ESTIMATES OF GROMOV’S BOX DISTANCE

KEI FUNANO

(Communicated by Jon G. Wolfson)

Abstract. In 1999, M. Gromov introduced the box distance function □\lambda on the space of all mm-spaces. In this paper, by using the method of T. H. Colding, we estimate □\lambda(S^n, S^m) and □\lambda(CP^n, CP^m), where S^n is the n-dimensional unit sphere in R^{n+1} and CP^n is the n-dimensional complex projective space equipped with the Fubini-Study metric. In particular, we give the complete answer to an exercise of Gromov’s green book. We also estimate □\lambda(SO(n), SO(m)) from below, where SO(n) is the special orthogonal group.

1. Introduction

In 1999, M. Gromov developed the theory of mm-spaces in [4, Chapter 3.2] by introducing two distance functions, called the box distance function □\lambda and the observable distance function H_{\lambda, L_1}, on the space X of all isomorphic classes of mm-spaces. Here, an mm-space is a triple (X, d_X, \mu_X), where d_X is a complete separable metric on a set X and \mu_X a finite Borel measure on (X, d_X). The notion of the distance function □\lambda is considered as a natural extension of the Gromov-Hausdorff distance function to the space X. On the other hand, the notion of the distance function H_{\lambda, L_1} is related to measure concentration. Roughly speaking, “measure concentration” amounts to saying that the push-forward measures f_n^* (\mu_n) on R concentrate to a point for any sequence of 1-Lipschitz functions f_n : (X_n, d_n, \mu_n) → R. For instance, the unit spheres in Euclidean spaces \{S^n\}_{n=1}^\infty, the complex projective spaces \{CP^n\}_{n=1}^\infty equipped with the Fubini-Study metrics, and the special orthogonal groups \{SO(n)\}_{n=1}^\infty have this property (we suppose that each space is equipped with its Riemannian volume measure normalized to have total volume 1). Gromov defined the distance H_{\lambda, L_1}(X, Y) by using the Hausdorff distance between the space of 1-Lipschitz functions on X and that on Y, and showed that a sequence \{X_n\}_{n=1}^\infty of mm-spaces concentrates if and only if the sequence \{X_n\}_{n=1}^\infty converges to a one-point space with respect to the distance function H_{\lambda, L_1}.

Received by the editors June 18, 2007.
2000 Mathematics Subject Classification. Primary 28E99, 53C23.
Key words and phrases. mm-space, box distance function, observable distance function.
This work was partially supported by research fellowships of the Japan Society for the Promotion of Science for Young Scientists.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

2911
The topology on $\mathcal{X}$ determined by $\square_\lambda$ is strictly stronger than that of $H_\lambda \mathcal{C}_1$. In fact, the sequences $\{S^n\}_{n=1}^{\infty}$, $\{CP^n\}_{n=1}^{\infty}$, and $\{SO(n)\}_{n=1}^{\infty}$ are all divergent with respect to the distance $\square_\lambda$ (see Proposition 3.1). This is related to the following exercise in Gromov’s book:

**Exercise** (cf. [1, Section 3.1.18]). Estimate the distance $\square_\lambda (S^n, S^m)$.

To solve the exercise, applying a method of [1, Lemma 5.10], we will estimate $\square_\lambda (M, N)$ from below for compact Riemannian manifolds $M$ and $N$ with positive Ricci curvatures and the volume measures satisfying a homogeneity condition (see Lemma 3.4). As a result, we get the following proposition:

**Proposition 1.1.** Assume that two sequences $\{n_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ of natural numbers satisfy $n_k \leq C_1 k$, $m_k \leq C_2 k$ and $|n_k - m_k| \geq C_3 k$, $k = 1, 2, \ldots$, for some positive constants $C_1, C_2, C_3$. Then, we have

$$\liminf_{k \to \infty} \square_\lambda (S^{n_k}, S^{m_k}), \liminf_{k \to \infty} \square_\lambda (CP^{n_k}, CP^{m_k}) \geq \min \left\{ \frac{1}{2}, \frac{C_3}{\sqrt{C_1 + C_2}} \right\}.$$ 

In particular, if in addition $|n_k - m_k| \geq C_4 k^\alpha$, $k = 1, 2, \ldots$, holds for some constant $C_4 > 0$ and a number $\alpha > 1$, then we have

$$\lim_{k \to \infty} \square_\lambda (S^{n_k}, S^{m_k}), \lim_{k \to \infty} \square_\lambda (CP^{n_k}, CP^{m_k}) = 1.$$ 

Note that $\text{diam}(X_1, \square_\lambda) = 1$, where $X_1$ is the space of all nn-spaces with Borel probability measures.

