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THE LIMITING SHAPE OF ONE-DIMENSIONAL TEICHMÜLLER SPACES

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Abstract. We show that the Bers embedding of the Teichmüller space of a once-punctured torus converges to the cardioid in the sense of Carathéodory up to rotation when the base torus goes to the boundary of its moduli space.

1. Main result

It is one of the great discoveries of L. Bers that the Teichmüller space of a (hyperbolic) Riemann surface can be realized as a bounded domain in a suitable complex Banach space through the Bers embedding. This method is, however, highly transcendental to visualize the domain even when the Teichmüller space is one-dimensional; see [3]. For instance, the fact that the Bers embedding of the one-dimensional Teichmüller space is a Jordan domain was finally proved by Minsky in his 1999 paper [10]. Miyachi [11] employed some techniques in [10] to show that one-dimensional Teichmüller spaces have inward pointing cusps and, in particular, are not quasidisks. Also, computer graphics of the Bers embeddings of one-dimensional Teichmüller spaces were recently given by [7] and [8]. The outer and inner radii of the Teichmüller space of a once-punctured square torus are given in a numerical way in [15] under some conjectures.

Though the internal geometry of the Teichmüller spaces is well understood, very little is known of the shape of the Bers embeddings of the Teichmüller spaces. In the present paper, we argue the limiting shape of the Bers embedding of the one-dimensional Teichmüller space when the base surface tends to the boundary of its moduli space. Our principal achievement can be presented somewhat loosely in the following form. (More precise formulations can be found in Theorems 1 and 2 below.)

Theorem. *The Bers embedding of a one-dimensional Teichmüller space converges in the sense of Carathéodory to a cardioid domain up to rotation when the base Riemann surface goes to the boundary of the moduli space.*

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Recall that a cardioid is a plane curve similar to

$$\{4e^{i\theta} - 2e^{2i\theta} : \theta \in \mathbb{R}\}.$$

A *cardioid domain* is a Jordan domain bounded by a cardioid.

The above result was announced in [16], which is an expository account and contains several computer graphics of Bers embeddings, but the proof is given only for a special case.

We briefly describe the organization of the present paper. In the next section, we set up the basic definitions in the Teichmüller theory and present a precise assertion of our main theorem (Theorem 1). Some preliminary results are given in Section 3. In Section 4, we present a more technical but useful assertion (Theorem 2) and deduce the main theorem from the assertion. Section 5 will be devoted to the proof of Theorem 2.

2. Precise formulation

Since we are concerned with the Bers embeddings of Teichmüller spaces in the present paper, the Teichmüller space will always be considered to be realized as the Bers embedding. For basic properties of Teichmüller spaces, we refer the reader to [4] or [12].

Let R be a fixed Riemann surface. In what follows, we will assume that R is hyperbolic, in other words, there exists a holomorphic universal covering projection p of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto R. The covering transformation group $\Gamma = \{\gamma \in \operatorname{Aut}(\mathbb{H}) : p \circ \gamma = p\}$ of $p : \mathbb{H} \to R$ is known to be a torsion-free Fuchsian group under the identification $\operatorname{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$. This Γ is called the Fuchsian model of R associated with p. Since the hyperbolic metric $\rho_{\mathbb{H}}(z)|dz| = |dz|/(2 \operatorname{Im} z)$ of $\mathbb{H}$ is invariant under the action of $\mathrm{PSL}(2, \mathbb{R})$, it descends to a metric $\rho_R(w)|dw|$ on R, which is again called the hyperbolic metric of R.

We denote by $B_2(R)$ the complex Banach space consisting of holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on R with finite norm $\|\varphi\|_R = \sup_{x \in R}(\rho_R^{-2}|\varphi|)(x)$. Here, we note that $\rho_R^{-2}|\varphi|$ can be regarded as a function on R. By the pullback $p^*\varphi = (\varphi \circ p)(p')^2$ by p, the space $B_2(R)$ is isometrically isomorphic to the closed subspace $B_2(\mathbb{H}, \Gamma) = \{\psi \in B_2(\mathbb{H}) : \forall \gamma \in \Gamma; (\psi \circ \gamma)(\gamma')^2 = \psi\}$ of $B_2(\mathbb{H})$.

Let R^* denote the mirror image of R and let $J_R : R \to R^*$ be the canonical reflection. Then $\hat{p}(z) = J_R(p(\bar{z}))$ is a holomorphic universal covering projection of the lower half-plane $\mathbb{H}^* = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ onto R^*. Note that the Fuchsian model Γ of R associated with p also acts on $\mathbb{H}^*$ and coincides with the covering transformation group of $\hat{p}$. We denote by $B_2(\mathbb{H}^*, \Gamma)$ the image of $B_2(R^*)$ under the pullback $\hat{p}^*$ by $\hat{p}$. Then the correspondence $\psi \mapsto \psi^*$ gives an isometric antilinear isomorphism of $B_2(\mathbb{H}, \Gamma)$ onto $B_2(\mathbb{H}^*, \Gamma)$, where $\psi^*(z) = \overline{\psi(\bar{z})}$.

