TRIANGULATED CATEGORIES OF GORENSTEIN CYCLIC QUOTIENT SINGULARITIES

KAZUSHI UEDA

(Communicated by Ted Chinburg)

Abstract. We prove there is an equivalence of derived categories between Orlov’s triangulated category of singularities for a Gorenstein cyclic quotient singularity and the derived category of representations of a quiver with relations, which is obtained from a McKay quiver by removing one vertex and half of the arrows. This result produces examples of distinct quivers with relations which have equivalent derived categories of representations.

Fix an integer $n$ greater than one. For a finite subgroup $G$ of $GL_n(C)$, let $\{\rho_i\}_{i=0}^{N-1}$ be the set of irreducible representations of $G$. Further, let $\rho_{\text{Nat}}$ be the natural $n$-dimensional representation of $G$ given by the inclusion. For $k,l = 0, \ldots, N-1$, define the natural numbers $a_{kl}$ by the decomposition

$$\rho_l \otimes \rho_{\text{Nat}} = \bigoplus_k \rho_k^{\otimes a_{kl}}$$

of the tensor product into the direct sum of irreducible representations. The McKay quiver of $G$ is the quiver whose set of vertices is $\{\rho_i\}_{i=0}^{N-1}$ and the number of whose arrows from the vertex $\rho_k$ to the vertex $\rho_l$ is $a_{kl}$.

Now assume that $G$ is a cyclic group generated by an element of the form $g = \text{diag}(\zeta^{a_1}, \ldots, \zeta^{a_n})$, where $a_1, \ldots, a_n$ are positive integers such that $\gcd(a_1, \ldots, a_n) = 1$ and $\zeta = \exp[2\pi \sqrt{-1}/(a_1 + \cdots + a_n)]$. Let $R = C[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables equipped with the $\mathbb{Z}$-grading given by $\deg x_i = a_i$, $i = 1, \ldots, n$. Define another $\mathbb{Z}$-graded ring $A(a_1, \ldots, a_n) = \bigoplus_{k \geq 0} A_k$ by

$$A_k = R_{k(a_1 + \cdots + a_n)}.$$

Then $A(a_1, \ldots, a_n)$ is the invariant ring of $R$ by the action of $G$ so that $\mathbb{C}^n/G = \text{Spec} A(a_1, \ldots, a_n)$.

In this case, the corresponding McKay quiver has $N = a_1 + \cdots + a_n$ vertices $\{\rho_k\}_{k=0}^{N-1}$ and $nN$ arrows $\{x_{i,k}\}_{i=1, \ldots, n} \rightarrow \{k=0, \ldots, N-1\}$, where $\rho_k$ is the one-dimensional representation of $G$ sending $g \in G$ to $\exp[-2k\pi \sqrt{-1}/(a_1 + \cdots + a_n)] \in C^\times$, and $x_{i,k}$ is an arrow from $\rho_k$ to $\rho_{k+a_i}$.

Next we introduce another quiver $Q(a_1, \ldots, a_n)$ obtained by removing the vertex $\rho_0$ and half of the arrows from the McKay quiver. The set of vertices of
$Q(a_1, \ldots, a_n)$ is $\{\rho_k\}_{k=1}^N$, and an arrow of the McKay quiver from $\rho_k$ to $\rho_l$ is an arrow of $Q_q$ if $0 < k < l$.

A **quiver with relations** is a pair $\Gamma = (Q, I)$ of a quiver $Q$ and a two-sided ideal $I$ of its path algebra $\mathbb{C}Q$. We equip $Q(a_1, \ldots, a_n)$ with the relations $I(a_1, \ldots, a_n)$ generated by $s_{j,k+a_i}x_{i,k} - s_{i,k+j}x_{j,k}$ for $1 \leq i < j \leq n$ and $k = 1, \ldots, N - a_i - a_j - 1$, and put $\Gamma(a_1, \ldots, a_n) = (Q(a_1, \ldots, a_n), I(a_1, \ldots, a_n))$. The main theorem is:

**Theorem 1.** For a sequence $a_1, \ldots, a_n$ of positive integers such that $\gcd(a_1, \ldots, a_n) = 1$, there exists an equivalence

$$D^b_{\mathbb{Sg}}(A(a_1, \ldots, a_n)) \cong D^b \operatorname{mod} \Gamma(a_1, \ldots, a_n)$$

of triangulated categories.