We estimate $\square_\lambda (SO(n), SO(m))$ from below by the difference of their diameters (see Lemma 5.8). Consequently, we obtain the following proposition:

**Proposition 1.2.** Assume that two sequences $\{n_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ of natural numbers satisfy $n_k \leq C_1 k$, $m_k \leq C_2 k$ and $|n_k - m_k| \geq C_3 k^\alpha$, $k = 1, 2, \ldots$, for some positive constants $C_1, C_2, C_3$. Then, we have

$$\liminf_{k \to \infty} \square_\lambda (SO(n_k), SO(m_k)) \geq \min \left\{ \frac{1}{2}, \frac{C_3}{\sqrt{C_1 + C_2}} \right\}.$$ 

In particular, if in addition $|n_k - m_k| \geq C_4 k^\alpha$, $k = 1, 2, \ldots$, holds for some constant $C_4 > 0$ and a number $\alpha > 1/2$, then we have

$$\liminf_{k \to \infty} \square_\lambda (SO(n_k), SO(m_k)) \geq \frac{1}{2}.$$ 

As it is related to the above Gromov’s exercise, we also prove the following proposition. This proposition is also mentioned by Gromov in [1, Section 3.1.3, Exercise (e)].

**Proposition 1.3.** We have

$$\square_\lambda (S^n, S^{n-1}), \square_\lambda (CP^n, CP^{n-1}) \to 0$$

as $n \to \infty$.

2. Preliminaries

2.1. Definition of Gromov’s box distance function $\square_\lambda$.

**Definition 2.1.** Let $\lambda \geq 0$ and $(X, \mu)$ be a measure space with $\mu(X) < +\infty$. For two maps $d_1, d_2 : X \times X \to \mathbb{R}$, we define a number $\square_\lambda (d_1, d_2)$ as the infimum of $\varepsilon > 0$ such that there exists a measurable subset $T_\varepsilon \subseteq X$ of measure at least $\mu(X) - \lambda \varepsilon$ satisfying $|d_1(x, y) - d_2(x, y)| \leq \varepsilon$ for any $x, y \in T_\varepsilon$. 

It is easy to see that this is a distance function on the set of all functions on $X \times X$, and the two distance functions $\Box_{\lambda}$ and $\Box_{\lambda'}$ are equivalent to each other for any $\lambda, \lambda' > 0$.

**Definition 2.2** (parameter). Let $X$ be an mm-space and $\mu(X) = m$. Then, there exists a Borel measurable map $\varphi : [0, m] \to X$ with $\varphi_*(\mathcal{L}) = \mu$, where $\mathcal{L}$ stands for the Lebesgue measure on $[0, m]$. We call $\varphi$ a parameter of $X$.

Note that if the support of $X$ is not one-point, then its parameter is not unique.

**Definition 2.3** (Gromov’s box distance function). If two mm-spaces $X, Y$ satisfy $\mu_X(X) = \mu_Y(Y) = m$, we define

$$\Box_{\lambda}(X, Y) := \inf \Box_{\lambda}(\varphi^*_X d_X, \varphi^*_Y d_Y),$$

where the infimum is taken over all parameters $\varphi_X : [0, m] \to X$, $\varphi_Y : [0, m] \to Y$, and $\varphi^*_X d_X$ is defined by $\varphi^*_X d_X(s, t) := dx(\varphi_X(s), \varphi_Y(t))$ for $s, t \in [0, m]$. If $\mu_X(X) < \mu_Y(Y)$, putting $m := \mu_X(X), m' := \mu_Y(Y)$, we define

$$\Box_{\lambda}(X, Y) := \Box_{\lambda} \left( X, \frac{m}{m'} Y \right) + m' - m,$$

where $(m/m')Y := (Y, d_Y, (m/m')\mu_Y)$.

We recall that two mm-spaces are isomorphic to each other if there is a measure preserving isometry between the supports of their measures. $\Box_{\lambda}$ is a distance function on $X$ for any $\lambda \geq 0$. See [2, Sections 1, 3] for a complete proof of that. Note that the distance functions $\Box_{\lambda}$ and $\Box_{\lambda'}$ are equivalent to each other for distinct $\lambda, \lambda' > 0$.

2.2. **Definition of observable distance functions** $H_{\lambda}\mathcal{L}_1$. For a measure space $(X, \mu)$ with $\mu(X) < +\infty$, we denote by $\mathcal{F}(X, \mathbb{R})$ the space of all functions on $X$. Given $\lambda \geq 0$ and $f, g \in \mathcal{F}(X, \mathbb{R})$, we put

$$\text{me}_\lambda(f, g) := \inf \{ \varepsilon > 0 : \mu(\{ x \in X \mid |f(x) - g(x)| \geq \varepsilon \}) \leq \lambda \varepsilon \}.$$ 

Note that this $\text{me}_\lambda$ is a distance function on $\mathcal{F}(X, \mathbb{R})$ for any $\lambda \geq 0$, and its topology on $\mathcal{F}(X, \mathbb{R})$ coincides with the topology of the convergence in measure for any $\lambda > 0$. Also, the distance functions $\text{me}_\lambda$ for all $\lambda > 0$ are mutually equivalent.

We recall that the Hausdorff distance between two closed subsets $A$ and $B$ in a metric space $X$ is defined by

$$d_H(A, B) := \inf \{ \varepsilon > 0 : \text{all } A \subseteq B_\varepsilon, B \subseteq A_\varepsilon \},$$

where $A_\varepsilon$ is a closed $\varepsilon$-neighborhood of $A$.

Let $(X, \mu)$ be a measure space with $\mu(X) < +\infty$. For a semi-distance $d$ on $X$, we indicate by $\text{Lip}_d$ the space of all 1-Lipschitz functions on $X$ with respect to $d$. Note that $\text{Lip}_d$ is a closed subset in $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$ for any $\lambda \geq 0$.