Set $\operatorname{Belt}(\mathbb{H}) = \{\mu \in L^\infty(\mathbb{H}) : \|\mu\|_\infty < 1\}$. For $\mu \in \operatorname{Belt}(\mathbb{H})$, the measurable Riemann mapping theorem (cf. [1]) ensures the existence of a unique quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0, f(1) = 1$ such that the complex dilatation $\mu[f] = f_{\bar{z}}/f_z$ of f satisfies

$$\mu[f] = \begin{cases} \mu & \text{a.e. on } \mathbb{H}, \\ 0 & \text{a.e. on } \mathbb{H}^*. \end{cases}$$

We write f^μ for the above f and call it the μ-conformal map. Note that f^μ is conformal on $\mathbb{H}^*$. Now define $\Phi[\mu]$ as the Schwarzian derivative S_{f^μ} of f^μ on $\mathbb{H}^*$. Here, the Schwarzian derivative S_f of a non-constant holomorphic function f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

By the Kraus-Nehari theorem, the image of $\mathrm{Belt}(\mathbb{H})$ under Φ is contained in $\{\varphi \in B_2(\mathbb{H}^*) : \|\varphi\|_{\mathbb{H}^*} \leq 6\}$. Moreover, it is known that the map $\Phi : \mathrm{Belt}(\mathbb{H}) \to B_2(\mathbb{H}^*)$ is a holomorphic submersion (see [12]).

Let Γ be a Fuchsian model of R and set

$$\mathrm{Belt}(\mathbb{H}, \Gamma) = \{\mu \in \mathrm{Belt}(\mathbb{H}) : \forall \gamma \in \Gamma;\ (\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu\}.$$

Then $\Phi(\mathrm{Belt}(\mathbb{H}, \Gamma)) \subset B_2(\mathbb{H}^*, \Gamma)$. We now define the (Bers embedded) Teichmüller space $T(R)$ of R by

$$T(R) = \{\varphi \in B_2(R^*) : \exists \mu \in \mathrm{Belt}(\mathbb{H}, \Gamma); \hat{p}^*\varphi = \Phi[\mu]\} = (\hat{p}^*)^{-1}(\Phi(\mathrm{Belt}(\mathbb{H}, \Gamma))).$$

The set $T(R)$ is known to be a bounded, contractible domain in the complex Banach space $B_2(R^*)$.

Remark. It is a routine task to observe that $T(R)$ does not depend on p (and thus, on Γ), which was already mentioned in [15]. Traditionally, the set $\Phi(\mathrm{Belt}(\mathbb{H}, \Gamma))$ is referred to as the Bers embedding of the Teichmüller space of R. We adopt, however, the above definition because of this fact.

From now on, we assume that the (complex) dimension of $T(R)$ is one. It is well known that $\dim T(R) = 1$ for a hyperbolic Riemann surface R if and only if R is of finite analytic type (1,1) or (0,4). Since any Riemann surface of analytic type (0,4) has a commensurable surface of analytic type (1,1) in a canonical way, we may restrict ourselves on the case of type (1,1) (see [7]). Section 4.6 of [12] gives nice information about $T(R)$ for this particular case.

A Riemann surface R of finite analytic type (1,1) is usually called a once-punctured torus. For such a surface, there exists a $\tau \in \mathbb{H}$ such that R is conformally equivalent to

$$R_\tau = \mathbb{C}/L_\tau - \{[0]\} = (\mathbb{C} - L_\tau)/L_\tau,$$

where L_τ is the lattice generated by 1 and τ; in other words, $L_\tau = \{m + n\tau : m, n \in \mathbb{Z}\}$, and $[z]$ denotes the equivalence class of z modulo L_τ. The parameter τ is sometimes called the Teichmüller parameter, and the Teichmüller space of a once-punctured torus can be identified with $\mathbb{H}$ through this parameter. It is well known that R_τ goes to the boundary of the moduli space precisely when the modular j-invariant $j(\tau)$ of τ tends to ∞. The differential $\mathrm{d}z^2$ on $\mathbb{C} - L_\tau$ descends to a holomorphic quadratic differential on R_τ, which is still denoted by $\mathrm{d}z^2$. The normalized differential

$$\varphi_\tau = (\|\mathrm{d}z^2\|_{R_\tau})^{-1}\mathrm{d}z^2$$

will be called the standard differential of R_τ. Note that these definitions make sense for $\tau \in \mathbb{H}^*$, as well, except for $j(\tau)$, which will not be used below.

The mirror image R_τ^* of R_τ is naturally identified with $R_{\bar{\tau}}$. As in the case of the upper and lower half-planes, we set $\varphi_\tau^* := \varphi_{\bar{\tau}} \in B_2(R_{\bar{\tau}}) = B_2(R_\tau^*)$.

We define the simply connected domain U_τ in $\mathbb{C}$ by

$$U_\tau = \{w \in \mathbb{C} : w\varphi_\tau^* \in T(R_\tau)\},$$

and we denote by w_0U_τ the set $\{w_0w : w \in U_\tau\}$. Then we may write $T(R_\tau) = U_\tau \cdot \varphi_\tau^*$.

