

**A NEW MAXIMUM PRINCIPLE
OF ELLIPTIC DIFFERENTIAL EQUATIONS
IN DIVERGENCE FORM**

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ABSTRACT. In this paper will be presented a new maximum principle of elliptic differential equations in divergence form which can be regarded as the counterpart of the Alexandroff-Bakelman-Pucci maximum principle of elliptic differential equations in nondivergence form.

1. INTRODUCTION

The Alexandroff-Bakelman-Pucci maximum principle is an important tool for studying elliptic differential equations in nondivergence form. In this paper, we shall find the counterpart of it for elliptic differential equations in divergence form.

Throughout this paper, we assume that $\Omega \subset R^n$ is a bounded domain, $f(x) \in L^p(\Omega)$ and $a^{ij}(x) \in L^\infty(\Omega)$, $i, j = 1, 2, \dots, n$ such that for $x \in \Omega$ a.e., $(a^{ij})_{n \times n}$ is symmetric and

$$(1.1) \quad \lambda I \leq (a^{ij})_{n \times n} \leq \Lambda I$$

in the sense of nonnegative definiteness, where $\lambda \leq \Lambda$ are two positive constants. We denote by B_r the open ball in R^n centered at the origin with radius r . For convenience, we extend $(a^{ij}(x))_{n \times n}$ to be the identity outside Ω and still denote the extended matrix by $(a^{ij}(x))_{n \times n}$. Since Ω is bounded, we assume that

$$\Omega \subset B_d$$

for $d > 0$. Let χ_E denote the character function of the set E , and define $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$ for any function f .

To compare with our new maximum principle, we first state our Alexandroff-Bakelman-Pucci maximum principle as the following (cf. [1], [2] for the proof).

Theorem 1.1 (Alexandroff-Bakelman-Pucci maximum principle). *Suppose $f \in L^p(\Omega)$ and $u \in W^{2,p}(\Omega)$ is a subsolution of the following:*

$$(1.2) \quad -a^{ij}(x)\partial_{ij}u(x) = f(x)$$

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in Ω . If $p \geq n$, then there exists a constant C depending only on λ, Λ, p and n such that

$$(1.3) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot d^{2-\frac{n}{p}} \cdot \left(\int_{\{\psi=\Gamma_{\psi}\} \cap \Omega} (f^+)^p \right)^{\frac{1}{p}},$$

where $\psi = u - \sup_{\partial\Omega} u^+$ and Γ_{ψ} is the concave envelope of ψ in B_{2d} (cf. [1] for its definition).

For the following elliptic differential equation in divergence form,

$$(1.4) \quad -\partial_i \left(a^{ij}(x) \partial_j u(x) \right) = f(x) \quad \text{in } \Omega,$$

the estimate of (1.3) cannot be obtained by exploiting the concave envelope Γ_{ψ} as Theorem 1.1 since the concave envelope is appropriate for (1.2) while not for (1.4). However, if we replace the concave envelope by another proper function, a similar estimate of (1.3) can still be obtained. In fact, the proper function is the solution of the following obstacle problem:

$$(1.5) \quad \begin{cases} \text{Find } u \in K_{\psi} \text{ such that} \\ J(u) = \inf_{w \in K_{\psi}} J(w), \end{cases}$$

where

$$J(w) = \int_{B_{2d}} a^{ij} \partial_i w \partial_j w$$

for any $w \in W_0^{1,2}(B_{2d})$ and

$$K_{\psi} = \{w \in W_0^{1,2}(B_{2d}) : w \geq \psi \text{ a.e. in } \Omega\}$$

for any fixed $\psi \in W^{1,2}(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. For Problem (1.5), we have the following well-known results (cf. [3], [4]):

Lemma 1.2. (i) *There exists a unique $\Upsilon_{\psi} \in K_{\psi}$ such that*

$$J(\Upsilon_{\psi}) = \inf_{w \in K_{\psi}} J(w).$$

(ii) *There exists a unique $\tilde{\Upsilon}_{\psi} \in K_{\psi}$ such that*

$$\int_{B_{2d}} a^{ij} \partial_i w \partial_j (w - \tilde{\Upsilon}_{\psi}) \geq 0, \quad \forall w \in K_{\psi}.$$

(iii) *There exists a unique $\hat{\Upsilon}_{\psi} \in K_{\psi}$ such that*

$$\int_{B_{2d}} a^{ij} \partial_i \hat{\Upsilon}_{\psi} \partial_j (w - \hat{\Upsilon}_{\psi}) \geq 0, \quad \forall w \in K_{\psi}.$$

(iv) $\Upsilon_{\psi} = \tilde{\Upsilon}_{\psi} = \hat{\Upsilon}_{\psi}$.

