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COMPLETE FORM OF FURUTA INEQUALITY

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Dedicated to the 20th anniversary of the birth of the Furuta inequality

Abstract. Let A and B be bounded linear operators on a Hilbert space satisfying $A \geq B \geq 0$. The well-known Furuta inequality is given as follows: Let $r \geq 0$ and $p > 0$; then $A^{\frac{r}{2}} A^{\min\{1,p\}} A^{\frac{r}{2}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\min\{1,p\}+r}{p+r}}$. In order to give a self-contained proof of it, Furuta (1989) proved that if $1 \geq r \geq 0$, $p > p_0 > 0$ and $2p_0 + r \geq p > p_0$, then $(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{p+r}{p_0+r}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{p+r}{p+r}}$.

This paper aims to show a sharpening of Furuta (1989): Let $r \geq 0$, $p_0 > 0$ and $s = \min\{p, 2p_0 + \min\{1, r\}\}$; then $(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{s+r}{p_0+r}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{s+r}{p+r}}$. We call it the complete form of Furuta inequality because the case $p_0 = 1$ of it implies the essential part $(p > 1)$ of Furuta inequality for $\frac{1+r}{s+r} \in (0, 1]$ by the famous Löwner-Heinz inequality. Afterwards, the optimality of the outer exponent of the complete form is considered. Lastly, we give some applications of the complete form to Aluthge transformation.

1. Introduction

A capital letter (such as T) means a bounded linear operator on a Hilbert space. $T \geq 0$ and $T > 0$ mean a positive operator and an invertible positive operator, respectively.

As an essential extension of the celebrated Löwner-Heinz inequality, $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for each $\alpha \in [0, 1]$, Furuta [4] showed the following operator inequality.

Theorem 1.1 ([4])**.** *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(1.1) \qquad \left(B^{r/2} A^p B^{r/2}\right)^{1/q} \geq \left(B^{r/2} B^p B^{r/2}\right)^{1/q},$$

$$(1.2) \qquad \left(A^{r/2} A^p A^{r/2}\right)^{1/q} \geq \left(A^{r/2} B^p A^{r/2}\right)^{1/q}$$

as long as real numbers p, r, q satisfy

$$(1.3) \qquad p \geq 0, \quad q \geq 1 \text{ with } (1 + r)q \geq p + r.$$

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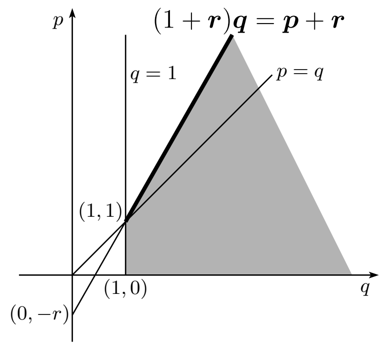

FIGURE 1. Domain of Furuta inequality.

Kamei [12] gave the first improvement of Furuta inequality. Tanahashi [14] showed that, for each $r \geq 0$, condition (1.3) is optimal for the validity of Furuta inequality. See [9] for details. It's well known that Furuta inequality has many applications. See [3], [7], [11] and [23]. It has been generalized to grand Furuta inequality and it extended the Ando-Hiai log-majorization theory [8, 15, 22]. It was used in the p-hyponormal operator theory ([2, 18, 21]).

In order to provide an elementary self-contained and alternative proof of Furuta inequality, Furuta [5] proved the following interesting inequality.

Theorem 1.2 ([5]). *Let $A \geq B \geq 0$, $1 \geq r \geq 0$ and $p > p_0 > 0$. If $2p_0 + r \geq p$, then*

$$(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{p+r}{p_0+r}} \geq (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{p+r}{p+r}}. \tag{1.4}$$

Moreover, [5] gave two examples to illustrate that the conditions $1 \geq r \geq 0$ and $2p_0 + r \geq p > p_0$ are essential for the validity of (1.4).

Here, we show the following refinement of Theorem 1.2.

