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PROPER HOLOMORPHIC MAPPINGS OF THE SPECTRAL UNIT BALL

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Abstract. We prove an Alexander type theorem for the spectral unit ball Ω_n showing that there are no non-trivial proper holomorphic mappings in Ω_n, $n \geq 2$.

Let $\mathcal{M}_n$ denote the space of $n \times n$ complex matrices.

In order to avoid some trivialities and ambiguities we assume in the whole paper that $n \geq 2$.

Let $\rho(A) := \max\{|\lambda| : \lambda \in \mathrm{Spec}(A)\}$ be the spectral radius of $A \in \mathcal{M}_n$. Denote also by $\mathrm{Spec}(A) := \{\lambda \in \mathbb{C} : \det(A - \lambda \mathbb{I}_n) = 0\}$ the spectrum of $A \in \mathcal{M}_n$, where the eigenvalues are counted with multiplicities ($\mathbb{I}_n$ denotes the identity matrix). We also denote the spectral unit ball by $\Omega_n := \{A \in \mathcal{M}_n : \rho(A) < 1\}$. Note that Ω_n is an unbounded pseudoconvex balanced domain in $\mathbb{C}^{n^2}$ with the continuous Minkowski functional equal to ρ. For $A \in \mathcal{M}_n$ denote $P_A(\lambda) := \det(\lambda \mathbb{I}_n - A) = \lambda^n + \sum_{j=1}^n (-1)^j \sigma_j(A) \lambda^{n-j}$, $A \in \mathcal{M}_n$. Denote also $\sigma := (\sigma_1, \ldots, \sigma_n)$. We put $\mathbb{G}_n := \sigma(\Omega_n)$. The domain $\mathbb{G}_n$ is called *the symmetrized polydisc*. Note that $\sigma \in \mathcal{O}(\mathcal{M}_n, \mathbb{G}_n)$. Denote also $\mathcal{J}_n := \pi_n(\{(\zeta_1, \ldots, \zeta_n) : \zeta_j = \zeta_k \text{ for some } j \neq k\})$, where $\pi_{n,j}(\zeta_1, \ldots, \zeta_n) := \sum_{1 \leq k_1 < \ldots < k_j \leq n} \zeta_{k_1} \cdot \ldots \cdot \zeta_{k_j}$, $\zeta_l \in \mathbb{D}$, $l = 1, \ldots, n$ ($\mathbb{D}$ denotes the unit disc in $\mathbb{C}$). Note that $\mathbb{G}_n \setminus \mathcal{J}_n$ is a domain and $\mathbb{G}_n \setminus \mathcal{J}_n$ is dense in $\mathbb{G}_n$.

Note that $\Omega_n = \bigcup_{z \in \mathbb{G}_n} \mathcal{T}_z$, where $\mathcal{T}_z := \{A \in \Omega_n : \sigma(A) = z\}$, $z \in \mathbb{C}^n$. The sets $\mathcal{T}_z$, $z \in \mathbb{C}^n$, are pairwise disjoint analytic sets. Note that if the matrix $A \in \mathcal{T}_z$ is non-derogatory, then A is a regular point of $\mathcal{T}_z$ (recall that in such a case $\operatorname{rank} \sigma'(A) = n$); it is the largest possible number. For a definition and basic properties of non-derogatory matrices see [Nik-Tho-Zwo 2007] and the references therein. One of the possible definitions of a non-derogatory matrix is that different blocks in the Jordan normal form correspond to different eigenvalues (or equivalently all eigenspaces are one-dimensional). We shall deliver some properties of the sets $\mathcal{T}_z$ (see Lemma 5, Lemma 6 and Corollary 7). It is also simple to see that $\mathcal{T}_0$ is a cone which contains at least $n^2 - n + 1$ linearly independent vectors: for instance the ones consisting of one 1 not lying on the diagonal (and with other entries equal to 0) and the matrix

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$A = (a_{j,k})_{j,k=1,\dots,n}$ such that $a_{1,1} = 1$, $a_{1,2} = -1$, $a_{2,1} = 1$, $a_{2,2} = -1$ (and with all other entries equal to 0). Consequently, we shall see that 0 is not a regular point of $\mathcal{T}_0$. On the other hand, the sets $\mathcal{T}_{\pi_n(\zeta_1,\dots,\zeta_n)}$, where the points $\zeta_1, \dots, \zeta_n \in \mathbb{D}$ are pairwise different, are submanifolds – it follows from the fact that in this case all elements of $\mathcal{T}_{\pi_n(\zeta_1,\dots,\zeta_n)}$ are non-derogatory.