**Definition 2.4.** For $\lambda \geq 0$ and two semi-distance functions $d, d'$ on $X$, we define

$$H_{\lambda}\mathcal{L}_1(d, d') := d_H(\text{Lip}_d, \text{Lip}_{d'})$$

where $d_H$ stands for the Hausdorff distance function in $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$.

This $H_{\lambda}\mathcal{L}_1$ is actually a distance function on the space of all semi-distance functions on $X$ for all $\lambda \geq 0$, and the two distance functions $H_{\lambda}\mathcal{L}_1$ and $H_{\lambda'}\mathcal{L}_1$ are equivalent to each other for any $\lambda, \lambda' > 0$. 


Lemma 2.5. For any two semi-distance functions $d, d'$ on $X$, we have
\[ H_\lambda \mathcal{L}_1(d, d') \leq \square_\lambda(d, d'). \]

Proof. For any $\varepsilon > 0$ with $\square_\lambda(X, Y) < \varepsilon$, there exists a measurable subset $T_\varepsilon \subseteq X$ such that $\mu(X \setminus T_\varepsilon) \leq \lambda \varepsilon$ and $|d(x, y) - d'(x, y)| \leq \varepsilon$ for any $x, y \in T_\varepsilon$. Given arbitrary $f \in \text{Lip}_1(d)$, we define $\bar{f} \in \mathcal{F}(X, \mathbb{R})$ by $\bar{f}(x) := \inf\{f(y) + d'(x, y) \mid y \in T_\varepsilon\}$. We easily see that $\bar{f} \in \text{Lip}_1(d')$ and $\bar{f}(x) \leq f(x)$ for any $x \in T_\varepsilon$. Taking any $x \in T_\varepsilon$, we have
\[
|f(x) - \bar{f}(x)| = f(x) - \bar{f}(x) \\
= \sup\{f(x) - f(y) - d'(x, y) \mid y \in T_\varepsilon\} \\
\leq \sup\{d(x, y) - d'(x, y) \mid y \in T_\varepsilon\} \\
\leq \varepsilon.
\]
Therefore, we get $m_\lambda(f, \bar{f}) \leq \varepsilon$, which implies $\text{Lip}_1(d) \subseteq (\text{Lip}_1(d'))_\varepsilon$. Similarly, we also have $\text{Lip}_1(d') \subseteq (\text{Lip}_1(d))_\varepsilon$, which yields $H_\lambda \mathcal{L}_1(d, d') \leq \varepsilon$. This completes the proof. \qed

Definition 2.6 (Observable distance function). If two mm-spaces $X, Y$ satisfy $\mu_X(X) = \mu_Y(Y) = m$, we define
\[ H_\lambda \mathcal{L}_1(X, Y) := \inf H_\lambda \mathcal{L}_1(\varphi_X^*, \varphi_Y^*) \]
where the infimum is taken over all parameters $\varphi_X : [0, m] \to X$, $\varphi_Y : [0, m] \to Y$. If $\mu_X(X) < \mu_Y(Y)$, putting $m := \mu_X(X), m' := \mu_Y(Y)$, we define
\[ H_\lambda \mathcal{L}_1(X, Y) := H_\lambda \mathcal{L}_1 \left( X, \frac{m}{m'} Y \right) + m' - m. \]

Hence, $H_\lambda \mathcal{L}_1$ is a distance function on $\mathcal{X}$ for any $\lambda \geq 0$. See [2] Section 3] for a complete proof. Note that the distance functions $H_\lambda \mathcal{L}_1$ and $H_{\lambda'} \mathcal{L}_1$ are equivalent to each other for any $\lambda, \lambda' > 0$.

For a Borel measure $\nu$ on $\mathbb{R}$ with $m := \nu(\mathbb{R}) < +\infty$ and $\kappa > 0$, we put
\[ \text{diam}((\nu, m - \kappa)) := \inf\{\text{diam} Y \mid Y \subseteq \mathbb{R} \text{ is a Borel subset such that } \nu_Y(Y) \geq m - \kappa\}, \]
and call it the partial diameter on $\nu$.

Definition 2.7 (Observable diameter). Let $(X, d, \mu)$ be an mm-space and let $m := \mu(X)$. For any $\kappa > 0$ we define the observable diameter of $X$ by
\[ \text{diam}(X \bigcap \text{Lip}_1[1], m, m - \kappa) := \sup\{\text{diam}(f, \mu, m - \kappa) \mid f : X \to \mathbb{R} \text{ is a 1-Lipschitz function}\}. \]

The idea of the observable diameter comes from quantum and statistical mechanics; that is, we think of $\mu$ as a state on a configuration space $X$ and $f$ is interpreted as an observable. We define a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces as a Lévy family if $\text{diam}(X_n \bigcap \text{Lip}_1[1], m_n - \kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$, where $m_n$ is the total measure of the mm-space $X_n$. This is equivalent to the fact that for any $\varepsilon > 0$ and any sequence $\{f_n : X_n \to \mathbb{R}\}_{n=1}^\infty$ of 1-Lipschitz functions, we have
\[ (\Diamond) \quad \mu_n(\{x \in X_n \mid |f_n(x) - m_{f_n}| \geq \varepsilon\}) \to 0 \text{ as } n \to \infty, \]
where $m_{f_n}$ is some constant determined by $f_n$. 
For a compact connected Riemannian manifold $M$, we denote by $\mu_M$ the Riemannian volume measure of $M$ normalized as $\mu_M(M) = 1$ and by $d_M$ the Riemannian distance of $M$. We shall consider $M$ as an mm-space $(M, d_M, \mu_M)$.