Theorem 1.3 (Complete form of Furuta inequality). *Let $A \geq B \geq 0$, $r \geq 0$, $p > p_0 > 0$ and $s = \min\{p, 2p_0 + \min\{1, r\}\}$. Then*

$$(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{s+r}{p_0+r}} \geq (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{s+r}{p+r}}. \tag{1.5}$$

Obviously, the case $p_0 = 1$ of Theorem 1.3 implies the essential part $(p > 1)$ of (1.2) for $\frac{1+r}{s+r} \in (0, 1]$ by the Löwner-Heinz inequality. So, we call it the complete form of Furuta inequality.

Afterwards, we investigate the optimality of the outer exponent of the complete form.

Theorem 1.4 (Optimality). *For each $\alpha > 1$, $r > 0$, $p > p_0 > 0$ and $s = \min\{p, 2p_0 + \min\{1, r\}\}$:*

(1) *If $2p_0 + \min\{1, r\} \geq p$, then there exist operators $A > 0$ and $B > 0$ that satisfy*

$$A \geq B, \quad (A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{(p+r)\alpha}{p_0+r}} \not\geq (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\alpha}. \tag{1.6}$$

(2) *If $r \geq 1$ and $2p_0 + 1 < p$, then there exist $A > 0$ and $B > 0$ that satisfy*

$$A \geq B, \quad (A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{(2p_0+1+r)\alpha}{p_0+r}} \not\geq (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{(2p_0+1+r)\alpha}{p+r}}. \tag{1.7}$$

Lastly, some applications of the complete form to Aluthge transformation are obtained.

2. Proofs

To give proofs, the following result is needed.

Lemma 2.1 ([6, 8]). *Let $\alpha \in R$ and X be invertible. Then*

$$(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X;$$

especially in the case $\alpha \geq 1$ the equality holds without invertibility of X.

Proof of Theorem 1.3. Step 1. To give a short proof of Theorem 1.2 by Löwner-Heinz inequality and Lemma 2.1.

Let $A \geq B \geq 0$, $1 \geq r \geq 0$ and $2p_0 + r \geq p > p_0 > 0$. By Lemma 2.1, (1.4) is equivalent to the following:

$$A^{\frac{r}{2}}B^{\frac{p_0}{2}}(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}})^{\frac{p-p_0}{p_0+r}}B^{\frac{p_0}{2}}A^{\frac{r}{2}} \geq A^{\frac{r}{2}}B^{\frac{p_0}{2}}B^{p-p_0}B^{\frac{p_0}{2}}A^{\frac{r}{2}}. \tag{2.1}$$

On the other hand, by $\frac{p-p_0}{p_0+r} \in (0,1]$ and the Löwner-Heinz inequality we have

$$(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}})^{\frac{p-p_0}{p_0+r}} \geq B^{p-p_0},$$

so that Theorem 1.2 follows.

Step 2. To prove case $1 \geq r \geq 0$ of Theorem 1.3. By Step 1, we assume that $2p_0 + r < p$.

In fact, take a positive integer n such that $p_n < p \leq p_{n+1}$ where $p_n = p_0 + (2^n - 1)(p_0 + \min\{1, r\})$.

If $n = 1$, then $p_1 < p \leq p_2 = 2p_1 + r$. The case $p = 2p_0 + r = p_1$ of (1.4) means $(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{p_1+r}{p_0+r}} \geq (A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}})^{\frac{p_1+r}{p_1+r}}$. By replacing p_0 with p_1 in (1.4), the following holds for $p_1 < p \leq p_2$:

$$(A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}})^{\frac{p+r}{p_1+r}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p+r}{p+r}}. \tag{2.2}$$

By repeating this technique in (1.4), the following holds for $p_n < p \leq p_{n+1}$:

$$(A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}})^{\frac{p+r}{p_n+r}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p+r}{p+r}}. \tag{2.3}$$

Therefore,

$$\begin{aligned}(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{p_1+r}{p_0+r}} &\geq (A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}})^{\frac{p_1+r}{p_1+r}} \\ &\geq \cdots \geq (A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}})^{\frac{p_1+r}{p_n+r}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p_1+r}{p+r}}.\end{aligned} \tag{2.4}$$