It is well-known that for a given mapping $F \in \mathcal{O}(\Omega_n, \Omega_n)$ there exists a mapping $\tilde{F} \in \mathcal{O}(\mathbb{G}_n, \mathbb{G}_n)$ such that $\sigma(F(A)) = \tilde{F}(\sigma(A))$ (see e.g. [Edi-Zwo 2005]).

If $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, then one may well-define the holomorphic mapping $F_f : \Omega_n \ni A \mapsto f(A) := \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} A^j \in \Omega_n$. Note that $\tilde{F}_f(\sigma(A)) = \sigma(F_f(A)) = \pi_n(f(\lambda_1), \dots, f(\lambda_n))$, where $\sigma(A) = \pi_n(\lambda_1, \dots, \lambda_n)$. In particular, $F_a \in \mathrm{Aut}(\Omega_n)$ for any $a \in \mathrm{Aut}\,\mathbb{D}$. On the other hand, the function F_B, where $B(\lambda) = \lambda^2$, $\lambda \in \mathbb{D}$, is a mapping of the form $\Omega_2 \ni A \mapsto A^2 \in \Omega_2$, which is not a proper holomorphic one – it maps $\mathcal{T}_0$ into 0.

The structure of the group of automorphisms of Ω_n has been been studied in several papers (see e.g. [Ran-Whi 1991] and [Ros 2003]). However, it is still not completely understood. Let us mention only that $\mathrm{Aut}(\Omega_n)$ is not transitive. Motivated by the results of the above-mentioned papers, we are going to examine the structure of the class of proper holomorphic self-mappings of the spectral unit ball. It turns out that we get an analogue of the theorem of Alexander on proper holomorphic self-mappings of the Euclidean ball in $\mathbb{C}^n$ stating that there are no non-trivial proper holomorphic self-maps in the unit ball $\mathbb{B}_n$, $n \geq 2$ (see [Ale 1977]).

In this paper we need some properties of proper holomorphic mappings between complex analytic sets that could be found in [Chi 1989] and [Loj 1991]. The book [Rud 1980] may serve as another reference on proper holomorphic mappings (mostly between open sets in $\mathbb{C}^n$).

Theorem 1. *Let $F : \Omega_n \mapsto \Omega_n$ be a proper holomorphic mapping, $n \geq 2$. Then F is an automorphism.*

The following necessary form of proper holomorphic mappings of the spectral ball, which is a simple consequence of the description of the set of proper holomorphic self-mappings of the symmetrized polydisc, will be crucial in our considerations and justifies the introduction of condition (1) below.

Proposition 2 (see Theorem 17 in [Edi-Zwo 2005])**.** *Let $F : \Omega_n \mapsto \Omega_n$ be a proper holomorphic mapping. Then there is a non-constant finite Blaschke product B such that $\sigma(F(A)) = \pi_n(B(\zeta_1), \dots, B(\zeta_n))$, where $A \in \Omega_n$ and $\sigma(A) = \pi_n(\zeta_1, \dots, \zeta_n)$, $\zeta_1, \dots, \zeta_n \in \mathbb{D}$.*

In view of Proposition 2 it is natural that we study below the holomorphic mappings $F : \Omega_n \mapsto \Omega_n$ such that there is a function $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with the property

(1) $\sigma(F(A)) = \pi_n(f(\zeta_1), \dots, f(\zeta_n))$, $A \in \Omega_n$, is such that $\sigma(A) = \pi_n(\zeta_1, \dots, \zeta_n)$.

We start with the following lemma.

