cdots + m_r + a)} = r! \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! \ell! \ell^{r+1}} \left(a - 1\right)^\ell \quad (a > -r; \ r \in \mathbb{N}),$$

Hoffman proved that (see [4])

\begin{equation}
\zeta_{MT,r}(1, \cdots, 1; s) = r! \sum_{m_1 > m_2 > \cdots > m_s \geq 1}^{\infty} \frac{1}{m_1^{s_1+1} m_2 \cdots m_s} 
\end{equation}

Analogously, Subbarao and Sitaramachandrarao (see [11]) established the following interesting result:

\begin{equation}
\zeta_{MT,r}(1, \cdots, 1; s) = (-1)^r r! \sum_{n=r}^{\infty} \frac{(-1)^n}{n!} \frac{s(n, r)}{n^s},
\end{equation}

where $s(n, k)$ are the Stirling numbers of the first kind defined by

\begin{equation}
z(z - 1) \cdots (z - n + 1) = \sum_{k=0}^{n} s(n, k) z^k
\end{equation}

or, equivalently, by the generating function:

\begin{equation}
\{\log(1 + z)\}^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{z^n}{n!} \quad (|z| < 1).
\end{equation}

A detailed bibliography for the Mordell-Tornheim zeta values is given by Hoffman [5].

In this paper we derive algorithmic formulas for the evaluation of

$$\zeta_{MT,r}(1, \cdots, 1; s) \quad \text{and} \quad \zeta_{MT,r}(0, \cdots, 0; s),$$

which involve (for example) the Stirling numbers $s(n, k)$ and the Riemann zeta function $\zeta(z)$. Our proof of the algorithmic formula for $\zeta_{MT,r}(1, \cdots, 1; s)$ is based upon the results given earlier by Choi and Srivastava [3] and by Subbarao and Sitaramachandrarao [11].
2. Preliminary results

First of all, we find integral representations for the functions

\[ \zeta_{MT,r}(0, \cdots, 0; s) \quad \text{and} \quad \zeta_{MT,r}(1, \cdots, 1; s). \]

Indeed, by applying the Eulerian integral:

\[ z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-zt} \, dt \quad (\Re(s) > 0; \Re(z) > 0), \]

where \( \Gamma(s) \) is the gamma function defined by

\[ \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt \quad (\Re(s) > 0), \]

we obtain

\[ \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(m_1 + \cdots + m_r)t}}{m_1^{s_1} \cdots m_r^{s_r}} \, dt, \]

which, in view of the monotone convergence theorem, yields

\[ \zeta_{MT,r}(s_1, \cdots, s_r; s) := \sum_{m_1, \cdots, m_r = 1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s} \]

\[ = \frac{1}{\Gamma(s)} \int_0^\infty \prod_{j=1}^r \left\{ \sum_{m_j = 1} \frac{e^{-m_j t}}{m_j^{s_j}} \right\} \, dt. \]

Thus, for \( s_1, \cdots, s_r \in \mathbb{C} \), we obtain

\[ \zeta_{MT,r}(s_1, \cdots, s_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \prod_{j=1}^r \left\{ \log(1 - e^{-t}) \right\} \, dt \quad (s_j \in \mathbb{C}; j = 1, \cdots, r), \]

where \( \log(z) \) is the polylogarithm function defined by [13] pp. 114 and 124:

\[ \log(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \quad (\nu \in \mathbb{C} \text{ when } |z| < 1; \Re(\nu) > 1 \text{ when } |z| = 1). \]

Hence we find that

\[ \zeta_{MT,r}(0, \cdots, 0; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \, dt \quad (\Re(s) > r; r \in \mathbb{N}) \]

and

\[ \zeta_{MT,r}(1, \cdots, 1; s) = \frac{(-1)^r}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^t - 1)^r} \, dt \quad (\Re(s) > 1; r \in \mathbb{N}). \]