Step 3. To show the essential part ($p > 1$) of (1.2). It is sufficient to prove that

$$A^{\frac{r}{2}}AA^{\frac{r}{2}} \geq A^{\frac{r}{2}}BA^{\frac{r}{2}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}. \tag{2.5}$$

(2.5) is the only theorem of [12]. ([5] gave a self-contained proof of the essential part ($p > 1$) of (1.2) by using Theorem 1.2 and the repeating method. Meanwhile, the case $p_0 = 1$ and $1 \geq r \geq 0$ of (1.5) implies the case $1 \geq r \geq 0$ of (2.5) for $\frac{1+r}{s+r} \in (0,1]$ by the Löwner-Heinz inequality.)

Step 4. To prove case $r > 1$ and $p_0 < p \leq 2p_0 + 1$ of Theorem 1.3.

Obviously, (2.5) implies

$$B^{\frac{p_0}{2}}BB^{\frac{p_0}{2}} \leq (B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}})^{\frac{1+p_0}{r+p_0}}. \tag{2.6}$$

By Lemma 2.1, (1.5) is equivalent to (2.1). On the other hand, by (2.6), $\frac{p-p_0}{1+p_0} \in (0,1]$ and the Löwner-Heinz inequality we have $(B^{\frac{p_0}{2}} A^r B^{\frac{p_0}{2}})^{\frac{p-p_0}{p_0+r}} \geq B^{p-p_0}$, so that Theorem 1.3 follows.

Step 5. To prove case $r > 1$ and $2p_0 + 1 < p$ of Theorem 1.3. The proof is similar to Step 2, so we omit it here. □

Remark 2.2. Steps 1–3 of the Proof of Theorem 1.3 show that the trivial part $(1 \geq p \geq 0)$ implies the essential part $(p > 1)$ of the Furuta inequality by the repeating method.

Remark 2.3. Though the best possible outer exponent of the latter inequality in (2.5) is no less than $s+r$ $(> 1+r)$ by Theorem 1.3, the best possible outer exponent of the first inequality in (2.5) is $1+r$ by Tanahsahi [14]. By (2.5), the outer exponent of the essential part of Furuta inequality is determined by the relations between $A^{\frac{r}{2}}AA^{\frac{r}{2}}$ and $A^{\frac{r}{2}}BA^{\frac{r}{2}}$, and the relations between $A^{\frac{r}{2}}BA^{\frac{r}{2}}$ and $A^{\frac{r}{2}}B^pA^{\frac{r}{2}}$ $(p > 1)$. So the best possible outer exponent of the essential part should be $1 + r$. This also implies that we call Theorem 1.3 the complete form of the Furuta inequality.

Proof of Theorem 1.4. The proof is similar to the proof of Theorem 3.1 of [20].

Step 1. To prove (1) of Theorem 1.4. Let

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b+\varepsilon+\delta \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

where $0 < b < 1 < a$, $0 < \delta = \frac{1-b}{a-1}\varepsilon$ as in [14]. Then $A \geq B > 0$. In order to show the assertion by contradiction, assume that (1) of Theorem 1.4 is not valid. Let

$$\gamma = a - b + \varepsilon - \delta, \quad U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix}.$$

Then

$$U = U^* = U^{-1}, \quad U^*AU = \begin{pmatrix} a+\varepsilon & 0 \\ 0 & b+\delta \end{pmatrix}.$$

Hence, by assumption,

$$\gamma^{-\frac{1}{q_2}} \begin{pmatrix} \widetilde{A_1} & \widetilde{A_3} \\ \widetilde{A_3} & \widetilde{A_2} \end{pmatrix}^{\frac{1}{q_2}} - \gamma^{-\frac{1}{q_1}} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}^{\frac{1}{q_1}} \leq 0, \tag{2.7}$$

