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SEPARATION OF SPECTRA FOR BLOCK TRIANGLES

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Abstract. Separation of the spectra of the diagonal elements of a block triangle corresponds to comparison with its fundamental projection.

Suppose

$$G = \begin{pmatrix} A & M \\ N & B \end{pmatrix} \equiv \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} : (a, m, n, b) \in A \times M \times N \times B \right\} \tag{0.1}$$

is a Banach algebra with block structure: thus [3] A and B are Banach algebras with identities and M and N are bimodules over A and B and everything is explained by formal matrix multiplication. An *upper triangle* in G is an element of the form

$$T = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}. \tag{0.2}$$

It is well known [2], [3], that of the three spectra

$$\sigma_G(T)\ ,\ \sigma_A(a)\ ,\ \sigma_B(b), \tag{0.3}$$

each is contained in the union of the other two: this extends more generally to "spectral triangles" [4]. Disjointness between the spectra of $a \in A$ and $b \in B$, or significant subsets of them, has consequences expressible [3] in terms of a comparison between the operators

$$S = \begin{pmatrix} a & am - mb \\ 0 & b \end{pmatrix},\ P = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}. \tag{0.4}$$

1. Theorem. *If $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(b))$ is holomorphic near the spectra of $a \in A$ and $b \in B$, then*

$$f \begin{pmatrix} a & am - mb \\ 0 & b \end{pmatrix} = \begin{pmatrix} f(a) & f(a)m - mf(b) \\ 0 & f(b) \end{pmatrix}. \tag{1.1}$$

In particular if and only if S has disjoint diagonal spectra

$$\sigma_A(a) \cap \sigma_B(b) = \emptyset, \tag{1.2}$$

then P is a holomorphic function of S:

$$P = f(S) \textit{ with } f \in \mathrm{Holo}\ \sigma(S)\ . \tag{1.3}$$

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Proof. We are writing $f \in \text{Holo}(K)$, for subsets $K \subseteq \mathbf{C}$, to mean that $f : D_f \to \mathbf{C}$ is defined and holomorphic on some neighbourhood $D_f \in \text{Nbd}(K)$. Generally,

$$f\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f(a) & f^\wedge(a;m;b) \\ 0 & f(b) \end{pmatrix} \tag{1.4}$$

with

$$f^\wedge(a;m;b) = \frac{1}{2\pi i}\oint_{\sigma(a)\cup\sigma(b)} f(z)(z-a)^{-1}m(z-b)^{-1}dz \ ; \tag{1.5}$$

now replace m by $am - mb = m(z-b) - (z-a)m$ for (1.1). For (1.3) take

$$f = \chi_K \ \textit{with}\ K = \sigma(a) \subseteq \sigma(a)\cup\sigma(b) \ ; \tag{1.6}$$

conversely if (1.3) holds, then

$$\sigma_A(a) \subseteq f^{-1}(1)\ ,\ \sigma_B(b) \subseteq f^{-1}(0) \tag{1.7}$$

are necessarily disjoint. □

If in particular $m = 0$, then $S = T$, as in Theorem 1 of [3].

If we write [3], for $f \in \text{Holo}(U, X)$ on open $U \subseteq \mathbf{C}$,

$$\Delta f(\alpha,\beta) = \delta_\alpha f(\beta) = \delta_\beta f(\alpha) = \begin{cases} (f(\beta) - f(\alpha))/(\beta - \alpha) & \beta \neq \alpha \\ f'(\alpha) & \beta = \alpha \end{cases}, \tag{1.8}$$

then if $\sigma(a) \cup \sigma(b) \subseteq U$, we can write

$$f^\wedge(a;m;b) = \Delta f(L_a, R_b)(m)\ , \tag{1.9}$$

at least if $\exists (L_a - R_b)^{-1} \in B(M)$. At another extreme, if $B = A$,

$$f^\wedge(a;m;a) = d_a f(m) \tag{1.10}$$

can be interpreted as a Fréchet differential.

If the projection P is a holomorphic function of the operator S, then it lies in the double commutant of S in G:

2. Theorem. *If $L_a - R_b \in B(M)$ and $L_b - R_a \in B(N)$ are left invertible, then we can factorize*

$$L_S - R_S = \mathbf{W}(L_P - R_P)\ \textit{with}\ \mathbf{W} \in B(G). \tag{2.1}$$

Proof. Replacing 2×2 matrices with 4×1 columns, write

$$\begin{aligned} (L_S - R_S)^\wedge &= \begin{pmatrix} L_a - R_a & 0 & L_{am-mb} & 0 \\ -R_{am-mb} & L_a - R_b & 0 & L_{am-mb} \\ 0 & 0 & L_b - R_a & 0 \\ 0 & 0 & -R_{am-mb} & L_b - R_b \end{pmatrix}, \\ (L_P - R_P)^\wedge &= \begin{pmatrix} 0 & 0 & L_m & 0 \\ -R_m & I & 0 & L_m \\ 0 & 0 & -I & 0 \\ 0 & 0 & -R_m & 0 \end{pmatrix}; \end{aligned} \tag{2.2}$$

now with

$$U(L_a - R_b) = I = V(L_b - R_a), \tag{2.3}$$

take

$$\mathbf{W}^\wedge = \begin{pmatrix} 0 & 0 & L_m V & 0 \\ -UR_m & U & U(L_m + R_m)V & UL_m \\ 0 & 0 & -V & 0 \\ 0 & 0 & -R_m V & 0 \end{pmatrix}. \tag{2.4}$$

□

Sufficient for the left invertibility of $L_a - R_b$ and $L_b - R_a$ is [1] a weaker spectral disjointness:

$$\sigma_A^{left}(a) \cap \sigma_B^{right}(b) = \emptyset = \sigma_A^{right}(a) \cap \sigma_B^{left}(b) \ ; \tag{2.5}$$

this disjointness is in turn necessary if [3] the algebra G is for example "ultraprime". It is sufficient for the double commutant condition that the operator $L_a - R_b$ is one-one:

3. Theorem. *If*

$$(L_a - R_b)^{-1}(0) = \{0\} \subseteq M, \tag{3.1}$$

then

$$(L_S - R_S)^{-1}(0) \subseteq (L_P - R_P)^{-1}(0) \subseteq G \ . \tag{3.2}$$

Proof. Generally,

$$\begin{aligned} (L_S - R_S)\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = O &\Longleftrightarrow a'a - aa' = b'b - bb' = n' = 0 \\ &= am' - m'b - a'(am - mb) + (am - mb)b' \\ &\Longrightarrow n' = mn' = 0 = (L_a - R_b)(m' - a'm + mb') \\ &\Longrightarrow n' = mn' = 0 = m' - a'm + mb', \end{aligned}$$

giving if $(L_a - R_b)^{-1}(0) = \{0\}$,

$$(L_P - R_P)\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = O.$$

□

The condition (3.2) says precisely that the commutant of S is a subalgebra of the commutant of P, or equivalently that P is in the double commutant of S. Sufficient ([3], Theorem 2) is that the left approximate point spectrum of $a \in A$ is disjoint from the right approximate point spectrum of $b \in B$. If the algebra G is "prime", then it is necessary that the corresponding point spectra are disjoint.

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