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TRIANGULATED CATEGORIES OF GORENSTEIN CYCLIC QUOTIENT SINGULARITIES

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Abstract. We prove there is an equivalence of derived categories between Orlov's triangulated category of singularities for a Gorenstein cyclic quotient singularity and the derived category of representations of a quiver with relations, which is obtained from a McKay quiver by removing one vertex and half of the arrows. This result produces examples of distinct quivers with relations which have equivalent derived categories of representations.

Fix an integer n greater than one. For a finite subgroup G of $GL_n(\mathbb{C})$, let $\{\rho_i\}_{i=0}^{N-1}$ be the set of irreducible representations of G. Further, let ρ_{Nat} be the natural n-dimensional representation of G given by the inclusion. For $k, l = 0, \dots, N-1$, define the natural numbers a_{kl} by the decomposition

$$\rho_l \otimes \rho_{\mathrm{Nat}} = \bigoplus_k \rho_k^{\oplus a_{kl}}$$

of the tensor product into the direct sum of irreducible representations. The *McKay quiver* of G is the quiver whose set of vertices is $\{\rho_i\}_{i=0}^{N-1}$ and the number of whose arrows from the vertex ρ_k to the vertex ρ_l is a_{kl} [5].

Now assume that G is a cyclic group generated by an element of the form $g = \mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_n})$, where $a_1, \dots, a_n$ are positive integers such that $\gcd(a_1, \dots, a_n) = 1$ and $\zeta = \exp[2\pi\sqrt{-1}/(a_1 + \cdots + a_n)]$. Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring in n variables equipped with the $\mathbb{Z}$-grading given by $\deg x_i = a_i$, $i = 1, \dots, n$. Define another $\mathbb{Z}$-graded ring $A(a_1, \dots, a_n) = \bigoplus_{k \geq 0} A_k$ by

$$A_k = R_{k(a_1 + \cdots + a_n)}. \tag{1}$$

Then $A(a_1, \dots, a_n)$ is the invariant ring of R by the action of G so that $\mathbb{C}^n/G = \mathrm{Spec}\, A(a_1, \dots, a_n)$.

In this case, the corresponding McKay quiver has $N = a_1 + \cdots + a_n$ vertices $\{\rho_k\}_{k=0}^{N-1}$ and nN arrows $\{x_{i,k}\}_{\substack{i=1,\dots,n \\ k=0,\dots,N-1}}$, where ρ_k is the one-dimensional representation of G sending $g \in G$ to $\exp[-2k\pi\sqrt{-1}/(a_1 + \cdots + a_n)] \in \mathbb{C}^\times$, and $x_{i,k}$ is an arrow from ρ_k to ρ_{k+a_i}.

Next we introduce another quiver $Q(a_1, \dots, a_n)$ obtained by removing the vertex ρ_0 and half of the arrows from the McKay quiver. The set of vertices of

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$Q(a_1,\dots,a_n)$ is $\{\rho_k\}_{k=1}^N$, and an arrow of the McKay quiver from ρ_k to ρ_l is an arrow of Q_g if $0<k<l$.

A *quiver with relations* is a pair $\Gamma=(Q,\mathcal{I})$ of a quiver Q and a two-sided ideal $\mathcal{I}$ of its path algebra $\mathbb{C}Q$. We equip $Q(a_1,\dots,a_n)$ with the relations $\mathcal{I}(a_1,\dots,a_n)$ generated by $x_{j,k+a_i}x_{i,k}-x_{i,k+a_j}x_{j,k}$ for $1\le i<j\le n$ and $k=1,\dots,N-a_i-a_j-1$, and put $\Gamma(a_1,\dots,a_n)=(Q(a_1,\dots,a_n),\mathcal{I}(a_1,\dots,a_n))$. The main theorem is:

Theorem 1. *For a sequence $a_1,\dots,a_n$ of positive integers such that* $\gcd(a_1,\dots,a_n)=1$*, there exists an equivalence*

$$D^{\mathrm{gr}}_{\mathrm{Sg}}(A(a_1,\dots,a_n))\cong D^b\operatorname{mod}\Gamma(a_1,\dots,a_n)$$

of triangulated categories.

Here, $D^b\operatorname{mod}\Gamma(a_1,\dots,a_n)$ is the bounded derived category of finite-dimensional right modules over the path algebra $\mathbb{C}\Gamma(a_1,\dots,a_n)=\mathbb{C}Q(a_1,\dots,a_n)/\mathcal{I}(a_1,\dots,a_n)$ with relations. The above theorem produces examples of distinct quivers with relations with equivalent derived categories of representations; e.g. when $n=3$, the $\mathbb{Z}$-graded rings $A(1,2,2)$ and $A(3,1,1)$ are isomorphic, although the quivers with relations $\Gamma(1,2,2)$ and $\Gamma(3,1,1)$ are not isomorphic.