We now recall the definition of the kernel convergence in the sense of Carathéodory. Let $a \in \mathbb{C}$ and let $\{D_n\}_{n=1}^\infty$ be a sequence of hyperbolic subdomains of $\mathbb{C}$ with $a \in D_n$. The kernel of the sequence with respect to a is the connected component of $W = \bigcup_{n=1}^\infty \operatorname{Int}\,(\bigcap_{k=n}^\infty D_k)$ containing a. Here, we define the kernel to be the empty set when $a \notin W$. We say that $\{D_n\}_{n=1}^\infty$ is *kernel convergent to D with respect to a* if D is the kernel of every subsequence of $\{D_n\}_{n=1}^\infty$. Let $f_n : \mathbb{D} \to D_n$ be the holomorphic universal covering projection of $\mathbb{D}$ onto D_n determined by $f_n(0) = a$ and $f_n'(0) > 0$. Hejhal [5] generalized the Carathéodory kernel convergence theorem by showing that $\{D_n\}_{n=1}^\infty$ is kernel convergent to a hyperbolic domain D if and only if $\{f_n\}_{n=1}^\infty$ converges to a non-constant map f uniformly on each compact subset of $\mathbb{D}$. Furthermore, in this case, f is the holomorphic universal covering projection of $\mathbb{D}$ onto D with $f(0) = a$ and $f'(0) > 0$. In particular, we see that the hyperbolic density ρ_{D_n} converges to ρ_D uniformly on each compact subset of D.

Let C be the domain bounded by the cardioid $\{4e^{i\theta} - 2e^{2i\theta} : \theta \in \mathbb{R}\}$. Observe that the function

$$g_0(t) = 4t - 2t^2$$

maps the unit disk conformally onto C and satisfies $g_0(0) = 0$ and $g_0'(0) = 4 > 0$. We are now ready to state our main result in a precise form.

Theorem 1. *Let τ_n be a sequence in the upper half-plane with $j(\tau_n) \to \infty$ as $n \to \infty$. Then there exists a sequence of real numbers θ_n such that $e^{i\theta_n}U_{\tau_n}$ is kernel convergent to the cardioid domain C with respect to the origin as $n \to \infty$. Furthermore, if $w_nU_{\tau_n}$ is kernel convergent to a proper subdomain of $\mathbb{C}$ with respect to the origin for a sequence of complex numbers w_n, then $e^{-i\theta_n}w_n$ tends to a non-zero complex number as $n \to \infty$.*

As we will observe below, when $\operatorname{Im}\tau_n \to +\infty$ the angles θ_n can be chosen to be 0.

3. Proof of Theorem 1

Let $\tau, \tau' \in \mathbb{H} = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$. It is well known that R_τ is conformally equivalent to $R_{\tau'}$ if and only if $\tau' = \gamma(\tau)$ for a $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$. For a given $\tau \in \mathbb{H}$, let $\tau' = \gamma(\tau)$, where $\gamma(\zeta) = (a\zeta + b)/(c\zeta + d)$ for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, and consider the function $h(z) = z/(c\bar{\tau} + d)$. Since $h(a\bar{\tau} + b) = \overline{\tau'}$ and $h(c\bar{\tau} + d) = 1$, the function h maps the lattice $L_{\bar{\tau}}$ onto $L_{\overline{\tau'}}$ and thus induces a conformal map $H : R_\tau^* \to R_{\tau'}^*$. The map H gives a conformal map $H^* : T(R_{\tau'}) \to T(R_\tau)$ by the pullback $H^*\varphi = \varphi \circ H \cdot (H')^2$ and therefore a conformal map $H^\star : U_{\tau'} \to U_\tau$ so that $H^\star(w)\varphi_\tau^* = H^*(w\varphi_{\tau'}^*) = wH^*(\varphi_{\tau'}^*)$. Since $h^*(\mathrm{d}z^2) = (\mathrm{d}(h(z))^2 = (c\bar{\tau} + d)^{-2}\mathrm{d}z^2 = \gamma'(\bar{\tau})\mathrm{d}z^2$ and since $H^* : B_2(R_{\tau'}^*) \to B_2(R_\tau^*)$ is a linear isometry, the relation $H^*\varphi_{\tau'}^* = e^{-i\sigma}\varphi_\tau^*$ holds, where $\sigma = \arg\gamma'(\tau)$. Hence, $H^\star(w) = e^{-i\sigma}w$ and

$$U_{\tau'} = e^{i\sigma}U_\tau, \quad \text{where } \sigma = \arg\gamma'(\tau). \tag{3.1}$$

It is a standard fact that the set

$$W = \{\zeta \in \mathbb{H} : |\operatorname{Re}\zeta| \le 1/2, |\zeta| \ge 1\}$$

is a (closed) fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$. The above argument enables us to assume that $\tau_n \in W$ in order to prove Theorem 1. In fact, the theorem follows from the following, slightly stronger assertion.

Theorem 2. *Let τ_n be a sequence of points in $\mathbb{H}$ with $\operatorname{Im}\tau_n \to +\infty$ as $n \to \infty$ and let $g_n : \mathbb{D} \to U_{\tau_n}$ be the conformal homeomorphism determined by $g_n(0) = 0$ and $g_n'(0) > 0$. Then the function $g_n(t)$ converges to $g_0(t) = 4t - 2t^2$ locally uniformly in $|t| < 1$ as $n \to \infty$.*

We postpone the proof of this theorem to Section 5. We remark that the author treats the special case when $\operatorname{Re}\tau_n = 0$ and gives the same assertion in [16].