Next, for \( m, r \in \mathbb{N} \), we introduce the following multiple sum:

\[ P_m(s_1, \cdots, s_r) := \sum_{k_1 + \cdots + k_r = m \atop k_1, \cdots, k_r \in \mathbb{N}} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}} \quad (m, r \in \mathbb{N}). \]
Using the Cauchy product of power series, we have
\[
\prod_{j=1}^{r} \left\{ \sum_{m_j=1}^{\infty} e^{-m_j t} \right\} = \sum_{m=1}^{\infty} \sum_{k_{r-1}=1}^{m} \sum_{k_{r-2}=1}^{m-k_{r-1}} \sum_{k_{r-3}=1}^{m-k_{r-1}-k_{r-2}+2} \cdots \sum_{k_1=1}^{m-k_{r-1} \cdots - k_2 + r - 2} \frac{e^{-(m+r-1)t}}{k_1^{s_1} k_2^{s_2} \cdots k_r^{s_r}} (m + r - 1 - k_1 - k_2 - \cdots - k_{r-1})^{s_r}.
\]

(2.8)

so that
\[
\prod_{j=1}^{r} \left\{ \sum_{m_j=1}^{\infty} e^{-m_j t} \right\} = \sum_{m=1}^{\infty} e^{-mt} P_m(s_1, \cdots, s_r).
\]

The relations (2.1), (2.2) and (2.9) yield the following result:
\[
\zeta_{MT,r}(s_1, \cdots, s_r; s) = \sum_{m=1}^{\infty} P_m(s_1, \cdots, s_r) \frac{z^n}{n!}.
\]

(2.10)

It is known that \(\text{cf. } [6, \text{p. 146, Equation (3)}]\); see also Equation (1.9) above)
\[
\{\log(1 - z)\}^r = r! \sum_{n=1}^{\infty} (-1)^n \frac{s(n, r)}{n!} z^n.
\]

(2.11)

Since
\[
\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n},
\]

we have
\[
\{\log(1 - z)\}^r = (-1)^r \sum_{n=1}^{\infty} P_n(1, \cdots, 1) z^n.
\]

(2.12)

Thus, by comparing the coefficients of \(z^n\) in (2.11) and (2.12), we obtain the following relationship:
\[
P_n(1, \cdots, 1) = (-1)^{n+r} r! \frac{s(n, r)}{n!}
\]

(2.13)

with the Stirling numbers \(s(n, r)\) of the first kind.

The Stirling numbers \(s(n, k)\) of the first kind satisfy a recurrence relation in the form (see \([13, \text{p. 56, Equation (3)}]\)):
\[
s(n + 1, k) = s(n, k - 1) - ns(n, k) \quad (k = 1, \cdots, n),
\]

(2.14)
which, in conjunction with the relationship (2.12), yields

(2.15) \( (n+1)P_{n+1}(\underbrace{1, \cdots, 1}_r) = rP_n(\underbrace{1, \cdots, 1}_r) + nP_n(\underbrace{1, \cdots, 1}_r) \).

3. Main results

By examining the Eulerian beta integral:

(3.1) \[ \int_0^1 (1-t)^{\lambda} t^{\mu-1} dt = \frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\mu \Gamma(1+\lambda+\mu)} \quad (\Re(\lambda) > -1; \Re(\mu) > 0), \]

in terms of the classical gamma function, Choi and Srivastava [3] introduced the following notation (see, for details, [3, p. 56]):

(3.2) \[ \frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\mu \Gamma(1+\lambda+\mu)} = \sum_{n=0}^{\infty} A_n(\mu)\lambda^n \quad (A_0(\mu) = 1), \]

(3.3) \[ A_p(\mu) = \sum_{n=0}^{\infty} \Lambda_n^{(p)} \mu^n, \]

and

(3.4) \[ \psi(1+\lambda) - \psi(1+\lambda+\mu) = \sum_{n=0}^{\infty} B_n(\mu)\lambda^n \]

\( (B_0(\mu) = \psi(1) - \psi(1+\mu)) \),

where

(3.5) \[ B_n(\mu) = (-1)^{n+1} \zeta(n+1) - \frac{\psi^{(n)}(1+\mu)}{n!} \quad (n \in \mathbb{N}) \]

and

(3.6) \[ (n+1)A_{n+1}(\mu) = \sum_{k=0}^{n} A_k(\mu)B_{n-k}(\mu) \quad (n \in \mathbb{N}_0) \]

in terms of the polygamma functions \( \psi^{(n)}(z) \quad (n \in \mathbb{N}_0) \) defined by

\[ \psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \{ \log \Gamma(z) \} = \frac{d^n}{dz^n} \{ \psi(z) \} \quad (n \in \mathbb{N}_0), \]

\[ \psi^{(0)}(z) = \psi(z) \]

being the relatively more familiar psi (or digamma) function.