Here, $D^b \operatorname{mod} \Gamma(a_1, \ldots, a_n)$ is the bounded derived category of finite-dimensional right modules over the path algebra $\mathbb{C}\Gamma(a_1, \ldots, a_n) = \mathbb{C}Q(a_1, \ldots, a_n)/I(a_1, \ldots, a_n)$ with relations. The above theorem produces examples of distinct quivers with relations equivalent to those of representations; e.g. when $n = 3$, the $\mathbb{Z}$-graded rings $A(1, 2, 2)$ and $A(3, 1, 1)$ are isomorphic, although the quivers with relations $\Gamma(1, 2, 2)$ and $\Gamma(3, 1, 1)$ are not isomorphic.

$D^b_{\mathbb{Sg}}(A(a_1, \ldots, a_n))$ is the triangulated category of singularities defined by Orlov [6] as the quotient category

$$D^b_{\mathbb{Sg}}(A(a_1, \ldots, a_n)) = D^b \operatorname{gr} A(a_1, \ldots, a_n)/D^b \operatorname{grproj} A(a_1, \ldots, a_n)$$

of the bounded derived category $D^b \operatorname{gr} A(a_1, \ldots, a_n)$ of finitely-generated $\mathbb{Z}$-graded $A(a_1, \ldots, a_n)$-modules by its full triangulated subcategory $D^b \operatorname{grproj} A(a_1, \ldots, a_n)$ consisting of perfect complexes. The $n = 2$ case in the above theorem is due to Takahashi [7] (see also Kajura, Saito, and Takahashi [4]).

The proof goes as follows: Let

$$qgr R := \operatorname{gr} R/\operatorname{tor} R$$

be the quotient category of the abelian category $\operatorname{gr} R$ of finitely-generated $\mathbb{Z}$-graded $R$-modules by its full subcategory $\operatorname{tor} R$ consisting of torsion modules, and let $\pi : \operatorname{gr} R \to qgr R$ be the natural projection functor. For $M \in \operatorname{gr} R$ and $l \in \mathbb{Z}$, $M(l)$ denotes the graded $R$-module shifted by $l$; $M(l)_k = M_{i+k}$. Define a shift operator $s : qgr R \to qgr R$ by $s(\pi M) = \pi M(a_1 + \cdots + a_n)$ and put $\mathcal{O} = \pi R$. Then one has $A(a_1, \ldots, a_n) = \bigoplus_{k=\mathbb{Z}}^\infty \operatorname{Hom}(\mathcal{O}, s^k(\mathcal{O}))$. Since $\gcd(a_1, \ldots, a_n) = 1$, the graded $R$-module $R(l)$ for any $l \in \mathbb{Z}$ is generated up to torsion modules by the subset $\bigcup_{j \in \mathbb{N}} R(l)_{j(a_1 + \cdots + a_n)}$ consisting of elements whose degrees are positive multiples of $a_1 + \cdots + a_n$. Hence $s$ is ample and one has

$$qgr R \cong qgr A(a_1, \ldots, a_n)$$


Since $s^{-1}(\mathcal{O})$ is the dualizing sheaf, $A(a_1, \ldots, a_n)$ is Gorenstein with Gorenstein parameter 1 (cf. [6] Lemma 2.11)). Therefore one has a semiorthogonal decomposition

$$D^b \operatorname{gr} A(a_1, \ldots, a_n) = \langle \mathcal{O}, D^b_{\mathbb{Sg}}(A(a_1, \ldots, a_n)) \rangle$$

by Orlov [6] Theorem 2.5(i)]. Here, $D^b \operatorname{gr} A(a_1, \ldots, a_n)$ denotes the bounded derived category of the abelian category $\operatorname{gr} A(a_1, \ldots, a_n)$. On the other hand, $D^b(qgr R)$ has a full strong exceptional collection $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(a_1 + a_2 + \cdots + a_n - 1))$ (see e.g. [2] Theorem 2.12)). Hence $D^b_{\mathbb{Sg}}(A(a_1, \ldots, a_n))$ is equivalent to the
full triangulated subcategory of $D^b \text{qgr} A(a_1, \ldots, a_n)$ generated by the exceptional collection $(\mathcal{O}(1), \ldots, \mathcal{O}(a_1 + \cdots + a_n - 1))$. By Bondal [3, Theorem 6.2], this subcategory is isomorphic to the derived category of finite-dimensional right modules over the total morphism algebra

$$\bigoplus_{i,j=1}^{a_1+\cdots+a_n-1} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j))$$

of this collection, which is isomorphic to $\mathbb{C}^\Gamma(a_1, \ldots, a_n)$.

References