Let us now deduce Theorem 1 from this theorem. As we saw above, we may assume that $\tau_n \in W$. The assumption $j(\tau_n) \to \infty$ now implies $\operatorname{Im}\tau_n \to +\infty$. We then apply Theorem 2 to see that g_n converges to g_0 locally uniformly on $\mathbb{D}$. The Carathéodory kernel convergence theorem tells us that $g_n(\mathbb{D}) = U_{\tau_n}$ is kernel convergent to $g_0(\mathbb{D}) = C$ with respect to the origin.

We next suppose that $w_n U_{\tau_n}$ is kernel convergent to a proper subdomain D of $\mathbb{C}$ with respect to the origin for a sequence of complex numbers w_n. Since the function $h_n(t) = w_n g_n(\bar{w}_n t/|w_n|)$ maps the unit disk univalently onto $w_n U_{\tau_n}$ in such a way that $h_n(0) = 0$ and $h_n'(0) = |w_n| g_n'(0) > 0$, the Carathéodory theorem again implies that h_n locally uniformly converges to the conformal map h of $\mathbb{D}$ onto D with $h(0) = 0$ and $h'(0) > 0$. In particular, $|w_n| = h_n'(0)/g_n'(0) \to h'(0)/g_0'(0) =: r$ as $n \to \infty$.

If $\{w_n\}$ were not convergent, then there would be at least two limit points $r\mathrm{e}^{\mathrm{i}\theta}$ and $r\mathrm{e}^{\mathrm{i}\psi}$ of the sequence $\{w_n\}$ for $\theta, \psi \in \mathbb{R}$ with $\theta - \psi \notin 2\pi\mathbb{Z}$. Then $h(t) = r\mathrm{e}^{\mathrm{i}\theta} g_0(\mathrm{e}^{-\mathrm{i}\theta} t) = r\mathrm{e}^{\mathrm{i}\psi} g_0(\mathrm{e}^{-\mathrm{i}\psi} t)$ and thus $\mathrm{e}^{\mathrm{i}\sigma} g_0(\mathrm{e}^{-\mathrm{i}\sigma} t) = g_0(t)$ would hold for $|t| < 1$ and $\sigma = \theta - \sigma$. This is, however, impossible. Thus the proof is complete.

Remark. The assertion similar to Theorem 2 is not valid when τ_n tends to a finite real number even non-tangentially. Indeed, for a rational number $x = q/p$, where p and q are coprime integers with $p \neq 0$, we can choose integers r and s so that $ps + qr = 1$. Then the transformation $\gamma(z) = (rz+s)/(-pz+q)$ belongs to $\mathrm{PSL}(2, \mathbb{Z})$ and sends x to ∞.

For a fixed number $\theta \in (0, \pi)$, let τ_n be a sequence in $\mathbb{H}$ approaching x in such a way that $\arg(\tau_n - x) = \theta$. Since $\arg \gamma'(\tau_n) = -2\arg(p\tau_n - q) = -2\theta$, the relation (3.1) yields $U_{\tau_n} = \mathrm{e}^{2\mathrm{i}\theta} U_{\tau_n'}$, where $\tau_n' = \gamma(\tau_n)$. Noting that $\operatorname{Im}\tau_n' \to +\infty$, we apply Theorem 2 to deduce that $U_{\tau_n'} \to C$ as $n \to \infty$. Hence, $U_{\tau_n} \to \mathrm{e}^{2\mathrm{i}\theta} C$, which means that the limit domain depends on the direction of convergence to x.

Moreover, a worse phenomenon may occur when τ approaches a boundary point of $\mathbb{H}$ tangentially. For instance, let $\tau_n = \tau_0 + n$ for a fixed $\tau_0 \in \mathbb{H}$. Then $\tau_n \to \infty$ tangentially as $n \to \pm\infty$. On the other hand, $R_{\tau_n} = R_{\tau_0}$ by definition. Therefore, recalling the remark in Section 2, we conclude that $U_{\tau_n} = U_{\tau_0}$ for each integer n. Therefore, U_{τ_n} does not converge to any cardioid domain as $n \to \infty$. Note that $j(\tau_n)(= j(\tau_0))$ does not converge to ∞ in this case.

4. Preliminary lemmas

The present section is devoted to some lemmas which will be used in the proof of Theorem 2.

Let

$$\psi_0(\zeta) = \frac{1}{4\zeta^2} \quad \text{and} \quad \mu_0(\zeta) = \frac{|\psi_0(\zeta)|}{\psi_0(\zeta)} = \frac{\zeta}{\bar{\zeta}} \quad \text{for } \zeta \in \mathbb{H}.$$

Then $\|\psi_0\|_{\mathbb{H}} = 1$, and its reflection is given by $\psi_0^*(\zeta) = 1/(4\zeta^2)$ for $\zeta \in \mathbb{H}^*$. The following lemma is essentially due to Kalme [6].