In the proof of one of our main results (see Theorem 1 below), we shall be using the following results of Choi and Srivastava [3, p. 56, Equation (2.3)] and Subbarao and Sitaramachandrarao [11, p. 250, Equation (3.4)]:

(3.7) \[ \int_0^1 \{ \log(1-t) \}^p \{ \log t \}^q \frac{dt}{t} = p! \cdot q! \cdot A^{(p)}_{q+1} \quad (p, q \in \mathbb{N}_0) \]

and

(3.8) \[ \sum_{m_1, \cdots, m_r, a=1}^{\infty} \frac{1}{m_1 \cdots m_r(m_1 + \cdots + m_r + a)} = (-1)^r \int_0^1 t^{a-1} \{ \log(1-t) \}^r \ dt, \]

respectively.
Remark 1. The above formulas (3.7) and (3.8) are the corrected versions of the corresponding results given in [3] and [11], respectively.

**Theorem 1.** The following recursion formula holds true for $\zeta_{MT,r}(1, \cdots, 1; s)$ with respect to $r$:

$$
\zeta_{MT,r}(1, \cdots, 1; s) = \frac{(s + r - 1)!}{s!} \zeta(s + r) - \sum_{n=1}^{s-1} \zeta(s - n + 1) \zeta_{MT,r-1}(1, \cdots, 1; n)
$$

$$
+ \sum_{k=1}^{r-2} \left[ \frac{(r - 1)!}{k!} \zeta(r-k) \zeta_{MT,k}(1, \cdots, 1; s) \right] - \left( r - 1 \right) \sum_{n=1}^{s} \frac{(s + r - n - k - 1)!}{(s-n)!} \zeta(s + r - n - k) \zeta_{MT,k}(1, \cdots, 1; n)
$$

$$
(3.9)
$$

(3.9)

$$
(r, s \in \mathbb{N}),
$$

where an empty sum is interpreted to be nil.

Remark 2. It is easily observed from the assertion (3.9) of Theorem 1 that

$$
\zeta_{MT,r}(1, \cdots, 1; 1) = r! \zeta(r + 1) \quad (r \in \mathbb{N})
$$

and

$$
\zeta_{MT,1}(1; s) = \zeta(s + 1) \quad (s \in \mathbb{N}).
$$

**Proof of Theorem 1.** Firstly, upon differentiating both members of (3.8) $s-1$ times with respect to $a$ and then setting $a = 0$, we get

$$
\zeta_{MT,r}(1, \cdots, 1; s) = \frac{(-1)^{s+r+1}}{(s-1)!} \int_{0}^{1} \{\log(1-t)^r (\log t)^{s-1} \} dt.
$$

Now, if we compare (3.12) with the case $p = r$ and $q = s - 1$ of (3.7), we readily arrive at the following relationship:

$$
\zeta_{MT,r}(1, \cdots, 1; s) = (-1)^{r+s+1} r! A^{(r)}_s
$$

with the coefficients $A^{(p)}_s$ occurring in (3.3).

Secondly, in view of the following known expansion formula [11] p. 85, Equation (6.3.14)]:

$$
\psi(z + 1) = \psi(1) + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) z^n}{n!} \quad (|z| < 1),
$$

we find from the equations (3.3) to (3.6) that

$$
(r + 1) \sum_{s=0}^{\infty} A^{(r+1)}_s \mu^s = \sum_{k=0}^{r-1} A_k(\mu) B_{r-k}(\mu) + A_r(\mu) B_0(\mu)
$$

$$
= \sum_{k=0}^{r-1} A_k(\mu) \left[ (-1)^{r-k+1} \zeta(r - k + 1) - \frac{\psi(r-k)(1+\mu)}{(r-k)!} \right]
$$

$$
+ A_r(\mu) \sum_{k=1}^{\infty} (-1)^k \zeta(k+1) \mu^k
$$

$$
(3.14)
$$

(3.14)