**Example 2.8.** Let $(M_n)_{n=1}^\infty$ be a sequence of compact connected Riemannian manifolds and assume that $\text{Ric}_{M_n} \geq \kappa_n \to +\infty$ as $n \to \infty$. Then, by virtue of Lévy-Gromov’s isoperimetric inequality, the sequence $(M_n)_{n=1}^\infty$ is a Lévy family (cf. [3] Section 1, Remark 2). For example, $(S^n)_{n=1}^\infty$ and $(\mathbb{C}P^n)_{n=1}^\infty$ are Lévy families. Recall that the Fubini-Study metric on $\mathbb{C}P^n$ is the unique Riemannian metric on $\mathbb{C}P^n$ such that the canonical projection $S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion. Since $\text{Ric}_{SO(n)} \geq (n-1)/4$, the sequence $(SO(n))_{n=1}^\infty$ is a Lévy family. Since the distance function induced from the Hilbert-Schmidt norm on $SO(n)$ is not greater than that of the Riemannian distance function, $(SO(n))_{n=1}^\infty$ is also a Lévy family with respect to the Hilbert-Schmidt norms.

**Example 2.9** (Hamming cube). Let $\mu_n$ be the normalized counting measure on $\{0,1\}^n$ and $d_n$ be the Hamming distance function on $\{0,1\}^n$; that is,

$$d_n ((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \frac{1}{n} \text{Card} (\{ i \in \{1, \ldots, n\} \mid x_i \neq y_i\}).$$

The mm-space $\{0,1\}^n$ is called the Hamming cube. The sequence $\{\{0,1\}^n\}_{n=1}^\infty$ is a Lévy family (cf. [3] Section 3.4.42]).

Gromov showed the following proposition by considering a constant $m_\mu_n$ in $(\diamondsuit)$ as a Lipschitz function from a one-point space $\{\ast_n\}$ with total measure $\mu_n(X_n)$.

**Proposition 2.10** (cf. [3] Section 3.4.45]). A sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces is a Lévy family if and only if $H_\lambda \mathcal{L}_{\text{Lip}}(X_n, \{\ast_n\}) \to 0$ as $n \to \infty$ for any $\lambda > 0$.

3. Estimates of Gromov’s box distance function

Let $X$ be a metric space. Denote by $B_X(x, r)$ the closed ball in $X$ centered at $x \in X$ with radius $r > 0$. A Borel measure $\mu$ on $X$ is said to be uniformly distributed if

$$0 < \mu(B_X(x, r)) = \mu(B_X(y, r)) < +\infty$$

for any $r > 0$ and $x, y \in X$.

From Lemma 2.3, we see that the topology on $X$ determined by $\Box_\lambda$ is not weaker than that of $H_\lambda \mathcal{L}_{\text{Lip}}$ for any $\lambda \geq 0$. For a Borel measure $\mu$ on a metric space, we denote by $\text{Supp} \mu$ its support.

**Proposition 3.1.** Let $\{(X_n, d_n, \mu_n)\}_{n=1}^\infty$ be a Lévy family such that $\mu_n$ is a uniformly distributed probability measure satisfying $X_n = \text{Supp} \mu_n$ and $\inf \text{diam} X_n > 0$. Then, the sequence $\{X_n\}_{n=1}^\infty$ does not converge with respect to the distance function $\Box_\lambda$ for any $\lambda \geq 0$.

**Proof.** Suppose that $\{X_n\}_{n=1}^\infty$ converges and let $X$ be its limit. Since $\{X_n\}_{n=1}^\infty$ is a Lévy family, by using Proposition 2.10, $X$ must be a one-point space. Fix $\varepsilon > 0$ with $\varepsilon < \min \{3, \inf \text{diam} X_n\}/3$. For any sufficiently large $n \in \mathbb{N}$, there exist a parameter $\varphi_n : [0, 1] \to X_n$ of $X_n$ and a Borel subset $T_n \subseteq [0, 1]$ such that $\mathcal{L}(T_n) > 1 - \varepsilon/2$ and $d_n (\varphi_n(s), \varphi_n(t)) < \varepsilon/2$ for any $s, t \in T_n$. Fix a point $t_n \in T_n$. For a compact connected Riemannian manifold $M$, we denote by $\mu_M$ the Riemannian volume measure of $M$ normalized as $\mu_M(M) = 1$ and by $d_M$ the Riemannian distance of $M$. We shall consider $M$ as an mm-space $(M, d_M, \mu_M)$. For a compact connected Riemannian manifold $M$, we denote by $\mu_M$ the Riemannian volume measure of $M$ normalized as $\mu_M(M) = 1$ and by $d_M$ the Riemannian distance of $M$. We shall consider $M$ as an mm-space $(M, d_M, \mu_M)$. For a compact connected Riemannian manifold $M$, we denote by $\mu_M$ the Riemannian volume measure of $M$ normalized as $\mu_M(M) = 1$ and by $d_M$ the Riemannian distance of $M$. We shall consider $M$ as an mm-space $(M, d_M, \mu_M)$.
Lemma 3.2. □