Now we can present our maximum principle for (1.4) as the following:

Theorem 1.3 (Maximum Principle). *Suppose $f \in L^p(\Omega)$ with $p > \frac{n}{2}$ if $n \geq 4$ or $p = 2$ if $n = 1, 2, 3$, and $u \in W^{1,2}(\Omega)$ is a subsolution of (1.4). Then there is a constant C depending only on λ, n and p such that*

$$(1.6) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot d^{2-\frac{n}{p}} \cdot \left(\int_{\{\psi=\Upsilon_{\psi}\} \cap \Omega} (f^+)^p \right)^{\frac{1}{p}},$$

where $\psi = u - \sup_{\partial\Omega} u^+$ and Υ_{ψ} is given by Lemma 1.2.

In section 2, we study the regularity of the solution of the obstacle problem (1.5). In section 3, we shall prove our main result, Theorem 1.3.

2. REGULARITY OF SOLUTIONS OF OBSTACLE PROBLEMS

Our regularity result for the obstacle problem (1.5) is the following:

Theorem 2.1. *Suppose $f \in L^2(\Omega)$ and $\psi \in W^{1,2}(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$ such that*

$$(2.1) \quad -\partial_i \left(a^{ij} \partial_j \psi \right) \leq f$$

in the weak sense in Ω . Then we have

$$(2.2) \quad 0 \leq -\partial_i \left(a^{ij}(x) \partial_j \Upsilon_\psi(x) \right) \leq f^+(x) \cdot \chi_E$$

in the weak sense in B_{2d} , where Υ_ψ is given by Lemma 1.2 and $E = \{x \in \Omega : \psi(x) = \Upsilon_\psi(x)\}$.

If a^{ij}, f and ψ are smooth and $-\partial_i \left(a^{ij} \partial_j \psi \right) = f$ in Ω , Lewy and Stampacchia [3] showed that

$$-\partial_i \left(a^{ij}(x) \partial_j \Upsilon_\psi(x) \right) = f^+(x) \cdot \chi_E$$

for $x \in B_{2d}$ a.e. The idea used in [3] with some appropriate changes is also applicable for the proof of Theorem 2.1. The first part of (2.2) is well known (cf. [3], [4]), and then we only need to show the second part of it, which needs the following two lemmas.

Lemma 2.2. *Suppose $w \in W^{1,2}(\Omega)$ such that $w \geq 0$ on $\partial\Omega$ and*

$$(2.3) \quad -\partial_i \left(a^{ij}(x) \partial_j w(x) \right) \geq f(x)$$

in the weak sense in Ω , where $f \in L^2(\Omega)$. If $w(x) \leq 0$ implies $f(x) \geq 0$ for $x \in \Omega$ a.e., then we have $w \geq 0$ a.e. in Ω .

Proof. Since $w(x) \geq 0$ on $\partial\Omega$, we have $w^- \in W_0^{1,2}(\Omega)$. From (2.3) and the assumption that $w(x) \leq 0$ implies $f(x) \geq 0$ for $x \in \Omega$ a.e., it follows that

$$\int_{\Omega} a^{ij} \partial_i w \partial_j w^- \geq \int_{\Omega} f w^- = \int_E f w^- \geq 0,$$

where $E = \{x \in \Omega : w(x) \leq 0\}$. Thereby,

$$\lambda \int_{\Omega} |Dw^-|^2 \leq \int_{\Omega} a^{ij} \partial_i w^- \partial_j w^- = \int_{\Omega} a^{ij} \partial_i (-w) \partial_j w^- \leq 0.$$

Hence $w^- = 0$ a.e. in Ω . □

Lemma 2.3. *Suppose $f \in L^2(\Omega)$ and θ is a Lipschitz function defined in $(-\infty, +\infty)$ such that $0 \leq \theta \leq 1$. Then the nonlinear equation*

$$(2.4) \quad -\partial_i \left(a^{ij}(x) \partial_j u(x) \right) = f^+(x) \cdot \theta(u - \psi) \cdot \chi_{\Omega}$$

has a solution in $W_0^{1,2}(B_{2d})$ for any $\psi \in L^2(\Omega)$.