$D^{\mathrm{gr}}_{\mathrm{Sg}}(A(a_1,\dots,a_n))$ is the *triangulated category of singularities* defined by Orlov [6] as the quotient category

$$D^{\mathrm{gr}}_{\mathrm{Sg}}(A(a_1,\dots,a_n))=D^b\operatorname{gr}A(a_1,\dots,a_n)/D^b\operatorname{grproj}A(a_1,\dots,a_n) \tag{2}$$

of the bounded derived category $D^b\operatorname{gr}A(a_1,\dots,a_n)$ of finitely-generated $\mathbb{Z}$-graded $A(a_1,\dots,a_n)$-modules by its full triangulated subcategory $D^b\operatorname{grproj}A(a_1,\dots,a_n)$ consisting of perfect complexes. The $n=2$ case in the above theorem is due to Takahashi [7] (see also Kajiura, Saito, and Takahashi [4]).

The proof goes as follows: Let

$$\operatorname{qgr}R:=\operatorname{gr}R/\operatorname{tor}R$$

be the quotient category of the abelian category $\operatorname{gr}R$ of finitely-generated $\mathbb{Z}$-graded R-modules by its full subcategory $\operatorname{tor}R$ consisting of torsion modules, and let $\pi:\operatorname{gr}R\to\operatorname{qgr}R$ be the natural projection functor. For $M\in\operatorname{gr}R$ and $l\in\mathbb{Z}$, $M(l)$ denotes the graded R-module shifted by l; $M(l)_k=M_{l+k}$. Define a shift operator $s:\operatorname{qgr}R\to\operatorname{qgr}R$ by $s(\pi M)=\pi M(a_1+\cdots+a_n)$ and put $\mathcal{O}=\pi R$. Then one has $A(a_1,\dots,a_n)=\bigoplus_{k=0}^\infty\operatorname{Hom}(\mathcal{O},s^k(\mathcal{O}))$. Since $\gcd(a_1,\dots,a_n)=1$, the graded R-module $R(l)$ for any $l\in\mathbb{Z}$ is generated up to torsion modules by the subset $\bigcup_{j\in\mathbb{N}}R(l)_{j(a_1+\cdots+a_n)}$ consisting of elements whose degrees are positive multiples of $a_1+\cdots+a_n$. Hence s is ample and one has

$$\operatorname{qgr}R\cong\operatorname{qgr}A(a_1,\dots,a_n)$$

by Artin and Zhang [1, Theorem 4.5].

Since $s^{-1}(\mathcal{O})$ is the dualizing sheaf, $A(a_1,\dots,a_n)$ is Gorenstein with Gorenstein parameter 1 (cf. [6, Lemma 2.11]). Therefore one has a semiorthogonal decomposition

$$D^b\operatorname{qgr}A(a_1,\dots,a_n)=\langle\mathcal{O},D^{\mathrm{gr}}_{\mathrm{Sg}}(A(a_1,\dots,a_n))\rangle$$

by Orlov [6, Theorem 2.5(i)]. Here, $D^b\operatorname{qgr}A(a_1,\dots,a_n)$ denotes the bounded derived category of the abelian category $\operatorname{qgr}A(a_1,\dots,a_n)$. On the other hand, $D^b(\operatorname{qgr}R)$ has a full strong exceptional collection $(\mathcal{O},\mathcal{O}(1),\dots,\mathcal{O}(a_1+a_2+\cdots+a_n-1))$ (see e.g. [2, Theorem 2.12]). Hence $D^{\mathrm{gr}}_{\mathrm{Sg}}(A(a_1,\dots,a_n))$ is equivalent to the

full triangulated subcategory of $D^b \operatorname{qgr} A(a_1, \dots, a_n)$ generated by the exceptional collection $(\mathcal{O}(1), \dots, \mathcal{O}(a_1 + \cdots + a_n - 1))$. By Bondal [3, Theorem 6.2], this subcategory is isomorphic to the derived category of finite-dimensional right modules over the total morphism algebra

$$\bigoplus_{i,j=1}^{a_1+\cdots+a_n-1} \operatorname{Hom}(\mathcal{O}(i), \mathcal{O}(j))$$

of this collection, which is isomorphic to $\mathbb{C}\Gamma(a_1, \dots, a_n)$.

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