Lemma 3. *Let $F \in \mathcal{O}(\Omega_n, \Omega_n)$ be such that $F(0) = 0$ and (1) is satisfied for $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ (then necessarily $f(0) = 0$) with $f'(0) \neq 0$. Then $F'(0)$ is a linear isomorphism (of $\mathcal{M}_n$).*

Proof. Put $\alpha := f'(0)$. Fix $V \in \mathcal{M}_n$. Let $\pi_n(\mu) = \sigma(V)$ for some $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. We first prove that

$$\frac{1}{\alpha} F'(0)(V) \in \sigma(V). \tag{2}$$

Actually, $\sigma(\zeta V) = \pi_n(\zeta\mu)$, $\zeta \in \mathbb{C}$. Consequently,

$$\sigma(F(\zeta V)) = \pi_n(f(\zeta\mu_1), \dots, f(\zeta\mu_n))$$

for sufficiently small $\zeta \in \mathbb{D}$ and then

$$\sigma\left(\frac{F(\zeta V)}{\zeta}\right) = \pi_n\left(\frac{(f(\zeta\mu_1), \dots, f(\zeta\mu_n))}{\zeta}\right).$$

Passing with ζ to 0 we get that $F'(0)(V) \in \mathcal{T}_{\pi_n(\alpha\mu)}$. Therefore, $\Phi := \frac{1}{\alpha}F'(0) : \mathcal{M}_n \mapsto \mathcal{M}_n$ is a linear mapping such that

$$\Phi(\mathcal{T}_z) \subset \mathcal{T}_z, \ z \in \mathbb{C}^n. \tag{3}$$

To finish the proof of the lemma it is sufficient to show that Φ is a monomorphism. Suppose that it does not hold. Then there is an $N \in \mathcal{M}_n$, $N \neq 0$, such that $\Phi(N) = 0$. Because of (3) we get that $N \in \mathcal{T}_0$. But then there is an $M \in \mathcal{T}_0$ such that $N + M \notin \mathcal{T}_0$. In particular,

$$\mathcal{T}_0 \not\ni \Phi(N + M) = \Phi(N) + \Phi(M) = \Phi(M) \in \mathcal{T}_0,$$

a contradiction. □

Lemma 4. *Let $F \in \mathcal{O}(\Omega_n, \Omega_n)$ be such that (1) is satisfied with $f(0) = 0$ and $f'(0) \neq 0$. Then $F^{-1}(0) \cap \mathcal{T}_0 \subset \{0\}$.*

Proof. Suppose that there is an $A \in \mathcal{T}_0$, $A \neq 0$, such that $F(A) = 0$. It follows from the Jordan decomposition theorem that there are linearly independent vectors $v_1, v_2 \in \mathcal{M}_n$ such that $A(v_2) = v_1$, $A(v_1) = 0$ (at the moment it is essential that $n \geq 2$). Let $(v_1, \dots, v_n)$ be a vector base of $\mathbb{C}^n$. Define the linear mapping $V : \mathbb{C}^n \mapsto \mathbb{C}^n$ (equivalently an element from $\mathcal{M}_n$) as follows: $V(v_2) := v_1$, $V(v_1) := v_2$ and $V(v_j) := 0$, $j = 3, \dots, n$. Then $(A + \zeta V)^2(v_j) = \zeta(1 + \zeta)v_j$, $j = 1, 2$. Consequently, the properties of the spectral radius imply that

$$|\zeta||1 + \zeta| \leq \rho((A + \zeta V)^2) \leq \rho^2(A + \zeta V).$$

For any $\zeta \in \mathbb{C}$ there are $\mu_j(\zeta) \in \mathbb{C}$, $j = 1, \dots, n$, such that $\pi_n(\mu_1(\zeta), \dots, \mu_n(\zeta)) = \sigma(A + \zeta V)$. Then $\sigma(F(A + \zeta V)) = \pi_n(f(\mu_1(\zeta)), \dots, f(\mu_n(\zeta)))$ for $\zeta \in \mathbb{D}$ small. We also know that $\max\{|\mu_j(\zeta)| : j = 1, \dots, n\} = \rho(A + \zeta V) \geq \sqrt{|\zeta|}\sqrt{|1 + \zeta|}$ and $\rho(A + \zeta V) \to 0$ as $\zeta \to 0$.

Note that $\rho(\frac{F(A+\zeta V)}{\zeta}) \to \rho(F'(A)(V))$ as $\zeta \to 0$. But on the other hand,

$$\rho\left(\frac{F(A + \zeta V)}{\zeta}\right) = \max\left\{\frac{|f(\mu_j(\zeta))|}{|\zeta|} : j = 1, \dots, n\right\},$$

which tends to infinity as $\zeta \to 0$ because $f'(0) \neq 0$, a contradiction. □

Note that the results proven so far have referred to a larger class of mappings than only proper ones. It is possible that they may have an application to the study of more general mappings than only the proper holomorphic ones.