Proof. We use the Schauder fixed point theory. For any $w \in L^2(B_{2d})$, the equation

$$(2.5) \quad -\partial_i \left(a^{ij}(x) \partial_j u(x) \right) = f^+(x) \cdot \theta(w - \psi) \cdot \chi_\Omega$$

has a unique weak solution $u \in W_0^{1,2}(B_{2d})$, and we denote $u = Tw$. Since $0 \leq \theta \leq 1$, there exists a constant C depending only on λ and d such that

$$\|Tw\|_{W^{1,2}(B_{2d})} \leq C \|f^+\|_{L^2(\Omega)}, \quad \forall w \in L^2(B_{2d}).$$

Thereby the compactness of T from $L^2(B_{2d})$ to $L^2(B_{2d})$ follows clearly. If we set $D = \{w \in L^2(B_{2d}) : \|w\|_{L^2(B_{2d})} \leq C \|f^+\|_{L^2(\Omega)}\}$, then $T(D) \subset T(L^2(B_{2d})) \subset D$. Since θ is Lipschitz continuous, T is continuous from $L^2(B_{2d})$ to $W_0^{1,2}(B_{2d})$ and then to $L^2(B_{2d})$. Therefore, according to the Schauder fixed point theory, there exists $u \in D$ such that $u = Tu \in W_0^{1,2}(B_{2d})$. \square

Proof of Theorem 2.1. Consider a sequence of functions $\{\theta_m\}_{m=1}^\infty$ defined by

$$\theta_m = \begin{cases} 1 & \text{for } t \leq \frac{1}{2m}, \\ 2 - 2mt & \text{for } \frac{1}{2m} \leq t \leq \frac{1}{m}, \\ 0 & \text{for } t \geq \frac{1}{m}, \end{cases}$$

and let $u_m \in W_0^{1,2}(B_{2d})$ be a solution of (2.4) with $\theta = \theta_m$.

We first claim that $u_m \geq \psi$ a.e. in Ω for any m . Indeed, the maximum principle implies $u_m \geq 0$ a.e. in B_{2d} . In view of $\psi \leq 0$ on $\partial\Omega$, we deduce $u_m - \psi \geq 0$ on $\partial\Omega$. By (2.1) and (2.4) with θ and u replaced by θ_m and u_m , respectively,

$$-\partial_i \left(a^{ij} \partial_j (u_m - \psi) \right) \geq f^+(x) \cdot \theta_m(u_m - \psi) - f$$

in the weak sense in Ω . Since $(f^+ \cdot \theta_m(u_m - \psi) - f)(x) \geq 0$ if $(u_m - \psi)(x) \leq 0$ for $x \in \Omega$ a.e., by Lemma 2.2, our claim is proved.

It is clear that $\{u_m\}$ is bounded in $W_0^{1,2}(B_{2d})$. Without loss of generality, we assume that

$$(2.6) \quad u_m \rightarrow \tilde{u} \text{ weakly in } W_0^{1,2}(B_{2d}), \text{ strongly in } L^2(B_{2d}) \text{ and a.e. in } B_{2d}.$$

We let $\tilde{E} = \{x \in \Omega : \tilde{u}(x) = \psi(x)\}$ and claim that

$$(2.7) \quad \limsup_{m \rightarrow \infty} \theta_m(u_m - \psi) \leq \chi_{\tilde{E}} \text{ a.e. in } \Omega.$$

In fact, by (2.6) and $u_m \geq \psi$ a.e. in Ω , there exists $K \subset \Omega$ with $|K| = 0$ such that $u_m(x) \rightarrow \tilde{u}(x)$ and $u_m(x) \geq \psi(x)$ for all $x \in \Omega \setminus K$. If $x \in \tilde{E}$, then (2.7) holds clearly since $\theta_m \leq 1$. Suppose $x \in \Omega \setminus \{\tilde{E} \cup K\}$ and then $\tilde{u}(x) > \psi(x)$. Since $u_m(x) \rightarrow \tilde{u}(x)$, there exists m_0 such that if $m \geq m_0$, $u_m(x) - \psi(x) > \frac{1}{m}$. Thereby $\theta_m(u_m(x) - \psi(x)) = 0$ for $m \geq m_0$. Hence, (2.7) holds.

