PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 9, September 2008, Pages 3025–3033
S 0002-9939(08)09293-9
Article electronically published on April 29, 2008

THE "FUNDAMENTAL THEOREM" FOR THE ALGEBRAIC K-THEORY OF SPACES. III. THE NIL-TERM

JOHN R. KLEIN AND E. BRUCE WILLIAMS

(Communicated by Paul Goerss)

Abstract. In this paper we identify the "nil-terms" for Waldhausen's algebraic K-theory of spaces functor as the reduced K-theory of a category of equivariant spaces equipped with a homotopically nilpotent endomorphism.

1. Introduction

This is the third in a series of papers which concerns the decomposition

$$A^{fd}(X \times S^1) \simeq A^{fd}(X) \times \mathcal{B}A^{fd}(X) \times N_- A^{fd}(X) \times N_+ A^{fd}(X)\,.$$

Here, $A^{fd}(X)$ is Waldhausen's algebraic K-theory of the space X and $\mathcal{B}A^{fd}(X)$ is a certain nonconnective delooping of it. The remaining factors on the right, called "nil-terms", are homotopy equivalent [H$_+$], [H$_+$2]. They have not been given a K-theoretic description thus far.

In this installment, we will identify the nil-terms as a shifted copy of the reduced K-theory of a category whose objects are equivariant spaces equipped with a homotopically nilpotent endomorphism.

Let X be a connected based space. Let $G_.$ denote the Kan loop group of the total singular complex of X, and let G denote the geometric realization of $G_.$. Then the classifying space BG has the weak homotopy type of X.

Define a category $\mathrm{nil}(X)$ in which an *object* consists of a pair

$$(Y, f)$$

such that Y is a based space with G-action and $f\colon Y \to Y$ is an equivariant map which is *homotopically nilpotent* under composition. Additionally, we assume that Y admits the structure of a based G-cell complex in which the action of G is free away from the basepoint. A *morphism* $(Y, f) \to (Z, g)$ is a based G-map $e\colon Y \to Z$ such that $g \circ e = e \circ f$.

There is a full subcategory $\mathrm{nil}_{fd}(X)$ of $\mathrm{nil}(X)$ whose objects are those Y which are *finitely dominated* in the sense that Y is a retract up to homotopy of an object which is built up from a point by attaching a finite number of free G-cells. A morphism of $\mathrm{nil}_{fd}(X)$ is a *weak equivalence* if and only if its underlying map of topological spaces is a weak homotopy equivalence. It is a *cofibration* if its underlying map of spaces is obtained up to isomorphism by attaching free G-cells.

Received by the editors May 7, 2007, and, in revised form, July 3, 2007.
2000 *Mathematics Subject Classification.* Primary 19D10; Secondary 19D35.

With the above structure, it turns out that $\mathrm{nil}_{fd}(X)$ is a category with cofibrations and weak equivalences. It therefore has a K-theory, which is denoted $K^{fd}(\mathrm{nil}(X))$.

The forgetful functor $(Y, f) \mapsto Y$ gives rise to a map on K-theories

$$K^{fd}(\mathrm{nil}(X)) \to A^{fd}(X)\,.$$

Here we are using the model for $A^{fd}(X)$ given by the algebraic K-theory of the category of finitely dominated based G-spaces ([W, §2.1], [H$_+$, 1.5]). Let

$$\widetilde{K}^{fd}(\mathrm{nil}(X))$$

denote the homotopy fiber of the map $K^{fd}(\mathrm{nil}(X)) \to A^{fd}(X)$.

Our main result establishes the other half of the "fundamental theorem" for $A^{fd}(X)$:

Main Theorem. *There is a homotopy equivalence of functors*

$$\widetilde{K}^{fd}(\mathrm{nil}(X)) \quad \simeq \quad \Omega N_+ A^{fd}(X)\,.$$

Remark. The above result is used in the paper [GKM], where it is shown that the homotopy groups of $N_+A^{fd}(X)$ are either trivial or infinitely generated. Another result of that paper determines the p-complete homotopy type of $N_+A^{fd}(*)$ in degrees $\leq 4p-7$, for p an odd prime.

2. Preliminaries

In what follows, we assume that the reader is familiar with the material of [H$_+$].