First we show simple results on the geometry of the sets $\mathcal{T}_z$.

Lemma 5. *The set of non-derogatory matrices is dense in $\mathcal{T}_z$ for any $z \in \mathbb{G}_n$.*

Proof. Fix $z \in \mathbb{G}_n$. Let $A \in \mathcal{T}_z$. Without loss of generality assume that A is not non-derogatory. Choose a vector base $\mathcal{B}$ in which A has Jordan normal form. Let us study two different blocks corresponding to the same λ (and the corresponding vectors from $\mathcal{B}$: $v_1, \ldots, v_k, w_1, \ldots, w_l$, $k, l \geq 1$). Let $Av_1 = \lambda v_1$, $Av_j = \lambda v_j + v_{j-1}$, $j = 2, \ldots, k$; and $Aw_1 = \lambda w_1$, $Aw_j = \lambda w_j + w_{j-1}$, $j = 1, \ldots, l$. For $\epsilon > 0$ define $Bv_1 := \lambda v_1 + \epsilon w_l$, and for all other elements of the base $\mathcal{B}$ define $Bv := Av$, $v \in \mathcal{B}$, $v \neq v_1$. This easily gives an approximation of A with matrices still in $\mathcal{T}_z$ having one block corresponding to the eigenvalue λ less than in the original matrix. Repeating this procedure for all Jordan blocks having the same eigenvalues, we easily construct a sequence of non-derogatory matrices in $\mathcal{T}_z$ tending to A. $\square$

Lemma 6. *The set of non-derogatory matrices in $\mathcal{T}_z$ is connected and open in $\mathcal{T}_z$ for any $z \in \mathbb{G}_n$.*

Proof. Fix $z \in \mathbb{G}_n$. The non-trivial part of the lemma is the connectedness. Let us fix a system of numbers $(\zeta_1, \ldots, \zeta_n)$ and the sequence of indices $1 = k_1 < k_2 < \ldots < k_{l+1} = n+1$ where $\zeta_{k_j} = \zeta_{k_j+1} = \ldots = \zeta_{k_{j+1}-1}$, $j = 1, \ldots, l$ (and such that no other equalities between different ζ_j's hold) and $\pi_n(\zeta_1, \ldots, \zeta_n) = z$. Now for any vector base $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ we define the matrix A (more precisely, an element in $\mathcal{T}_z \subset \mathcal{M}_n$) as follows: $Av_t = \zeta_j v_t + v_{t-1}$, $j = 1, \ldots, l$, $k_j + 1 \leq t < k_{j+1}$, $Av_{k_j} = \zeta_j v_{k_j}$, $j = 1, \ldots, l$. Note that the above mapping is continuous and its image equals the set of non-derogatory matrices in $\mathcal{T}_{\pi_n(\zeta_1, \ldots, \zeta_n)}$. This together with the fact that the set of all vector basis is connected in $\mathcal{M}_n$ completes the proof. $\square$

As a simple corollary of the results on the set of non-derogatory matrices in the sets $\mathcal{T}_z$ we get the following.

Corollary 7. *For any $z \in \mathbb{G}_n$ the set $\mathcal{T}_z$ is an analytic irreducible set of codimension n.*

At the moment we are ready to move to the proof of our main result.

Proof of Theorem 1. First recall that when $F : \Omega_n \mapsto \Omega_n$ is a proper holomorphic mapping, there is a finite non-constant Blaschke product B such that $\sigma(F(A)) = \pi_n(B(\zeta_1), \ldots, B(\zeta_n))$, where $\sigma(A) = \pi_n(\zeta_1, \ldots, \zeta_n)$. In particular, $F(\mathcal{T}_{\pi_n(\zeta_1, \ldots, \zeta_n)}) \subset \mathcal{T}_{\pi_n(B(\zeta_1), \ldots, B(\zeta_n))}$, $\zeta_j \in \mathbb{D}$, $j = 1, \ldots, n$. But the properness of F implies even that the equality $F(\mathcal{T}_{\pi_n(\zeta_1, \ldots, \zeta_n)}) = \mathcal{T}_{\pi_n(B(\zeta_1), \ldots, B(\zeta_n))}$, $\zeta_j \in \mathbb{D}$, holds – it is sufficient to note that $\mathcal{T}_z$ is always connected. Even more, $F_{|\mathcal{T}_{\pi_n(\zeta_1, \ldots, \zeta_n)}} : \mathcal{T}_{\pi_n(\zeta_1, \ldots, \zeta_n)} \mapsto \mathcal{T}_{\pi_n(B(\zeta_1), \ldots, B(\zeta_n))}$ is open and proper for any $\zeta_j \in \mathbb{D}$, $j = 1, \ldots, n$.