Lemma 3. *$\Phi[-t\mu_0] = g_0(t)\psi_0^*$ for $|t| < 1$, where $g_0(t) = 4t - 2t^2$.*

Proof. We define the map $f_t : \mathbb{C} \to \mathbb{C}$ for $t \in \mathbb{D}$ by

$$f_t(z) = \begin{cases} z\bar{z}^{-t}, & z \in \overline{\mathbb{H}}, \\ z^{1-t}, & z \in \mathbb{H}^*. \end{cases}$$

By definition, f_t is analytic on $\mathbb{H}^*$ and satisfies $f_t(0) = 0$ and $f_t(1) = 1$. A straightforward computation further gives $\mu[f_t] = -t\mu_0$ on $\mathbb{H}$ and $S_{f_t} = g_0(t)\psi_0^*$ on $\mathbb{H}^*$. One can also check that f_t is univalent on $\mathbb{H}^*$ (see [14]). Therefore f_t is quasiconformal and thus $f_t = f^{-t\mu_0}$. We now conclude that $\Phi[-t\mu_0] = S_{f_t} = g_0(t)\psi_0^*$. □

Let $L = L(\omega_1, \omega_2)$ be the lattice generated by complex numbers ω_1 and ω_2, which are linearly independent on $\mathbb{R}$, and consider the once-punctured torus $R = \mathbb{C}/L - \{\pi(0)\} = (\mathbb{C} - L)/L$. Here, we denote by π the canonical projection $\mathbb{C} \to \mathbb{C}/L$. The images of the half-periods form the set A of Weierstrass points of R. It is known that every simple closed geodesic of R passes through exactly two Weierstrass points (see [9]). Since A consists of three points, any pair of simple closed geodesics of R intersect at a Weierstrass point. In the following, we need more detailed information about the intersection point.

Define the simple closed curves α_0 and β_0 on R by $\alpha_0(s) = \pi(s\omega_1 + z_0)$ and $\beta_0(s) = \pi(s\omega_2 + z_0)$ for $0 \le s \le 1$, respectively, where $z_0 = (\omega_1 + \omega_2)/2$. Let α and β be the (oriented) hyperbolic geodesics on R which are freely homotopic to α_0 and β_0 in R, respectively. Since α_0 and β_0 intersect only at the point $\pi(z_0)$ transversally, the intersection of α and β consists of only one point. That point is nothing but $\pi(z_0)$ by the following lemma.

Lemma 4. *The half-period $(\omega_1 + \omega_2)/2$ projects on the intersection point of α and β.*

Proof. Let w_0 be the intersection point of α and β. The mapping $z \mapsto -z$ induces an elliptic involution $h : R \to R$. Note that the fixed points of h consist of $\pi(\omega_1/2)$, $\pi(\omega_2/2)$ and $\pi(z_0)$. Since $h_*\alpha_0$ and $h_*\beta_0$ are equal to α_0^{-1} and β_0^{-1}, respectively, one has the relations $h(|\alpha|) = |\alpha|$ and $h(|\beta|) = |\beta|$. Here, $|\gamma|$ denotes the image of a curve γ. In particular, we have $h(|\alpha| \cap |\beta|) = |\alpha| \cap |\beta|$, which means that h fixes the intersection point w_0 of α and β. Therefore, w_0 is one of the points $\pi(\omega_1/2)$, $\pi(\omega_2/2)$ and $\pi(z_0)$.

To prove the assertion, we may restrict ourselves on the case when $(\omega_1, \omega_2) = (\tau, 1)$ for some $\tau \in \mathbb{H}$. We write $\pi_\tau(z)$ for $\pi(z)$ and $w_0(\tau)$ for w_0 to indicate the parameter τ. Obviously $w_0(\tau)$ is continuous in $\tau \in \mathbb{H}$. On the other hand, the points $\pi_\tau(\omega_1/2)$, $\pi_\tau(\omega_2/2)$ and $\pi_\tau((\omega_1 + \omega_2)/2)$ do not pairwise collide and $w_0(\mathrm{i}) = \pi_{\mathrm{i}}((\mathrm{i} + 1)/2)$ by obvious symmetry. The continuity of $w_0(\tau)$ in τ now implies that $w_0(\tau) = \pi_\tau((\tau + 1)/2)$ for all $\tau \in \mathbb{H}$. Thus the proof is complete. □

5. Proof of Theorem 2

First, we give an outline of the proof. Let ψ_n be a lift of the standard differential φ_{τ_n} by a universal covering projection $q_n : \mathbb{H} \to R_{\tau_n}$. The conformal map of $\mathbb{D}$ onto $T(R_{\tau_n})$ is given by $\Phi(-t\mu_n)$, $|t| < 1$, by the Teichmüller theorem, where $\mu_n = |\psi_n|/\psi_n$. By a suitable choice of q_n, we will easily see that $\psi_n(\zeta)$ converges to c/ζ^2 locally uniformly on $\mathbb{H}$ for a constant c (at least after passing to a subsequence). If we had $c \neq 0$, then we would have $\Phi(-t\mu_n) \to \Phi(-t\mu_0) = g_0(t)\psi_0^*$ (recall

Lemma 3). The most difficult part of the proof is to guarantee $c \neq 0$. We thus need a suitable choice of q_n and a careful analysis of it.