The spaces in this paper are to be given the compactly generated topology. Products are taken in the compactly generated sense. Let $M.$ be a simplicial monoid, and let $M = |M.|$ denote its geometric realization. Let $\mathbb{T}(M)$ denote the category of based (left) M-spaces and based M-maps. We say that a based morphism $Y \to Z$ of $\mathbb{T}(M)$ is *weak equivalence* if (and only if) it is a weak homotopy equivalence of underlying topological spaces. Similarly, we say that is a *fibration* if it is a Serre fibration after forgetting actions. A morphism is a cofibration if and only if it satisfies the left lifting property with respect to the acyclic fibrations (i.e., those fibrations which are weak equivalences). Then $\mathbb{T}(M)$ is a Quillen model category (see, e.g., [VS]).

Then every object of $\mathbb{T}(M)$ is fibrant, and the cofibrant objects are precisely the retracts of those objects which are built up from a point by *cell attachments,* where the cell of dimension n is given by

$$D^n \times M$$

with action defined by left translation.

Recall from [H$_+$] that $\mathbb{C}(M)$ denotes the full subcategory of $\mathbb{T}(M)$ consisting of the cofibrant objects. Then $\mathbb{C}(M)$ is a *category with cofibrations and weak equivalences* in the sense of Waldhausen [W]. For objects Y and Z of $\mathbb{C}(M)$, we let

$$[Y, Z]$$

denote the homotopy classes of morphisms in $\mathbb{C}_{fd}(G)$, i.e., the based equivariant homotopy classes.

We next recall the various finiteness notions. An object of $\mathbb{C}(M)$ is *finite* if it is built up from a point by finitely many cell attachments (up to isomorphism). An object of $\mathbb{C}(M)$ is said to be *homotopy finite* if there exists a weak equivalence to a

finite object. An object of $\mathbb{C}(M)$ is said to be *finitely dominated* if it is a retract of a homotopy finite object. Let $\mathbb{C}_{fd}(M)$ denote the full subcategory of $\mathbb{C}(M)$ whose objects are finitely dominated.

We let $h\mathbb{C}_{fd}(M)$ denote the subcategory of $\mathbb{C}_{fd}(M)$ defined by the weak equivalences. Then the associated K-theory space is given by

$$A^{fd}(*;M) \quad := \quad \Omega|h\mathcal{S}_{\cdot}\mathbb{C}_{fd}(M)|\,,$$

where the right side is the based loop space of the geometric realization of Waldhausen's $\mathcal{S}_{\cdot}$-construction of $\mathbb{C}_{fd}(M)$ ([W, p. 330]). If M is the realization of a simplicial group, then $A^{fd}(*;M)$ is one of the definitions of $A^{fd}(BM)$ (cf. [W, p. 379], [H_+, 1.6]).

The category $\mathrm{nil}_{fd}(X)$ has *objects* specified by pairs (Y,f) with $Y \in \mathbb{C}_{fd}(G)$ and object $f\colon Y \to Y$ a morphism which is homotopically nilpotent under composition, i.e., the associated homotopy class

$$[f] \in [Y,Y]$$

is nilpotent in the sense that some iterate $[f^{\circ k}] = [f]^{\circ k}$ is trivial.

A *morphism* $(Y,f) \to (Z,g)$ of $\mathrm{nil}_{fd}(X)$ is a map $e\colon Y \to Z$ such that $g \circ e = e \circ f$. A *cofibration* of $\mathrm{nil}_{fd}(X)$ is a morphism $(Y,f) \to (Z,g)$ such that $Y \to Z$ is a cofibration of $\mathbb{C}_{fd}(G)$. A *weak equivalence* is a morphism whose underlying map of spaces is a weak homotopy equivalence.