We claim that for any $\lambda_0 \in \mathbb{D}$ such that $B'(\lambda_0) \neq 0$ (note that such points exist) the function

$$F_{|\mathcal{T}_{\pi_n(\lambda_0, \ldots, \lambda_0)}} \text{ is injective.} \tag{4}$$

Actually, making use of the automorphisms of Ω_n and the properties of Blaschke products, we may assume that $\lambda_0 = 0$, $B(0) = 0$ and $B'(0) \neq 0$. It follows from Lemma 4 that $F^{-1}(0) \cap \mathcal{T}_0 = \{0\}$. In particular, $F(0) = 0$. Now Lemma 3 applies, and we get that $F'(0)$ is an isomorphism. Consequently, F is locally invertible near 0. Note that there is a neighborhood $\mathcal{V}$ of 0 such that $\#F^{-1}(A) \cap \mathcal{T}_0 = 1$ for any $A \in \mathcal{V} \cap \mathcal{T}_0$. Otherwise there would exist $\mathcal{T}_0 \ni A^\nu, \tilde{A}^\nu$ such that $A^\nu \neq \tilde{A}^\nu$ and $F(A^\nu) = F(\tilde{A}^\nu) \to 0$. But the properness of F implies that (taking if necessary a subsequence) either both sequences $(A^\nu), (\tilde{A}^\nu)$ converge to 0 or at least one of the sequences converges to a non-zero element $\tilde{A}$ from $\mathcal{T}_0$ such that $F(\tilde{A}) = 0$. In the

first case we contradict the local invertibility of F near 0, and in the second case we get two points in $F^{-1}(0) \cap \mathcal{T}_0$, a contradiction, too.

Now the analyticity of the set $\{A \in \mathcal{T}_0 : \#F^{-1}(A) \cap \mathcal{T}_0 = 1\}$ (see e.g. [Loj 1991], Section V.7.1) (the mapping $F_{|\mathcal{T}_0} : \mathcal{T}_0 \mapsto \mathcal{T}_0$ is proper and open) and the fact that $\mathcal{T}_0$ is a cone show that the mapping $F_{|\mathcal{T}_0} : \mathcal{T}_0 \mapsto \mathcal{T}_0$ is a one-to-one mapping.

Now we prove the following property.

(5) Let $z^\nu \to z^0 \in \mathbb{G}_n$, where $z^\nu \in \mathbb{G}_n$ be such that $F_{|\mathcal{T}_{z^\nu}}$ is not injective for any ν; then $F_{|\mathcal{T}_{z^0}}$ is not injective.

Actually, to prove (5) note that because of the properties of proper holomorphic mappings we may assume that there are two sequences of non-derogatory matrices (A^ν), $(\tilde{A}^\nu)$ with $A^\nu \neq \tilde{A}^\nu$ lying in $\mathcal{T}_{z^\nu}$, $F(A^\nu) = F(\tilde{A}^\nu)$ and tending to matrices $A, \tilde{A} \in \mathcal{T}_{z^0}$ such that A is non-derogatory and $F_{|\mathcal{T}_{z^0}}$ is locally invertible in A. In the case $A \neq \tilde{A}$ we are done, so assume that $A = \tilde{A}$. The local invertibility of $F_{|\mathcal{T}_{z^0}}$ near A implies that there is a neighborhood $\mathcal{U}$ of A such that $F_{|\mathcal{T}_{z^\nu} \cap \mathcal{U}}$ is invertible for ν large enough, which contradicts the equality $F(A^\nu) = F(\tilde{A}^\nu)$.

We claim that

(6) for any $z \in \mathbb{G}_n$ the mapping $F_{|\mathcal{T}_z}$ is injective.