Let us start with the choice of q_n. Suppose that we are given a sequence of points $\tau_n = \xi_n + \mathrm{i}\eta_n$ in $\mathbb{H}$ with $\eta_n \to +\infty$ as $n \to \infty$. Consider the modified lattice $\Lambda_n = \eta_n^{-1} L_{\tau_n} = \{(m + n\tau_n)/\eta_n : m, n \in \mathbb{Z}\}$ and its complement $\Omega_n = \mathbb{C} - \Lambda_n$. We denote by $\pi_n : \Omega_n \to R_{\tau_n}$ the mapping $z \mapsto \eta_n z$ followed by the canonical projection $\mathbb{C} - L_{\tau_n} \to R_{\tau_n}$. Let α_n and β_n be the (oriented) hyperbolic geodesics of R_{τ_n} constructed in the previous section for $(\omega_1, \omega_2) = (\tau_n, 1)$. Lemma 4 means that the intersection of the curves α_n and β_n consists of one point $w_n = \pi_n(z_n)$, where $z_n = (\tau_n + 1)/2\eta_n$. We parametrize α_n and β_n so that w_n is the common initial point of them.

Let p_n be a holomorphic universal covering projection of the upper half-plane $\mathbb{H}$ onto Ω_n with $p_n(\mathrm{i}) = z_n$. Then $q_n = \pi_n \circ p_n$ is a holomorphic universal covering projection of $\mathbb{H}$ onto R_{τ_n}. Let Γ_n be the covering transformation group of q_n, that is, $\Gamma_n = \{\gamma \in \mathrm{PSL}(2, \mathbb{R}) : q_n \circ \gamma = q_n\}$. We denote by $\hat{\alpha}_n$ and $\hat{\beta}_n$ the lifts of α_n and β_n via the covering map q_n with initial point i. Then the terminal points of them can be expressed as $A_n(\mathrm{i})$ and $B_n(\mathrm{i})$ for unique elements A_n and B_n of Γ_n. Note that A_n and B_n form a set of free generators of Γ_n. By composing a rotation of $\mathbb{H}$ about i with p_n, we can assume that the attracting fixed point of A_n is ∞. Since the axis of A_n passes through i, the repelling fixed point of A_n must be 0, and thus A_n has the form $A_n(z) = M_n z$ for a constant $M_n > 1$. Note that $(1/2)\log M_n$ is the hyperbolic length of α_n in R_{τ_n}. By the choice of A_n, we have the identity

$$p_n(M_n\zeta) = p_n(\zeta) + \frac{1}{\eta_n}. \tag{5.1}$$

Let $\rho_n(z)|\mathrm{d}z|$ denote the hyperbolic metric of the domain Ω_n and $\psi_n(\zeta)\mathrm{d}\zeta^2$ be the pullback of the standard differential φ_{τ_n} by the covering projection q_n. Note that $\|\psi_n\|_{\mathbb{H}} = \|\varphi_{\tau_n}\|_{R_{\tau_n}} = 1$. We now summarize facts about these particular universal projections.

Lemma 5. *As $n \to \infty$,*

(i) $M_n \to 1$,
(ii) $p_n(\zeta) - z_n + \mathrm{i}/2 \to (1/\pi)\log\zeta$ *locally uniformly on* $\operatorname{Im}\zeta > 0$,
(iii) $\rho_n(z) \to \dfrac{\pi}{2\sin(\pi \operatorname{Im} z)}$ *locally uniformly on* $0 < \operatorname{Im} z < 1$, *and*
(iv) $\psi_n(\mathrm{i}) \to -1/4$.

Proof. We first collect necessary information about the hyperbolic metric. We denote by $\lambda_{a,b}(z)|\mathrm{d}z|$ the hyperbolic metric of the twice-punctured plane $\mathbb{C} - \{a, b\}$ and set $\delta_n(z) = \min_{a \in \Lambda_n} |z - a|$. For any pair of distinct points $a, b \in \Lambda_n$, we have $\mathbb{D}(z, \delta_n(z)) \subset \Omega_n \subset \mathbb{C} - \{a, b\}$, where $\mathbb{D}(z, r) = \{w : |w - z| < r\}$. The domain monotonicity of the hyperbolic metric thus yields the double inequality

$$\lambda_{a,b}(z) \leq \rho_n(z) \leq \rho_{\mathbb{D}(z,\delta_n(z))}(z) = \frac{1}{\delta_n(z)}. \tag{5.2}$$

It is known (see [2]) that there is a universal constant $K > 0$ such that

$$\frac{K}{\left|\log\frac{|z-a|}{|b-a|}\right| + 1} \leq |z - a|\lambda_{a,b}(z), \quad z \in \mathbb{C} - \{a, b\}. \tag{5.3}$$

By (5.2), we have $\rho_n(s+\mathrm{i}/2) \le 2$ for $s \in \mathbb{R}$. Thus,

$$\frac{1}{2}\log M_n = \int_{\alpha_n} \rho_{R_{\tau_n}}(z)|\mathrm{d}z| \le \int_{\alpha_{n,0}} \rho_{R_{\tau_n}}(z)|\mathrm{d}z| = \int_0^{1/\eta_n} \rho_n(s+\mathrm{i}/2)\mathrm{d}s \le \frac{2}{\eta_n},$$

where $\alpha_{n,0}(s) = \pi_n(z_n + s/\eta_n),\ 0 \le s \le 1$. By assumption, $\eta_n \to \infty$ and hence $M_n \to 1$ as $n \to \infty$.