Lemma 2.1. *With respect to the above conventions,* $\mathrm{nil}_{fd}(X)$ *is a category with cofibrations and weak equivalences.*

Proof. The nontrivial thing to be verified is that the cobase change axiom holds. Given a diagram

$$(B,f_1) \leftarrow (A,f_0) \rightarrowtail (C,f_2)$$

we define the pushout to be $(B \cup_A C, f)$, where f denotes $f_1 \cup_{f_0} f_2$. Choose a positive integer k such that $[f_i]^{\circ k}$ is trivial, for $i = 0,1,2$. It will be sufficient to check that $[f]$ is nilpotent. Let us rename $g_i = f_i^{\circ k}$ and $g = f^{\circ k}$. Then, using the model structure, one has a Barratt-Puppe cofiber sequence

$$B \vee C \xrightarrow{\ j\ } B \cup_A C \xrightarrow{\delta} \Sigma A$$

in $\mathbb{T}(M)$, where $\vee$ means wedge and Σ is suspension. Consequently, there is an exact sequence of pointed sets

$$[\Sigma A, B \cup_A C] \xrightarrow{\ \delta^*\ } [B \cup_A C, B \cup_A C] \xrightarrow{\ j^*\ } [B \vee C, B \cup_A C]\,.$$

Then

$$j^*([g]) = [g \circ j] = [g_1 \vee g_2] = 0\,,$$

so there is a homotopy class

$$\gamma \in [\Sigma A, B \cup_A C]$$

such that $[g] = \delta^*(\gamma) = \gamma \circ [\delta]$. Then

$$[g]^{\circ 2} = \gamma \circ [\delta] \circ \gamma \circ [\delta]$$

is trivial because $[\delta]\circ\gamma\circ[\delta] = [\delta]\circ[g]$ coincides with $[\Sigma g_0]\circ[\delta]$, and $[\Sigma g_0]$ is trivial. $\square$

3. Another look at the projective line

Let $\mathbb{N}_-$ denote the monoid of negative integers with generator t^{-1} and $\mathbb{N}_+$ denote the monoid of positive integers with generator t. Let G be the realization of a simplicial group G..

Recall that the *mapping telescope* of an object $Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+)$ is the object $Y_+(t^{-1}) \in \mathbb{C}_{fd}(G \times \mathbb{Z})$ defined by taking the categorical colimit of the sequence

$$\cdots \xrightarrow{t} Y_+ \xrightarrow{t} Y_+ \xrightarrow{t} \cdots .$$

Similarly, if $Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-)$ is an object, we have a mapping telescope $Y_-(t)$ given by the colimit of

$$\cdots \xrightarrow{t^{-1}} Y_+ \xrightarrow{t^{-1}} Y_+ \xrightarrow{t^{-1}} \cdots .$$

Define $\mathbb{D}_{fd}(G \times \mathbb{Z})$ to be the category whose *objects* are diagrams

$$Y_- \to Y \leftarrow Y_+$$

in which $Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-)$, $Y \in \mathbb{C}_{fd}(G \times \mathbb{Z})$ and $Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+)$, and where the maps $Y_- \to Y$ and $Y_+ \to Y$ are required to be based and equivariant. Moreover, the induced morphisms

$$Y_-(t) \to Y(t) \cong Y \quad \text{and} \quad Y_+(t^{-1}) \to Y(t^{-1}) \cong Y$$

are required to be cofibrations of $\mathbb{C}_{fd}(G \times \mathbb{Z})$. We take the liberty of specifying the object as a diagram or as a triple (Y_-, Y, Y_+).

A *morphism* $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{D}_{fd}(G \times \mathbb{Z})$ is a morphism $Y_- \to Z_-$, a morphism $Y \to Z$ and a morphism $Y_+ \to Z_+$ so that the evident diagram commutes. A *cofibration* is a morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ in which

- each of the maps
$$Y_- \to Z_-, \quad Y_+ \to Z_+ \quad \text{and} \quad Y \to Z$$
is a cofibration (of $\mathbb{C}_{fd}(G \times \mathbb{N}_-)$, $\mathbb{C}_{fd}(G \times \mathbb{N}_+)$ resp. $\mathbb{C}_{fd}(G \times \mathbb{Z})$), and
- the induced maps
$$Y \cup_{Y_-(t)} Z_-(t) \to Z \quad \text{and} \quad Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \to Z$$
are cofibrations of $\mathbb{C}_{fd}(G \times \mathbb{Z})$.