There exists a point \( x_n \in X_n \) such that \( d_n(\varphi_n(t_n), x_n) \geq \text{diam } X_n/3 > \varepsilon \), and hence \( B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cap B_{X_n}(x_n, \varepsilon/2) = \emptyset \). Therefore, we get

\[
1 \geq \mu_n(B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cup B_{X_n}(x_n, \varepsilon/2)) \\
= 2\mu_n(B_{X_n}(\varphi_n(t_n), \varepsilon/2)) \\
= 2\mathcal{L}(\varphi_n^{-1}(B_{X_n}(\varphi_n(t_n), \varepsilon/2))) \geq 2\mathcal{L}(T_n) \geq 2 - \varepsilon > 1,
\]

which gives a contradiction. This completes the proof. □

From Proposition 3.1, we see that many Lévy families such as \( \{\mathbb{S}^n\}_{n=1}^\infty \), \( \{\mathbb{C}P^n\}_{n=1}^\infty \), \( \{SO(n)\}_{n=1}^\infty \), and \( \{\{0,1\}^n\}_{n=1}^\infty \) have no convergent subsequences with respect to the distance function \( d_n \). Therefore, the distance function \( d_n \) determines the topology on \( X \) strictly stronger than that of the distance function \( H_X \mathcal{L}_1 \) for any \( \lambda > 0 \). However, since the proof of Proposition 3.1 is by contradiction, we do not estimate \( \mathbb{L}_n(X_n, X_m) \) from below for \( n, m \in \mathbb{N} \).

The proof of the following lemma is an analogue of the proof of [11] Lemma 5.10.

**Lemma 3.2.** Let \( (X, d_X, \mu_X), (Y, d_Y, \mu_Y) \) be mm-spaces and assume that \( \mu_X, \mu_Y \) are uniformly distributed Borel probability measures. Denote by \( v_X(r) \) (respectively, \( v_Y(r) \)) the measure of a closed ball of \( X \) (respectively, \( Y \)) with radius \( r > 0 \) and assume that \( v_X(a + c) \leq (1 - c)v_Y(a/2) \) for some \( a, c > 0 \) with \( c < 1 \). Then, we have \( \mathbb{L}_n(X, Y) \geq c \).

**Proof.** Let us prove the lemma by contradiction. Suppose that \( \mathbb{L}_n(X, Y) < c \).

Then, there exist a compact subset \( T \subseteq [0,1] \) and two parameters \( \varphi_X : [0,1] \to X \), \( \varphi_Y : [0,1] \to Y \) such that

1. \( \mathcal{L}(T) > 1 - c \),
2. \( \varphi_X|_T : T \to X \), \( \varphi_Y|_T : T \to Y \) are continuous,
3. \( d_X(\varphi_X(s), \varphi_X(t)) - d_Y(\varphi_Y(s), \varphi_Y(t)) < c \) for any \( s, t \in T \).

By (1) and (2), \( \varphi_Y(T) \) is compact. Put

\[
l := \max\{k \in \mathbb{N} \mid \text{there exist points } p_i \in \varphi_Y(T), i = 1, \ldots, k, \text{ such that } B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset \text{ for any } i, j \text{ with } i \neq j \}.
\]

Then, there exist points \( p_i \in \varphi_Y(T), i = 1, \ldots, l \), such that \( B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset \) for any \( i, j \) with \( i \neq j \). Hence, we get

\[
1 \geq \mu_Y\left(\bigcup_{i=1}^l B_Y(p_i, a/2)\right) = \sum_{i=1}^l \mu_Y(B_Y(p_i, a/2)) = l \cdot v_Y(a/2).
\]

It also follows from the definition of \( l \) that \( \varphi_Y(T) \subseteq \bigcup_{i=1}^l B_Y(p_i, a) \). For any \( i = 1, \ldots, l \), we fix \( t_i \in T \) with \( p_i = \varphi_Y(t_i) \).

**Claim 3.3.**

\[
\varphi_X(T) \subseteq \bigcup_{i=1}^l B_X(\varphi_X(t_i), a + c).
\]
Proof: Take an arbitrary \( q = \varphi_X(s) \in \varphi_X(T) \), \( s \in T \). Since \( \varphi_Y(s) \in \varphi_Y(T) \subseteq \bigcup_{i=1}^l B_Y(p_i, a) \), there exists \( i \) with \( 1 \leq i \leq l \) such that \( d_Y(\varphi_Y(s), p_i) \leq a \). Therefore, by using (2), we obtain

\[
d_X(\varphi_X(s), \varphi_X(t_i)) < d_Y(\varphi_Y(s), p_i) + c \leq a + c.
\]

This completes the proof of the claim. \( \square \)

Applying Claim 3.3 we get

\[
1 \leq \sum_{i=1}^l \frac{\mu_X(B_X(\varphi_X(t_i), a + c))}{\mu_X(\varphi_X(T))} = l \cdot \frac{v_X(a + c)}{v_Y(a/2) \cdot \mu_X(\varphi_X(T))} \leq \frac{v_X(a + c)}{v_Y(a/2) \cdot \mu_X(\varphi_X(T))}.
\]