As $n \to \infty$, the lattice Λ_n tends to the set $\{x + \mathrm{i}n : x \in \mathbb{R}, n \in \mathbb{Z}\}$ in the sense of Hausdorff. Therefore, Ω_n is kernel convergent to the parallel strip $\Sigma = \{z : 0 < \operatorname{Im} z < 1\}$ with respect to $\mathrm{i}/2$. As is explained in Section 2, a theorem of Hejhal [5] implies that $\tilde{p}_n = p_n - z_n + \mathrm{i}/2$ converges to a conformal mapping p of $\mathbb{H}$ onto Σ locally uniformly. It is easy to see that $(\tilde{p}_n(\zeta_n) - \tilde{p}_n(\zeta))/(\zeta_n - \zeta) \to p'(\zeta)$ whenever $\zeta_n \to \zeta$ as $n \to \infty$. On the other hand, by (5.1), we have

$$\frac{\tilde{p}_n(M_n\mathrm{i}) - \tilde{p}_n(\mathrm{i})}{M_n\mathrm{i} - \mathrm{i}} = \frac{1}{\eta_n(M_n - 1)\mathrm{i}}.$$

Thus, $1/\eta_n(M_n - 1) \to \mathrm{i}p'(\mathrm{i})$. In particular, we see that $\mathrm{i}p'(\mathrm{i})$ is a positive number, which together with $p(\mathrm{i}) = \mathrm{i}/2$ implies $p(\zeta) = (\log\zeta)/\pi$. We note that we have $\eta_n(M_n - 1) \to \pi$ as a by-product. Since $\Omega_n + z_n - \mathrm{i}/2 = \Omega_n + \varepsilon_n$, where $\varepsilon_n \in [0, 1/\eta_n)$ with $\xi_n/2 - \varepsilon_n\eta_n \in \mathbb{Z}$, Ω_n is also kernel convergent to Σ with respect to $\mathrm{i}/2$. Thus, we see that $\rho_n(z)$ converges to $\rho_\Sigma(z) = \pi/2\sin(\pi \operatorname{Im} z)$ locally uniformly in Σ.

We now show (iv). Set $\tilde{\psi}_n = p_n^*(\mathrm{d}z^2) = (p_n')^2$. Then, $\|\tilde{\psi}_n\|_{\mathbb{H}} = \|\mathrm{d}z^2\|_{\Omega_n} = 1/m_n^2$, where $m_n = \inf_{z\in\Omega_n} \rho_n(z)$. Therefore, we have the relation $\psi_n = \tilde{\psi}_n/\|\tilde{\psi}_n\|_{\mathbb{H}} = (m_n p_n')^2$. Since $\{x + \mathrm{i}y : 0 \le x < \eta_n, 0 \le y < 1\} - \{0\}$ is a fundamental set for the lattice Λ_n and since $\rho_n(\tau_n/\eta_n - z) = \rho_n(z)$ for $z \in \Omega_n$, the infimum (indeed the minimum) is attained in the set $Q_n = \{x + \mathrm{i}y : 0 \le x < 1/\eta_n, 0 \le y \le 1/2\} - \{0\}$. Let w_n be a point in Q_n with $m_n = \rho_n(w_n)$. As we saw, $\rho_n(\mathrm{i}/2) \le 2$, thus $m_n \le 2$.

Choose a sufficiently small number $\delta \in (0, 1/2)$ so that the inequality

$$x(-\log x + 1) \le K/8$$

holds for $0 < x \le 2\delta$. Then $\rho_n(z) \ge 4$ for $z \in Q_n$ with $\operatorname{Im} z \le \delta$, provided that $1/\eta_n < \delta$. Indeed, applying (5.2) and (5.3) to the choice $a, b \in \Lambda_n$ such that $|a - \operatorname{Re} z| \le 1/2\eta_n$, $\operatorname{Im} a = 0$, $\operatorname{Im} b = 1$ and $0 \le \operatorname{Re} b \le 1/\eta_n$ yields the inequality

$$\rho_n(z) \ge \frac{K}{|z-a|(\left|\log|(z-a)/b|\right| + 1)} = \frac{K/|b|}{|(z-a)/b|(\left|\log|(z-a)/b|\right| + 1)}.$$

Since $|(z-a)/b| \le |z-a| \le |\operatorname{Re} z - a| + \operatorname{Im} z \le 2\delta (< 1)$, by the choice of δ, we have $\rho_n(z) \ge 8/|b| > 4$.

Therefore, for a large enough n, the minimum of ρ_n is attained in the closed parallel strip $\delta \le \operatorname{Im} z \le 1 - \delta$, where $\rho_n(z)$ converges to $\pi/2\sin(\pi \operatorname{Im} z)$ uniformly. Thus $w_n \to \mathrm{i}/2$ and $m_n = \rho_n(w_n) \to \pi/2$ as $n \to \infty$. In view of the convergence $p_n'(\zeta) \to 1/\pi\zeta$ by (ii), we finally obtain $\psi_n(\mathrm{i}) = (m_n p_n'(\mathrm{i}))^2 \to -1/4$ as $n \to \infty$. $\square$

The following simple fact was effectively used by Nakanishi [13, Proposition 3.1]. For convenience of the reader, we also give a proof.