The *projective line* $\mathbb{P}_{fd}(G)$ of $[\mathrm{H}_+]$ is given by the full subcategory of $\mathbb{D}_{fd}(G \times \mathbb{Z})$ whose objects (Y_-, Y, Y_+) satisfy an auxiliary condition, viz., that the induced maps $Y_-(t) \to Y$ and $Y_+(t^{-1}) \to Y$ are weak homotopy equivalences. A *cofibration* is a morphism which is a cofibration of $\mathbb{D}_{fd}(G \times \mathbb{Z})$. A *weak equivalence* is a morphism in which $Y_- \to Z_-$, $Y \to Z$ and $Y_+ \to Z_+$ are weak homotopy equivalences of spaces.

Let $\mathbb{D}_{fd}(G \times \mathbb{N}_-) \subset \mathbb{D}_{fd}(G \times \mathbb{Z})$ denote the full subcategory whose objects (Y_-, Y, Y_+) satisfy the condition that $Y_-(t) \to Y$ is a weak equivalence. Similarly, define $\mathbb{D}_{fd}(G \times \mathbb{N}_+)$ to be the full subcategory whose objects (Y_-, Y, Y_+) satisfy the condition that $Y_+(t^{-1}) \to Y$ is a weak equivalence.

A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{D}_{fd}(G \times \mathbb{N}_+)$ is a *weak equivalence* if the map $Y_+ \to Z_+$ is a weak homotopy equivalence. It is a *cofibration* if it is so when considered in $\mathbb{D}_{fd}(G \times \mathbb{Z})$.

Let $\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G) \subset \mathbb{P}_{fd}(G)$ denote the full subcategory with objects (Y_-, Y, Y_+) such that Y_+ is acyclic.

Proposition 3.1. *There is a homotopy fiber sequence*

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|\,.$$

Proof. Define a coarser notion of weak equivalence on the projective line by specifying a morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ to be an $h_{\mathbb{N}_+}$-*equivalence* if (and only if) the map $Y_+ \to Z_+$ is a weak equivalence. Application of the *fibration theorem* [W, 1.6.5] shows that the sequence

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)|$$

is a fibration up to homotopy.

Let $\mathbb{P}_{fd}(G) \to \mathbb{D}_{fd}(G \times \mathbb{N}_+)$ denote the inclusion functor. By [H$_+$, §4] we have that the induced map

$$|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)| \to |h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

induces an isomorphism on homotopy groups in degrees > 1. Hence, the homotopy fiber of the induced map of loop spaces

$$\Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

is homotopically trivial.

It follows that the homotopy fiber of the map

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

is identified with the homotopy fiber of the map

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)|\,.$$

The result follows. □

4. The "characteristic sequence"

Let $(Y, f) \in \mathrm{nil}_{fd}(X)$ be an object, and let $Y \otimes \mathbb{N}_- \in \mathbb{C}_{fd}(G)$ be the object given by

$$(Y \times \mathbb{N}_-)/(* \times \mathbb{N}_-)\,.$$

Then f induces a self-map of $Y \otimes \mathbb{N}_-$ which is given by $(y, r) \mapsto (f(y), r)$. We will denote this self-map also by f.

Let Y_f be the *homotopy coequalizer* of the pair of maps

$$Y \otimes \mathbb{N}_- \underset{t^{-1}}{\overset{f}{\rightrightarrows}} Y \otimes \mathbb{N}_-\,,$$

where t^{-1} denotes the map $(y, r) \mapsto (y, r-1)$. (Recall that the homotopy coequalizer of a pair of morphisms $\alpha, \beta\colon U \to V$ is defined to be the quotient of the disjoint union $V \amalg (U \times [0,1])$ which is given by identifying $(u, 0)$ with $\alpha(u)$, $(u, 1)$ with $\beta(u)$ and $* \times [0,1]$ with the basepoint of V.)

If we give Y the structure of a based $(G \times \mathbb{N}_-)$-space by letting $\mathbb{N}_-$ act by means of f, then we also have a $(G \times \mathbb{N}_-)$-equivariant map

$$\pi_f\colon Y \otimes \mathbb{N}_- \to Y$$

which is given by $(y, r) \mapsto f^{-r}(y)$. Then π_f coequalizes f and t^{-1}, so by the universal property of the homotopy coequalizer, there is an induced map

$$Y_f \to Y\,,$$

which is $(G \times \mathbb{N}_-)$-equivariant.