Put $U := \{z \in \mathbb{G}_n : F_{|\mathcal{T}_z} \text{ is injective}\}$. The fact that $F_{|\mathcal{T}_0}$ is injective shows that U is not empty. The property (5) shows that U is open. To see that U is closed in $\mathbb{G}_n$ take a sequence $U \ni z^\nu \to z \in \mathbb{G}_n$. Suppose that $z \notin U$. Then there are different non-derogatory matrices $A_1, \ldots, A_k \in \mathcal{T}_z$, $C \in \mathcal{T}_w$ with $k \geq 2$ such that $F^{-1}(C) \cap \mathcal{T}_z = \{A_1, \ldots, A_k\}$. We may choose arbitrarily small open connected neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_k, \mathcal{V}$ of $A_1, \ldots, A_k, C$ such that $\mathcal{U}_l \cap \mathcal{U}_p = \emptyset$ for $l \neq p$, $\mathcal{U}_j \cap \mathcal{T}_{\tilde{z}}$ is connected, $\mathcal{V} \cap \mathcal{T}_{\tilde{w}}$ is connected for any $\tilde{z}, \tilde{w} \in \mathbb{G}_n$, $j = 1, \ldots, k$, and $F^{-1}(\mathcal{V}) \subset \bigcup_{j=1}^k \mathcal{U}_j$. Consequently, for any ν there are pairwise disjoint sets $F(\mathcal{U}_j \cap \mathcal{T}_{z^\nu}) \cap \mathcal{V}$, $j = 1, \ldots, k$, that are open in $\mathcal{V} \cap \mathcal{T}_{w^\nu}$ and that are non-empty for ν large enough. Now the properness of F shows that for $\mathcal{V}$ sufficiently small the sets $F(\mathcal{U}_j \cap \mathcal{T}_{z^\nu})$, $j = 1, \ldots, k$, cover the whole set $\mathcal{V} \cap \mathcal{T}_{w^\nu}$ for ν large enough, thus contradicting the connectedness of $\mathcal{V} \cap \mathcal{T}_{w^\nu}$.

Since $\mathbb{G}_n$ is connected we get that $U = \mathbb{G}_n$, so (6) is satisfied.

Let m denote the degree of B. We claim that $m = 1$. Suppose that $m \geq 2$. Note that taking instead of F the composition of many F's we may assume that $m \geq n$. There is a point $\zeta_0 \in \mathbb{D}$ such that $B^{-1}(\zeta_0) = \{\zeta_1, \ldots, \zeta_m\}$. Composing, if necessary, with automorphisms of Ω_n, we may assume that $\zeta_0 = 0$. Recall that $\mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}$ is an $n^2 - n$ dimensional submanifold. Choose $A \in \mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}$ such that $F(A) = 0$. Let $f := F_{|\mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}} : \mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)} \mapsto \mathcal{T}_0$. Then f is a holomorphic bijective mapping. Let us fix a regular point C in $\mathcal{T}_0$. Then the function

$$\varphi : \mathbb{C} \ni \lambda \mapsto f^{-1}(\lambda C) \in \mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}$$

is holomorphic on $\mathbb{C} \setminus \{0\}$ (the points λC, $\lambda \in \mathbb{C} \setminus \{0\}$, are regular in $\mathcal{T}_0$) and continuous at 0 with $\varphi(0) = A$ (use the properness and injectivity of f). Consequently, φ is holomorphic on $\mathbb{C}$. Note that $(F \circ \varphi)(\lambda) = \lambda C$, $\lambda \in \mathbb{C}$, so the tangent space to $\mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}$ at A, i.e. $T_A(\mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)})$, is mapped onto $H := F'(A)(T_A(\mathcal{T}_{\pi_n(\zeta_1,\ldots,\zeta_n)}))$, which contains the vector $F'(A)(\varphi'(0)) = (F \circ \varphi)'(0) = C$. Consequently, H contains all regular points of $\mathcal{T}_0$, so it contains the whole $\mathcal{T}_0$, which

contains $n^2 - n + 1$ linearly independent vectors, contradicting the fact that H is at most an $n^2 - n$ dimensional vector space.

Consequently, we have proven that $\#F^{-1}(C) = 1$ for $C \in \Omega_n$, showing that F is an automorphism. $\square$

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