Since \( \mu_X(\varphi_X(T)) \geq L(\varphi_X^{-1}(\varphi_X(T))) \geq L(T) > 1 - c \), we obtain

\[
1 \leq \frac{v_X(a + c)}{v_Y(a/2) \cdot \mu_X(\varphi_X(T))} < \frac{v_X(a + c)}{v_Y(a/2) \cdot (1 - c)} \leq 1,
\]

which is a contradiction. This completes the proof of Lemma 3.2. \( \square \)

For a compact Riemannian manifold \( M \), we denote by \( \text{vol}(M) \) the total Riemannian volume of \( M \). We indicate by \( \Gamma \) the Gamma function.

**Lemma 3.4.** Let \( M \) (respectively, \( N \)) be an \( m \) (respectively, \( n \))-dimensional compact Riemannian manifold having a uniformly distributed Riemannian measure. Assume that \( \text{Ric}_M \geq (m - 1)\kappa_1 > 0 \) and \( \text{Ric}_N \geq 0 \), and put \( a_N := \text{vol}(N) / \text{vol}(\mathbb{S}^n) \). If a positive number \( c \) with \( c < 1 \) satisfies

\[
c^{n-m} \leq (1 - c)^{na_N(\kappa_1)^{m/2} \Gamma(\frac{m+1}{2}) \Gamma(\frac{m}{2}) \Gamma(\frac{n+1}{2})} \quad \text{and} \quad c \sqrt{\kappa_1} \leq \pi,
\]

then we have \( \square_1(M, N) \geq c \).

**Proof.** For \( r > 0 \), we put \( v_M(r) := \mu_M(B_M(x, r)) \) for \( x \in M \) and \( v_N(r) := \mu_N(B_N(y, r)) \) for \( y \in N \). From the Bishop-Gromov volume comparison theorem, we get

\[
v_M(c/2) = \frac{\text{vol}(B_M(x, c/2))}{\text{vol}(M)} \geq \frac{\text{vol}(\mathbb{S}^{m-1})}{\text{vol}(\mathbb{S}^m)} \int_0^{(c \sqrt{\kappa_1})/2} \sin^{m-1} \theta d\theta.
\]

From \( c \sqrt{\kappa_1} \leq \pi \), we have \( \sin \theta \geq (\pi \theta)/2 \) for any \( \theta \in [0, (c \sqrt{\kappa_1})/2] \). Hence, we obtain

\[
v_M(c/2) \geq \frac{2^{m-1} \text{vol}(\mathbb{S}^{m-1})}{\pi^{m-1} \text{vol}(\mathbb{S}^m)} \int_0^{(c \sqrt{\kappa_1})/2} \theta^{m-1} d\theta = \frac{c^m \kappa_1^{m/2} \text{vol}(\mathbb{S}^{m-1})}{2^{m-1} \pi^{m-1} \text{vol}(\mathbb{S}^m)}
\]

Let \( \kappa_2 \) be a positive number such that \( \text{Ric}_N \geq (n - 1)\kappa_2 \). We also obtain from the Bishop inequality that

\[
v_N(2c) \leq \frac{\text{vol}(\mathbb{S}^n)}{a_N(\kappa_2)^{n/2} \text{vol}(\mathbb{S}^n)} \cdot \frac{\text{vol}(\mathbb{S}^{n-1})}{a_N(\kappa_2)^{n/2} \text{vol}(\mathbb{S}^{n-1})} \int_0^{2c \sqrt{\kappa_2}} \sin^{n-1} \theta d\theta < \frac{(2c)^n \text{vol}(\mathbb{S}^{n-1})}{a_N \pi \text{vol}(\mathbb{S}^n)}.
\]

Recall that \( \text{vol}(\mathbb{S}^n) = 2 \pi^{(n+1)/2} / \Gamma((n + 1)/2) \). Therefore, combining the above calculations with Lemma 3.2 we complete the proof. \( \square \)
In the proof of Lemma 3.4, we use only the Bishop inequality and the Bishop-Gromov volume comparison theorem. Therefore, Lemma 3.4 also holds for general mmm-spaces satisfying the Bishop inequality and the Bishop-Gromov volume comparison theorem. Therefore, Lemma 3.4 by

\[
(1 - c) \frac{n_k \Gamma \left( \frac{m_k + 1}{2} \right) \Gamma \left( \frac{n_k}{2} \right)}{m_k 2^{m_k} \sqrt{\left( \frac{n_k}{2} \right)^{m_k}}} \geq (1 - c) 2^{-C_1 k} \pi^{-C_2 k + 1} \frac{n_k \Gamma \left( \frac{n_k}{2} \right)}{m_k 2^{m_k} \sqrt{\left( \frac{n_k}{2} \right)^{m_k}}}.\]

Therefore, if

\[
c \leq \left\{ (1 - c) \frac{n_k \Gamma \left( \frac{n_k}{2} \right)}{m_k 2^{m_k} \sqrt{\left( \frac{n_k}{2} \right)^{m_k}}} \right\} \frac{1}{\pi^{C_2 k - C_1 k + 1}},\]

then we obtain from Lemma 3.4 that \( \square \left(S^{n_k}, S^{m_k}\right) \geq c \). Since

\[
\left\{ (1 - c) \frac{n_k \Gamma \left( \frac{n_k}{2} \right)}{m_k 2^{m_k} \sqrt{\left( \frac{n_k}{2} \right)^{m_k}}} \right\} \frac{1}{\pi^{C_2 k - C_1 k + 1}} \to 1 \text{ as } k \to \infty,
\]
we have completed the proof for \( \{S^n\}_{n=1}^\infty \).