Since for any $\phi \in C_0^1(B_{2d})$ with $\phi \geq 0$,

$$\int_{B_{2d}} a^{ij} \partial_i u_m \partial_j \phi = \int_{B_{2d}} f^+ \cdot \theta_m(u_m - \psi) \cdot \chi_\Omega \cdot \phi,$$

by taking the limit in both sides and in view of (2.6), (2.7), we deduce

$$(2.8) \quad \int_{B_{2d}} a^{ij} \partial_i \tilde{u} \partial_j \phi \leq \int_{B_{2d}} f^+ \cdot \chi_{\tilde{E}} \cdot \phi.$$

If we can show that $\tilde{u} = \Upsilon_\psi$, then the proof of Theorem 2.1 will be complete. Indeed, for each m , since θ_m is decreasing, we have for any $v \in W_0^{1,2}(B_{2d})$,

$$\int_{B_{2d}} \left\{ a^{ij} \partial_i (v - u_m) \partial_j (v - u_m) - f^+ \cdot (\theta_m(v - \psi) - \theta_m(u_m - \psi)) \cdot \chi_\Omega \cdot (v - u_m) \right\} \geq 0.$$

Combining with (2.4), where θ and u are replaced by θ_m and u_m respectively, we obtain

$$(2.9) \quad \int_{B_{2d}} \left\{ a^{ij} \partial_i v \partial_j (v - u_m) - f^+ \cdot \theta_m(v - \psi) \cdot \chi_\Omega \cdot (v - u_m) \right\} \geq 0.$$

For any fixed positive integer m_0 , in (2.9), take $v - \psi > \frac{1}{m_0}$ a.e. in Ω and then

$$\int_{B_{2d}} a^{ij} \partial_i v \partial_j (v - u_m) \geq 0$$

for $m > m_0$. By (2.6),

$$(2.10) \quad \int_{B_{2d}} a^{ij} \partial_i v \partial_j (v - \tilde{u}) \geq 0.$$

Since m_0 is arbitrary, we have that (2.10) holds for any $v \in K_\psi$. Then Lemma 1.2 implies $\tilde{u} = \Upsilon_\psi$. \square

3. PROOF OF THEOREM 1.3

To prove Theorem 1.3, we also need the following theorem, which can be proved by Moser iteration (cf. [2]) or by the rearrangement technique (cf. [5]).

Theorem 3.1. *Suppose $f \in L^p(\Omega)$ with $p > \frac{n}{2}$ and $u \in W^{1,2}(\Omega)$ is a subsolution of (1.4). Then there is a constant C depending only on λ, n and p such that*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \cdot r^{2-\frac{n}{p}} \cdot \|f\|_{L^p(\Omega)},$$

where r satisfies $|\Omega| = \omega_n r^n$ and ω_n denotes the volume of the unit ball in R^n .

Remark 3.2. It is clear that the estimate of u by Theorem 1.3 is more accurate than by Theorem 3.1.

Proof of Theorem 1.3. Set $\psi = u - \sup_{\partial\Omega} u^+$, and then (2.1) holds. Let Υ_ψ be given by Lemma 1.2. From Theorem 2.1, it follows that

$$-\partial_i \left(a^{ij} \partial_j \Upsilon_\psi \right) \leq f^+ \cdot \chi_{\{\psi = \Upsilon_\psi\} \cap \Omega}$$

in the weak sense in B_{2d} . In view of Theorem 3.1, there exists a constant C depending only on λ, n and p such that

$$\sup_\Omega \Upsilon_\psi \leq C \cdot d^{2-\frac{n}{p}} \cdot \|f^+ \cdot \chi_{\{\psi = \Upsilon_\psi\} \cap \Omega}\|_{L^p(\Omega)}.$$

Combining with

$$\sup_\Omega u - \sup_{\partial\Omega} u^+ \leq \sup_\Omega \psi \leq \sup_\Omega \Upsilon_\psi,$$

we have that (1.6) holds. \square

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