where

$$\begin{aligned} &1/q_1 = \frac{(s+r)\alpha}{p_0+r}, 1/q_2 = \frac{(s+r)\alpha}{p+r}, \\ &A_1 = (a+\varepsilon)^r(a-b-\delta+\varepsilon b^{p_0}), \\ &A_2 = (b+\delta)^r\big(\varepsilon + b^{p_0}(a-b-\delta)\big), \\ &A_3 = (a+\varepsilon)^{r/2}(b+\delta)^{r/2}(1-b^{p_0})\sqrt{\varepsilon(a-b-\delta)}, \\ &\widetilde{A_1} = (a+\varepsilon)^r(a-b-\delta+\varepsilon b^p), \\ &\widetilde{A_2} = (b+\delta)^r\big(\varepsilon + b^p(a-b-\delta)\big), \\ &\widetilde{A_3} = (a+\varepsilon)^{r/2}(b+\delta)^{r/2}(1-b^p)\sqrt{\varepsilon(a-b-\delta)}. \end{aligned} \tag{2.8}$$

For sufficiently large $a > 1$, let

$$D = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}, \quad V = \frac{1}{\sqrt{A_3^2 + \varepsilon_1^2}} \begin{pmatrix} A_3 & \varepsilon_1 \\ \varepsilon_1 & -A_3 \end{pmatrix},$$
$$\widetilde{D} = \begin{pmatrix} \widetilde{A_1} & \widetilde{A_3} \\ \widetilde{A_3} & \widetilde{A_2} \end{pmatrix}, \quad \widetilde{V} = \frac{1}{\sqrt{\widetilde{A_3}^2 + \widetilde{\varepsilon_1}^2}} \begin{pmatrix} \widetilde{A_3} & \widetilde{\varepsilon_1} \\ \widetilde{\varepsilon_1} & -\widetilde{A_3} \end{pmatrix},$$

where $2\varepsilon_1 = -(A_1 - A_2) + \sqrt{(A_1 - A_2)^2 + 4A_3^2}$, $2\widetilde{\varepsilon_1} = -(\widetilde{A_1} - \widetilde{A_2}) + \sqrt{(\widetilde{A_1} - \widetilde{A_2})^2 + 4\widetilde{A_3}^2}$. Then

$$V = V^* = V^{-1}, \quad V^* D V = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix},$$
$$\widetilde{V} = \widetilde{V}^* = \widetilde{V}^{-1}, \quad \widetilde{V}^* \widetilde{D} \widetilde{V} = \begin{pmatrix} \widetilde{A_1} + \widetilde{\varepsilon_1} & 0 \\ 0 & \widetilde{A_2} - \widetilde{\varepsilon_1} \end{pmatrix}.$$

Hence, by (2.7),

$$\begin{aligned} &\gamma^{-\frac{1}{q_2}} \frac{1}{(A_3^2 + \varepsilon_1^2)(\widetilde{A_3}^2 + \widetilde{\varepsilon_1}^2)} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} \\ &-\gamma^{-\frac{1}{q_1}} \begin{pmatrix} (A_1 + \varepsilon_1)^{\frac{1}{q_1}} & 0 \\ 0 & (A_2 - \varepsilon_1)^{\frac{1}{q_1}} \end{pmatrix} \le 0, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} B_1 &= (\widetilde{A_1} + \widetilde{\varepsilon_1})^{\frac{1}{q_2}} (A_3 \widetilde{A_3} + \varepsilon_1 \widetilde{\varepsilon_1})^2 + (\widetilde{A_2} - \widetilde{\varepsilon_1})^{\frac{1}{q_2}} (\widetilde{A_3} \varepsilon_1 - A_3 \widetilde{\varepsilon_1})^2, \\ B_2 &= (\widetilde{A_1} + \widetilde{\varepsilon_1})^{\frac{1}{q_2}} (\widetilde{A_3} \varepsilon_1 - A_3 \widetilde{\varepsilon_1})^2 + (\widetilde{A_2} - \widetilde{\varepsilon_1})^{\frac{1}{q_2}} (A_3 \widetilde{A_3} + \varepsilon_1 \widetilde{\varepsilon_1})^2, \\ B_3 &= \big((\widetilde{A_1} + \widetilde{\varepsilon_1})^{\frac{1}{q_2}} - (\widetilde{A_2} - \widetilde{\varepsilon_1})^{\frac{1}{q_2}}\big)(A_3 \widetilde{A_3} + \varepsilon_1 \widetilde{\varepsilon_1})(\widetilde{A_3} \varepsilon_1 - A_3 \widetilde{\varepsilon_1}). \end{aligned} \tag{2.10}$$