Lemma 6. *Let Γ_n be a sequence of Fuchsian groups acting on $\mathbb{H}$, each of which contains a hyperbolic element of the form $\zeta \mapsto M_n\zeta$ such that $M_n \to 1$ as $n \to \infty$. Further, let φ_n be an element of $B_2(\mathbb{H}, \Gamma_n)$ such that φ_n converges locally uniformly to a holomorphic function φ_∞ on $\mathbb{H}$. Then $\varphi_\infty(\zeta) = c/\zeta^2$ for some constant c.*

Proof. Let $\varphi_n(\zeta) = P_n(\zeta)/\zeta^2$ and $\varphi_\infty(\zeta) = P_\infty(\zeta)/\zeta^2$. Then $P_n \to P_\infty$ locally uniformly. Since $\varphi_n(M_n\zeta)M_n^2 = \varphi_n(\zeta)$, the relation $P_n(M_n^k \mathrm{i}) = P_n(\mathrm{i})$ holds for any $k \in \mathbb{Z}$. Since the set $\{M_n^k : k \in \mathbb{Z}\}$ tends to $\mathbb{R}_+$ as $n \to \infty$ in the sense of Hausdorff, we see that P_∞ is constant along the positive imaginary axis, which implies that P_∞ is constant by the identity theorem. □

By employing the lemma, we are now able to show the following.

Lemma 7. *The pulled-back quadratic differential $\psi_n(\zeta)$ converges to $\psi_0(\zeta) = 1/4\zeta^2$ locally uniformly on the upper half-plane $\operatorname{Im}\zeta > 0$ as $n \to \infty$.*

Proof. First note that $\{\psi_n\}$ forms a normal family on $\mathbb{H}$ by the condition $\|\psi_n\|_{\mathbb{H}} = 1$. By a standard argument, it is enough to see that the limit function of any convergent subsequence of ψ_n is equal to ψ_0. After renumbering, we may thus assume that ψ_n converges to a holomorphic function ψ_∞ locally uniformly on $\mathbb{H}$. Then, Lemma 6 forces $\psi_\infty(\zeta)$ to be of the form c/ζ^2 for some constant c. Lemma 5 (iv) now implies that $c/\mathrm{i}^2 = -1/4$, namely, $c = 1/4$. Thus the proof is complete. □

Let us continue the proof of Theorem 2. Consider the Teichmüller differential $\mu_n = |\psi_n|/\psi_n$ on $\mathbb{H}$. Then $-t\mu_n \in \operatorname{Belt}(\mathbb{H}, \Gamma_n)$ for each $t \in \mathbb{D}$. Since ψ_n^* forms a basis of the vector space $B_2(\mathbb{H}^*, \Gamma_n)$, we can write

$$\Phi(-t\mu_n) = h_n(t)\psi_n^*, \quad t \in \mathbb{D},$$

where $h_n : \mathbb{D} \to \mathbb{C}$ turns to be a holomorphic map by the holomorphy of the Bers projection Φ. Since $\psi_n^* = \hat{q}_n^* \varphi_{\tau_n}$, we further see that $h_n(t) \in U_{\tau_n}$ for $t \in \mathbb{D}$. Teichmüller's theorem (see [12, §2.6.4]) implies that $h_n : \mathbb{D} \to U_{\tau_n}$ is injective and proper. Since $T(R_{\tau_n})$ is one-dimensional, h_n is biholomorphic. Obviously, $h_n(0) = 0$. Therefore, $h_n(t) = g_n(\mathrm{e}^{\mathrm{i}\theta_n} t)$, where $\theta_n = \arg h_n'(0) \in (-\pi, \pi]$.

Lemma 7 now implies that $\mu_n \to \mu_0 = |\psi_0|/\psi_0$ pointwise as $n \to \infty$. The following assertion follows from Theorems 8 and 9 in the paper [1] of Ahlfors and Bers (see also [16] for details).

Lemma 8. *Let μ_n be a sequence of measurable functions on $\mathbb{C}$ such that $\|\mu_n\|_\infty \le 1$ and $\mu_n \to \mu$ a.e. as $n \to \infty$. Then the normalized $t\mu_n$-conformal map $f^{t\mu_n}(z)$ converges to $f^{t\mu}(z)$ as $n \to \infty$ locally uniformly in $(t, z) \in \mathbb{D} \times \mathbb{C}$.*

This implies that $f^{-t\mu_n}(z)$ converges to $f^{-t\mu_0}(z)$ locally uniformly in $(t, z) \in \mathbb{D} \times \mathbb{C}$. By the Weierstrass double series theorem, we see that the Schwarzian derivative $\Phi[-t\mu_n](z) = h_n(t)\psi_n^*(z)$ of $f^{-t\mu_n}(z)$ converges to $\Phi[-t\mu_0](z) = g_0(t)\psi_0^*(z)$, that of $f^{-t\mu_0}(z)$, as $n \to \infty$ locally uniformly on $\mathbb{D} \times \mathbb{H}^*$ (see also Lemma 3). Hence, we conclude that $h_n(t)$ converges to $g_0(t)$ locally uniformly on $\mathbb{D}$. In particular, $h_n'(0) \to g_0'(0) = 4$, and hence, $\theta_n = \arg h_n'(0) \to 0$ as $n \to \infty$. Therefore, we finally see that $g_n(t) = h_n(\mathrm{e}^{-\mathrm{i}\theta_n} t) \to g_0(t)$ locally uniformly on $\mathbb{D}$ as $n \to \infty$. The proof of Theorem 2 is now complete.

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