Next, consider \( \{CP^n\}_{n=1}^\infty \). It is well-known that \( vol(CP^n) = \pi^n/n! \) and the sectional curvature of \( CP^n \) is bounded from below by \( 1 \) (cf. [3, Section 3.D.2, 3.H.3]). Hence, we get

\[
a_{CP^n} = \frac{\Gamma \left( \frac{n + 1}{2} \right)}{2 \sqrt{\pi n!}}.
\]

For any \( 0 < c < 1 \), we have \( e^{2n_k - 2m_k} \leq e^{2C_5 k} \). Substituting \( n := 2n_k \) and \( m := 2m_k \), we calculate the right-hand side of the inequality of Lemma 3.4 by

\[
(1 - c) \frac{\Gamma \left( m_k + \frac{1}{2} \right)}{2 \sqrt{\pi m_k} 2^{\left( m_k + 1 \right) \frac{1}{2} - \left( m_k + 1 \right) \frac{1}{2}}} \geq (1 - c) \frac{1}{2 \sqrt{\pi C_2 k}} \cdot 2^{-2C_1 k - 1} \pi^{-2C_2 k + 1}.
\]

So, if

\[
c \leq \left\{ (1 - c) \frac{1}{2 \sqrt{\pi C_2 k}} \right\} \frac{1}{\pi^{C_2 k - C_1 k + 1}},\]

then we get by using Lemma 3.4 that \( \square \left(CP^{m_k}, CP^{n_k}\right) \geq c \). Since

\[
\left\{ (1 - c) \frac{1}{2 \sqrt{\pi C_2 k}} \right\} \frac{1}{\pi^{C_2 k - C_1 k + 1}} \to 1 \text{ as } k \to \infty,
\]
we complete the proof of the proposition. \( \Box \)

**Lemma 3.6** (J. Christensen, cf. [6, Section 3.3]). Let \( X \) be a metric space and \( \mu, \nu \) be uniformly distributed Borel measures on \( X \). Then, there exists a positive number \( c > 0 \) such that \( c \mu \leq c \nu \).

**Proof of Proposition 1.3**. We identify \( S^{n-1} \) with \( \{ (x_1, \cdots, x_n, 0) \in S^n \mid (x_1, \cdots, x_n) \in S^{n-1} \} \). Given an arbitrary \( \varepsilon > 0 \), since the sequence \( \{S^n\}_{n=1}^\infty \) is a Lévy family, we have \( r_n := \mu_n \left( \left( S^{n-1} \right)_\varepsilon \right) \to 1 \) as \( n \to \infty \). Hence, there is \( m \in \mathbb{N} \) such that \( 1 - r_n < \varepsilon \) for any \( n \geq m \). Suppose that \( n \geq m \). Taking two parameters \( \Phi_1 : [0, r_n] \to (S^{n-1})_\varepsilon \)
and \( \Phi_2 : (r_n, 1) \to S^n \setminus (S^{n-1})_\varepsilon \), we define a Borel measurable map \( \Phi : [0, 1] \to S^n \) by

\[
\Phi(t) := \begin{cases} 
\Phi_1(t), & t \in [0, r_n], \\
\Phi_2(t), & t \in (r_n, 1].
\end{cases}
\]

The map \( \Phi \) is a parameter of \( S^n \). Let \( \psi : S^n \setminus \{(0, \cdots, 0, 1), (0, \cdots, 0, -1)\} \to S^{n-1} \) be the projection; that is, \( \psi(x) \) is the unique element of \( S^{n-1} \) satisfying \( d_n(x, \psi(x)) = d_n(x, S^{n-1}) \). Put \( \varphi_1 := \psi \circ \Phi_1 : [0, r_n] \to S^{n-1} \).

Claim 3.7. \( \varphi_1^*(\mathcal{L}) = r_n \mu_{n-1} \).

Proof. Take any Borel subset \( A \subseteq S^{n-1} \). For any \( g \in SO(n-1) \), we have

\[
\varphi_1^*(\mathcal{L})(gA) = r_n \mu_n(\psi^{-1}(gA)) = r_n \mu_n(\psi^{-1}(A)) = \varphi_1^*(\mathcal{L})(A).
\]

Hence, \( \varphi_1^*(\mathcal{L}) \) is an \( SO(n-1) \)-invariant Borel measure. From Lemma 3.6 we complete the proof of the claim. \( \square \)