Similar to the arguments in [20],

$$\begin{aligned} &\big(\gamma^{-(\frac{1}{q_2} - \frac{1}{q_1})} (\widetilde{A_1} + \widetilde{\varepsilon_1})^{\frac{1}{q_2}} - (A_1 + \varepsilon_1)^{\frac{1}{q_1}}\big)\big(\gamma^{-(\frac{1}{q_2} - \frac{1}{q_1})} (\widetilde{A_2} - \widetilde{\varepsilon_1})^{\frac{1}{q_2}} - (A_2 - \varepsilon_1)^{\frac{1}{q_1}}\big) \\ &\times (A_3 \widetilde{A_3} + \varepsilon_1 \widetilde{\varepsilon_1})^2 + (\widetilde{A_3} \varepsilon_1 - A_3 \widetilde{\varepsilon_1})^2 \big(\gamma^{-(\frac{1}{q_2} - \frac{1}{q_1})} (\widetilde{A_1} + \widetilde{\varepsilon_1})^{\frac{1}{q_2}} - (A_2 - \varepsilon_1)^{\frac{1}{q_1}}\big) \\ &\times \big(\gamma^{-(\frac{1}{q_2} - \frac{1}{q_1})} (\widetilde{A_2} - \widetilde{\varepsilon_1})^{\frac{1}{q_2}} - (A_1 + \varepsilon_1)^{\frac{1}{q_1}}\big) \ge 0 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &a^r \big(\frac{(a^r - b^r)(b^{p_0} - b^p)}{(a^r - b^{p+r})(a^r - b^{p_0+r})}\big)^2 (a^{r/q_2} - b^{(s+r)\alpha})(a^{r/q_1} - b^{(s+r)\alpha}) \\ &\le (a^{r/q_2} - a^{r/q_1}) b^{(s+r)\alpha - r} \big[(a^r - b^r)\big(\frac{1}{q_2} \frac{1 - b^p}{a^r - b^{p+r}} - \frac{1}{q_1} \frac{1 - b^{p_0}}{a^r - b^{p_0+r}}\big) \\ &\qquad - (\frac{1}{q_1} - \frac{1}{q_2})(a - b) \frac{r}{b} \frac{1 - b}{a - 1}\big]. \end{aligned} \tag{2.12}$$

By (2.12) we have the following:

(2.13)
$$\begin{aligned}&\Big(\frac{(1-a^{-r}b^{r})(b^{p_0}-b^{p})}{(1-a^{-r}b^{p+r})(1-a^{-r}b^{p_0+r})}\Big)^2(1-a^{-r/q_2}b^{(p+r)\alpha})(1-a^{-r/q_1}b^{(p+r)\alpha})\\&\le a^{r(1-1/q_2)}(a^{r/q_2-r/q_1}-1)b^{(p+r)\alpha-r}\Big[(1-a^{-r}b^{r})\\&\times\Big(\frac{1}{q_2}\frac{1-b^{p}}{1-a^{-r}b^{p+r}}-\frac{1}{q_1}\frac{1-b^{p_0}}{1-a^{-r}b^{p_0+r}}\Big)-\Big(\frac{1}{q_1}-\frac{1}{q_2}\Big)(1-a^{-1}b)\frac{r}{b}\frac{1-b}{1-a^{-1}}\Big].\end{aligned}$$

Letting $a\to\infty$, for $r(1-\frac{1}{q_2})<0$, $r(\frac{1}{q_2}-\frac{1}{q_1})<0$, we have
$$(b^{p_0}-b^{p})^2\le 0.$$
This is a contradiction.