$$
(\Re(\mu) > 0 \text{ and } |\mu| < 1).
$$

(3.14)
Now, by applying the known result [1, p. 86, Equation (6.4.9)]:

\[ \psi^{(n)}(z + 1) = (-1)^{n+1} \sum_{k=0}^{\infty} (-1)^k \frac{(n+k)!}{k!} \zeta(n+k+1) z^k \quad (|z| < 1) \]

in (3.14), we obtain

\[ (r + 1) \sum_{s=0}^{\infty} \Lambda_s^{(r+1)} \mu^s = \mu \left( \sum_{s=0}^{\infty} \Lambda_s^{(r)} \mu^s \right) \sum_{s=0}^{\infty} (-1)^s \zeta(s+1) \mu^s \]

\[ + \sum_{\ell=0}^{r-1} \left( \sum_{k=0}^{\infty} (-1)^{-k+1} \zeta(r-k+1) \Lambda_\ell^{(k)} \right) \mu^\ell \]

\[ + \frac{1}{(r-k)!} \left( \sum_{s=0}^{\infty} \Lambda_s^{(k)} \mu^s \right) \sum_{s=0}^{\infty} (-1)^{r-k+s} \zeta(r-k+s+1) \mu^s, \]

where \((\kappa)_n\) represents the Pochhammer symbol (or the shifted factorial) given by

\[ (\kappa)_n = \begin{cases} 1 & \text{if } n = 0, \\ \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} & \text{if } n \in \mathbb{N}. \end{cases} \]

Hence we have

\[ (r + 1) \sum_{s=0}^{\infty} \Lambda_s^{(r+1)} \mu^s = \mu \left( \sum_{s=0}^{\infty} \Lambda_s^{(r)} \mu^s \right) \sum_{s=0}^{\infty} (-1)^s \zeta(s+1) \mu^s \]

\[ + \sum_{\ell=0}^{r-1} \left( \sum_{k=0}^{\infty} (-1)^{-k+1} \zeta(r-k+1) \Lambda_\ell^{(k)} \right) \mu^\ell \]

\[ + \frac{1}{(r-k)!} \left( \sum_{s=0}^{\infty} \Lambda_s^{(k)} \mu^s \right) \sum_{s=0}^{\infty} (-1)^{r-k+s} \zeta(r-k+s+1) \mu^s, \]

so that

\[ (r + 1) \sum_{s=0}^{\infty} \Lambda_s^{(r+1)} \mu^s = \mu \sum_{s=0}^{\infty} \left( \sum_{n=0}^{s} \Lambda_n^{(r)} (-1)^{n-s+1} \zeta(s-n+2) \right) \mu^s \]

\[ + \sum_{s=0}^{r-1} \left( \sum_{\ell=0}^{\infty} (-1)^{r-k+1} \zeta(r-k+1) \Lambda_\ell^{(k)} \right) \mu^\ell \]

\[ + \frac{1}{(r-k)!} \sum_{s=0}^{\infty} \left( \sum_{n=0}^{s} \Lambda_n^{(k)} (-1)^{r-k-s-n} (s-n+1) \zeta(r-k+s-n+1) \right), \]
that is, that
\[
(r + 1) \sum_{s=0}^{\infty} \Lambda_s^{(r+1)} \mu^s = \sum_{s=1}^{\infty} \left( \sum_{n=0}^{s-1} \Lambda_n^{(r)} (-1)^{n-s} \zeta(s-n+1) \right) \mu^s + \sum_{s=1}^{\infty} \left( \sum_{k=0}^{r-1} (-1)^{r-k+1} \zeta(r-k+1) \Lambda_s^{(k)} \right) \mu^s
\]
\[
+ \sum_{s=1}^{\infty} \left[ \sum_{k=0}^{r-1} \frac{1}{(r-k)!} \left( \sum_{n=0}^{s} \Lambda_n^{(k)} (-1)^{r-k+s-n} (s-n+1)_{r-k} \right) \right] \mu^s.
\]
(3.17)