Lemma 4.1. *The map $Y_f \to Y$ induces an isomorphism in reduced singular homology.*

Proof. Let $p\colon S^1 \to S^1 \vee S^1$ be the pinch map, and let $\rho\colon S^1 \to S^1$ be the reflection map. Then the composite

$$S^1 \xrightarrow{\ p\ } S^1 \vee S^1 \xrightarrow{\mathrm{id}\vee\rho} S^1 \vee S^1$$

will be denoted by $(1,-1)$.

The homotopy coequalizer induces a homotopy cofiber sequence

$$\Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{t^{-1}-f} \Sigma(Y \otimes \mathbb{N}_-) \to \Sigma Y_f$$

where the first map is defined to be the composite

$$\Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{(1,-1)\wedge\mathrm{id}} \Sigma(Y \otimes \mathbb{N}_-) \vee \Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{t^{-1}\vee f} \Sigma(Y \otimes \mathbb{N}_-)\,.$$

Taking reduced singular chains, we get an induced homotopy cofiber sequence of chain complexes

$$C_*(Y) \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t_*^{-1}-f_*} C_*(Y) \otimes \mathbb{Z}[t^{-1}] \longrightarrow C_*(Y_f)\,. \tag{1}$$

(Recall that a sequence $A \xrightarrow{i} B \xrightarrow{j} C$ of chain complexes is a homotopy cofiber sequence when the composite $j \circ i\colon A \to C$ is equipped with a null homotopy such that the induced map from the mapping cone $T_{i\circ j}$ to C is a quasi-isomorphism.)

Now, for any $\mathbb{Z}$-module M equipped with a self-map $f\colon M \to M$, we have an exact sequence of $\mathbb{Z}[t^{-1}]$-modules

$$0 \longrightarrow M \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t^{-1}-f} M \otimes \mathbb{Z}[t^{-1}] \longrightarrow M_f \longrightarrow 0 \tag{2}$$

in which M_f denotes M considered as a $\mathbb{Z}[t^{-1}]$-module where t^{-1} acts via f (see [B, p. 630]). This implies that the sequence (1) becomes exact when $C_*(Y_f)$ is replaced by $C_*(Y)$ by means of the chain map $C_*(Y_f) \to C_*(Y)$ which is induced by the map $Y_f \to Y$. Consequently, the five lemma implies that the chain map $C_*(Y_f) \to C_*(Y)$ is a quasi-isomorphism. □

Remark 4.2. The sequence (1) is a chain complex version of the so-called, "characteristic sequence" (2) of the module M. Consequently, it is not inappropriate to think of the homotopy coequalizer diagram

$$Y \otimes \mathbb{N}_- \underset{t^{-1}}{\overset{f}{\rightrightarrows}} Y \otimes \mathbb{N}_- \longrightarrow Y_f$$

as a kind of nonlinear version of the characteristic sequence (of the object Y).

Preliminary identification of $K(\mathrm{nil}_{fd}(X))$. Define an exact functor

$$\mathrm{nil}_{fd}(X) \xrightarrow{\ \Phi\ } \mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)$$

by

$$(Y, f) \quad \mapsto \quad (Y_f, Y_f(t), *)\,,$$

where Y_f is defined above.

In the other direction, define an exact functor

$$\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G) \xrightarrow{\Psi} \mathrm{nil}_{fd}(X)$$

by

$$(Y_-, Y, Y_+) \quad \mapsto \quad (Y_-, t^{-1}).$$

To see that Ψ is well-defined, let (Y_-, Y, Y_+) be an object of $\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)$. Then Y_+ and Y are acyclic. Hence Y_- has an acyclic mapping telescope. This implies that there exists a $k \in \mathbb{N}_-$ such that $t^k : Y_- \to Y_-$ is (equivariantly) null homotopic (this follows for finite objects by the "small object" argument, and hence for finitely dominated ones since a retract of a null homotopic morphism is again null homotopic; compare [H_+, p. 40 bottom]).