Step 2. To prove (2) of Theorem 1.4. In fact, similar to the proof of Step 1, (2.12) follows. Then

$$\begin{aligned}&a^{r}\Big(\frac{(a^{r}-b^{r})(1-b^{p-p_0})}{(a^{r}-b^{p+r})(a^{r}-b^{p_0+r})}\Big)^2(a^{r/q_2}-b^{(2p_0+1+r)\alpha})(a^{r/q_1}-b^{(2p_0+1+r)\alpha})\\&\le (a^{r/q_2}-a^{r/q_1})b^{(2p_0+1+r)\alpha-r-2p_0-1}\Big[b(a^{r}-b^{r})\\&\times\Big(\frac{1}{q_2}\frac{1-b^{p}}{a^{r}-b^{p+r}}-\frac{1}{q_1}\frac{1-b^{p_0}}{a^{r}-b^{p_0+r}}\Big)-\Big(\frac{1}{q_1}-\frac{1}{q_2}\Big)(a-b)r\frac{1-b}{a-1}\Big].\end{aligned}\tag{2.14}$$

Letting $b\to 0+$, for $(2p_0+1+r)\alpha-r-2p_0-1>0$, $p-p_0>0$, we have
$$a^{r/q_2+r/q_1-r}\le 0.$$
This is a contradiction. □

Remark 2.4. It is not known whether the outer exponent of the case $2p_0+r<p$ and $1>r>0$ of Theorem 1.3 is the best possible in the sense of Theorem 1.4. This problem closely relates the problem of the structure on powers of operators (Remark 3.3 of [21]) by Lemma 3.5 of [18].

3. Applications

For $q>0$, T is called a q-hyponormal operator if $(T^*T)^q\ge(TT^*)^q$, where T^* is the adjoint operator of T. If $q=1$, T is called a hyponormal operator and if $q=1/2$, T is called a semi-hyponormal operator. See Martin–Putinar [13] and Xia [16] for related topics and basic properties of hyponormal operators.

Aluthge [1] introduced Aluthge transformation $\widetilde{T}=|T|^{1/2}U|T|^{1/2}$ where the polar decomposition of T is $T=U|T|$. For each $p>0$ and $r>0$, $\widetilde{T}_{p,r}=|T|^{p}U|T|^{r}$ is called the generalized Aluthge transformation.

Recently, many researchers obtained the interesting order properties of the generalized Aluthge transformation of q–hyponormal operators.

Theorem 3.1 ([1, 10, 19]). *If T is a q-hyponormal operator and*
$$\gamma=\min\{q+p,q+r,p+r\},$$
then
$$\big((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\big)^{\frac{\gamma}{p+r}}\ge\big(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\big)^{\frac{\gamma}{p+r}}.$$

Moreover, Theorem 3.2 below implies that the outer exponent in Theorem 3.1 is the best possible.

Theorem 3.2 ([17])**.** *For each $p > 0$, $r > 0$, $q > 0$ and $\alpha > 1$, there exists a q-hyponormal operator T such that*

$$\big((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\big)^{\frac{\gamma}{p+r}\alpha} \not\geq \big(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\big)^{\frac{\gamma}{p+r}\alpha}.$$

The complete form of Furuta inequality can be utilized to give the following result.

Theorem 3.3. *Let T be a q-hyponormal operator, $r > 0$, $r_0 > 0$, $p > 0$, $p_0 > 0$, $s(q) = \min\{p, 2p_0 + \min\{q, r\}\}$ and $\widetilde{s}(q) = \min\{r, 2r_0 + \min\{q, p\}\}$.*

(1) *If $p > p_0$, then*

$$\big((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\big)^{\frac{s(q)+r}{p+r}} \geq \big((\widetilde{T}_{p_0,r})^*\widetilde{T}_{p_0,r}\big)^{\frac{s(q)+r}{p_0+r}}, \tag{3.1}$$

$$\big((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\big)^{\frac{\min\{p,q\}+r}{p+r}} \geq (T^*T)^{\min\{p,q\}+r}. \tag{3.2}$$

(2) *If $r > r_0$, then*

$$\big(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\big)^{\frac{\widetilde{s}(q)+p}{r+p}} \leq \big(\widetilde{T}_{p,r_0}(\widetilde{T}_{p,r_0})^*\big)^{\frac{\widetilde{s}(q)+p}{r_0+p}}, \tag{3.3}$$

$$\big(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\big)^{\frac{\min\{r,q\}+p}{r+p}} \leq (T^*T)^{\min\{r,q\}+p}. \tag{3.4}$$

Remark 3.4. Obviously, Theorem 3.1 is a direct result of (3.2) and (3.4) for $\gamma = \min\big\{\min\{p, q\} + r,\ \min\{r, q\} + p\big\}$. It is clear that the case $p_0 = q$ of (3.1) implies the essential part ($p > q$) of (3.2) for $\frac{q+r}{s(q)+r} \in (0, 1]$ by the Löwner-Heinz inequality. Similarly, the case $r_0 = q$ of (3.3) implies the essential part ($r > q$) of (3.4).