Taking a parameter \( \phi : (0, 1] \to S^{n-1} \) of \( S^{n-1} \), we define a Borel measurable map \( \varphi_2 : (r_n, 1) \to S^{n-1} \) by \( \varphi_2(t) := \phi((t - r_n)/(1 - r_n)) \). Then, we have \( \varphi_2^*(\mathcal{L}) = (1 - r_n)\mathcal{L} \). Therefore, defining a Borel measurable map \( \varphi : [0, 1] \to S^{n-1} \) by

\[
\varphi(t) := \begin{cases} 
\varphi_1(t), & t \in [0, r_n], \\
\varphi_2(t), & t \in (r_n, 1],
\end{cases}
\]

we see that the map \( \varphi \) is a parameter of \( S^{n-1} \). Since

\[
|dn(\Phi(s), \Phi(t)) - dn-1(\varphi(s), \varphi(t))| = |dn(\Phi_1(s), \Phi_1(t)) - dn-1(\varphi_1(s), \varphi_1(t))| \leq 2\varepsilon
\]

for any \( s, t \in [0, r_n] \), we get

\[
\square_1(S^n, S^{n-1}) \leq \square_1(\Phi^* d_n, \varphi^* d_{n-1}) \leq \max\{2\varepsilon, 1 - r_n\} = 2\varepsilon.
\]

Consequently, we obtain \( \square_1(S^n, S^{n-1}) \to 0 \) as \( n \to \infty \). A similar argument shows that \( \square_1(CP^n, CP^{n-1}) \to 0 \) as \( n \to \infty \). This completes the proof of Proposition 1.3 \( \square \)

Lemma 3.8. For any \( n, m \in \mathbb{N} \), we have

\[
\square_1(SO(n), SO(m)) \geq c(n, m) := \min\left\{ \frac{1}{2}, |\text{diam } SO(n) - \text{diam } SO(m)| \right\}.
\]

Proof. Suppose that \( n > m \) and \( \square_1(SO(n), SO(m)) < c(n, m) \). There exist a compact subset \( T \subseteq [0, 1] \) and two parameters \( \varphi_n : [0, 1] \to SO(n) \), \( \varphi_m : [0, 1] \to SO(m) \) such that

1. \( \mathcal{C}(T) > 1 - c(n, m) \geq 1/2 \),
2. \( \varphi_n|_T : T \to SO(n) \), \( \varphi_m|_T : T \to SO(m) \) are continuous,
3. \( |d_n(\varphi_n(s), \varphi_n(t)) - d_n(\varphi_m(s), \varphi_m(t))| < c(n, m) \) for any \( s, t \in T \).

Claim 3.9. There exist \( s_0, t_0 \in T \) such that \( d_n(\varphi_n(s_0), \varphi_n(t_0)) = \text{diam } SO(n) \).

Proof. Take \( A_0, B_0 \in SO(n) \) such that \( \text{diam } SO(n) = d_n(A_0, B_0) \) and define a map \( \psi : SO(n) \to SO(n) \) by \( \psi(A) := AA_0^{-1}B \). Then, \( \psi_*(\mu_n) = \mu_n \) and \( d_n(A, \psi(A)) = \).
diam \( SO(n) \) for any \( A \in SO(n) \). Suppose that \( d_n(\varphi_n(s), \varphi_n(t)) < \text{diam} \ SO(n) \) for any \( s, t \in T \). Then, we get \( \psi(\varphi_n(T)) \cap \varphi_n(T) = \emptyset \), which leads to
\[
\mu_n(\psi(\varphi_n(T)) \cap \varphi_n(T)) = \mu_n(\psi(\varphi_n(T))) + \mu_n(\varphi_n(T))
\]
\[
= \mu_n(\psi^{-1}(\psi(\varphi_n(T)))) + \mu_n(\varphi_n(T))
\]
\[
\geq 2\mu_n(\varphi_n(T)) > 1.
\]
This is a contradiction, and thus we complete the proof of the claim.

By Claim 3.9, we obtain
\[
\text{diam} \ SO(n) - \text{diam} \ SO(m) \leq |d_n(\varphi_n(s_0), \varphi_n(t_0)) - d_m(\varphi_m(s_0), \varphi_m(t_0))| < c(n, m),
\]
which is a contradiction. This completes the proof of Lemma 3.8.

Proof of Proposition 1.2. An easy calculation shows that
\[
2\sqrt{n} - 1 \leq \text{diam} \ SO(n) \leq 2\sqrt{n}.
\]
Therefore, supposing \( n_k \geq m_k \), we have
\[
\text{diam} \ SO(n_k) - \text{diam} \ SO(m_k) \geq 2\sqrt{n_k} - 1 - 2\sqrt{m_k}
\]
\[
= 2\frac{n_k - m_k - 1}{\sqrt{n_k - 1} + \sqrt{m_k}} \geq 2\frac{C_3 - 1/\sqrt{k}}{C_1 - 1/k + \sqrt{C_2}}.
\]
Thus, applying Lemma 3.8, we complete the proof.

ACKNOWLEDGEMENTS

The author would like to thank Professor Takashi Shioya and Mr. Masayoshi Watanabe for valuable discussions and many fruitful suggestions. He also thanks Professor Hajime Urakawa for his encouragement.

REFERENCES

2. K. Funano, A note for Gromov’s distance functions on the space of mm-spaces, available online at \texttt{http://front.math.ucdavis.edu/0706.2647}.

Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
E-mail address: sadm23@math.tohoku.ac.jp