We note here that the following recurrence relations hold true for \( \Lambda_s^{(r)} \):
(3.18) \( \Lambda_0^{(n)} = 0 \), \( \Lambda_0^{(0)} = 1 \) and \( \Lambda_n^{(0)} = 0 \) \((n \in \mathbb{N})\)
and
\[
r\Lambda_s^{(r)} = \sum_{n=0}^{s-1} (-1)^{n-s} \zeta(s-n+1) \Lambda_n^{(r-1)} + \sum_{k=0}^{r-2} (-1)^{r-k} \zeta(r-k) \Lambda_s^{(k)}
\]
(3.19)
\[
- \frac{1}{(r-k)!} \left( \sum_{n=0}^{s} (-1)^{r-k+s-n} \Lambda_n^{(k)} (s-n+1)_{r-k-1} \zeta(r-k+s-n) \right)
\]
Formula (3.19) is obtained by comparing the coefficients of \( \mu^s \) in (3.17).

Finally, using (3.18) and (3.19) in conjunction with (3.13), we have the assertion (3.9) of Theorem 1. The proof of Theorem 1 is thus completed. \( \square \)

Remark 3. The equations in (3.18) follow from (3.2) and (3.17).

Example. For \( s \in \mathbb{N} \setminus \{1\} \) and \( r = 2, 3 \), Theorem 1 would provide the following special cases:
(3.20) \( \zeta_{MT,2}(1, 1; s) = (s+1)\zeta(s+2) - \sum_{n=1}^{s-1} \zeta(s-n+1)\zeta(n+1) \)
and
\[
\zeta_{MT,3}(1, 1; s) = (s+1)(s+2)\zeta(s+3) + 2\zeta(2)\zeta(s+1)
\]
\[- \sum_{n=1}^{s-1} (n+1)\zeta(s-n+1)\zeta(n+2)
\]
\[
+ \sum_{n=2}^{s-1} \zeta(s-n+1) \sum_{m=1}^{n-1} \zeta(n-m+1)\zeta(m+1)
\]
(3.21)
\[- 2 \sum_{n=1}^{s} (s-n+1)\zeta(s-n+2)\zeta(n+1).
\]
Equation (3.20) is a known result given in [11, p. 247, Equation (2.8)]. Equation (3.21), on the other hand, is equivalent to another known result [7, p. 128,
Corollary 4.3]. Thus, for \( r \in \mathbb{N} \setminus \{1, 2\} \), the assertion (3.9) of Theorem 1 is presumably new.

We next establish a recursion formula given by Theorem 2 below.

**Theorem 2.** The following recursion formula holds true for \( \zeta_{MT,r}(0, \cdots, 0; s) \):

\[
(3.22) \quad \zeta_{MT,r}(0, \cdots, 0; s) = \frac{1}{(r-1)!} \sum_{m=1}^{r} s(r, m) \zeta(s-m+1) \quad (\Re(s) > r; \ r \in \mathbb{N}),
\]

where \( s(n,k) \) denotes the Stirling numbers of the first kind given by (1.8) and (1.9).

**Proof.** By integration by parts, it is easily seen that

\[
\frac{s-1}{r} \int_{0}^{\infty} \frac{t^{s-2}}{(e^t - 1)^r} dt = \int_{0}^{\infty} \frac{e^{t}s^{-1}}{(e^t - 1)^{r+1}} dt \quad (\Re(s) > r+1; \ r \in \mathbb{N}),
\]

so that

\[
(3.23) \quad \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{(e^t - 1)^{r+1}} dt = \frac{1}{r\Gamma(s-1)} \int_{0}^{\infty} \frac{t^{s-2}}{(e^t - 1)^r} dt - \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{(e^t - 1)^r} dt
\]

(\( \Re(s) > r+1; \ r \in \mathbb{N} \)).

Thus, by using the expression (2.5) in (3.23), we get the following recursion formula:

\[
(3.24) \quad \zeta_{MT,r+1}(0, \cdots, 0; s) = \frac{1}{r} \zeta_{MT,r}(0, \cdots, 0; s-1) - \zeta_{MT,r}(0, \cdots, 0; s).
\]

Since

\[\zeta_{MT,1}(0; s) = \zeta(s),\]

the proof of Theorem 2 can be completed by mathematical induction on \( r \) in light of (2.14) and (3.24). \( \square \)

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