To give proofs, we write out a variant on the complete form.

Theorem 3.5 (Variant on the complete form)**.** *Let $A \geq 0$, $B \geq 0$ such that $A^q \geq B^q$, $r \geq 0$ and $p > p_0 > 0$. Then*

$$(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{s(q)+r}{p_0+r}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{s(q)+r}{p+r}}. \tag{3.5}$$

Theorem 3.6 (Optimality)**.** *For each $\alpha > 1$, $q > 0$, $r > 0$ and $p > p_0 > 0$:*

(1) *If $2p_0 + \min\{q, r\} \geq p$, then there exist operators $A > 0$ and $B > 0$ that satisfy*

$$A^q \geq B^q,\ (A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{(p+r)\alpha}{p_0+r}} \not\geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\alpha}. \tag{3.6}$$

(2) *If $r \geq q$ and $2p_0 + q < p$, then there exist $A > 0$ and $B > 0$ that satisfy*

$$A^q \geq B^q,\ (A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{(2p_0+q+r)\alpha}{p_0+r}} \not\geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{(2p_0+q+r)\alpha}{p+r}}. \tag{3.7}$$

Proof of Theorem 3.3. We only need to prove (3.1) by Remark 3.4.

By the property of the polar decomposition of T, (3.1) is equivalent to

$$\big(|T^*|^r|T|^{2p}|T^*|^r\big)^{\frac{s(q)+r}{p+r}} \geq \big(|T^*|^r|T|^{2p_0}|T^*|^r\big)^{\frac{s(q)+r}{p_0+r}}.$$

This is a direct result of Theorem 3.5 by q-hyponormality of T. □

Corresponding to Theorem 3.2, we write the following result.

Theorem 3.7. *For each $\alpha > 1$, $q > 0$, $r > 0$, $r_0 > 0$, $p > 0$ and $p_0 > 0$:*

(1) *There exists a q-hyponormal operator T such that*

$$\big((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\big)^{\frac{\min\{p,q\}+r}{p+r}\alpha} \not\geq (T^*T)^{(\min\{p,q\}+r)\alpha}.$$

(2) *If* $2p_0+\min\{q,r\}\geq p$ *or* $r\geq q$ *and* $2p_0+q<p$*, there exists a* q*-hyponormal operator* T *such that*

$$\left((\widetilde{T}_{p,r})^*\widetilde{T}_{p,r}\right)^{\frac{s(q)+r}{p+r}\alpha}\ngeq\left((\widetilde{T}_{p_0,r})^*\widetilde{T}_{p_0,r}\right)^{\frac{s(q)+r}{p_0+r}\alpha}.$$

(3) *There exists a* q*-hyponormal operator* T *such that*

$$\left(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\right)^{\frac{\min\{r,q\}+p}{r+p}\alpha}\nleq(T^*T)^{(\min\{r,q\}+p)\alpha}.$$

(4) *If* $2r_0+\min\{q,p\}\geq r$ *or* $p\geq q$ *and* $2r_0+q<r$*, there exists a* q*-hyponormal operator* T *such that*

$$\left(\widetilde{T}_{p,r}(\widetilde{T}_{p,r})^*\right)^{\frac{\tilde{s}(q)+p}{r+p}\alpha}\nleq\left(\widetilde{T}_{p,r_0}(\widetilde{T}_{p,r_0})^*\right)^{\frac{\tilde{s}(q)+p}{r_0+p}\alpha}.$$

The proof is quite similar to the corresponding parts of [17, 18, 21, 23], so we omit it here.

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