Let Z denote the quotient

$$Y_-/t^k(Y_-)$$

considered as an object of $\mathbb{C}(G)$. Then Z is finitely dominated. This is a consequence of a cell-by-cell induction when Y_- is a finite object of $\mathbb{C}(G \times \mathbb{N}_-)$. It is true for homotopy finite objects because the functor $Y_+ \mapsto Y_+/t^k(Y_-)$ preserves weak equivalences. It is therefore also true when Y_- is finitely dominated since this functor also preserves retracts (cf. [H_+, p. 41 top]).

Since t^k is G-equivariantly null homotopic, the identity map $Y_- \to Y_-$ factors through Z up to homotopy. It follows that Y_- is also finitely dominated when considered as an object of $\mathbb{C}(G)$. This shows that (Y_-, t^{-1}) is an object of $\mathrm{nil}_{fd}(X)$.

Lemma 4.3. *The functors Ψ and Φ induce mutually inverse homotopy equivalences on K-theory.*

Proof. The composite $\Psi \circ \Phi$ is given by

$$(Y, f) \quad \mapsto \quad (Y_f, t^{-1})$$

and Lemma 4.1 implies that there is a morphism $(Y_f, t^{-1}) \to (Y, f)$ which is a weak equivalence after taking a suitable number of suspensions. Since suspension induces a homotopy equivalence on the level of K-theory [W, 1.6.2], it follows that $\Psi \circ \Phi$ induces a homotopy equivalence.

The composite $\Phi \circ \Psi$ is given by

$$(Y_-, Y, Y_+) \quad \mapsto \quad (Y_-, Y_-(t), *).$$

This admits an evident equivalence to the identity functor. Consequently $\Phi \circ \Psi$ induces a map which is homotopic to the identity on the level of K-theory. □

5. Proof of the main theorem

By Lemma 4.3, we have a homotopy equivalence,

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \quad \simeq \quad \Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)|.$$

Plugging this into Proposition 3.1, we obtain a homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|.$$

Let $\epsilon : \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \to \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)|$ denote the *augmentation* map of [H_+, 7.1], which is induced by

$$(Y_-, Y, Y_+) \quad \mapsto \quad Y/\mathbb{Z},$$

where $Y/\mathbb{Z}$ denotes the orbit space under the $\mathbb{Z}$-action. Recall that the *nil-term* $N_+A^{fd}(X)$ was defined to be the homotopy fiber of ϵ. Similarly, ϵ restricts to a map on $\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|$. Denote the homotopy fiber of this restriction by $\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon$. Consequently, we have an induced homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \to N_+A^{fd}(X)\,.$$

In was shown in [H_+, 7.6] that the second of these maps,

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \to N_+A^{fd}(X),$$

is null homotopic. Moreover, it was shown in [H_+, 7.5] that there is a homotopy equivalence

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \quad \simeq \quad \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)|$$

induced by the *global sections* functor $\Gamma\colon \mathbb{P}_{fd}(G) \to \mathbb{C}_{fd}(G)$ defined by

$$(Y_-, Y, Y_+) \quad \mapsto \quad CY_- \cup Y \cup CY_+\,,$$

where CY_- denotes the cone on Y_-.

Assembling this information, we have a homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \xrightarrow{\alpha} \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)| \xrightarrow{\beta} N_+A^{fd}(X) \tag{3}$$

where α is induced by the functor $(Z, f) \mapsto \Sigma Z$. Since the suspension functor $\Sigma\colon \mathbb{C}_{fd}(G) \to \mathbb{C}_{fd}(G)$ induces a homotopy equivalence (by [W, 1.6.2]), we see that the homotopy fiber of α is homotopy equivalent to the homotopy fiber of the map α' which is induced by the forgetful map $(Z, f) \mapsto Z$.

On the one hand, the homotopy fiber of α' is $\widetilde{K}^{fd}(\mathrm{nil}(X))$, by definition. On the other hand, the homotopy fiber sequence (3) implies that the homotopy fiber of α is homotopy equivalent to $\Omega N_+A^{fd}(X)$. We conclude that there is a homotopy equivalence

$$\widetilde{K}^{fd}(\mathrm{nil}(X)) \quad \simeq \quad \Omega N_+A^{fd}(X)\,.$$

This completes the proof of the theorem.

Acknowledgements

An early draft of this paper was circulated in the mid 1990s. A revived interest in understanding nil phenomena [*], [F] was an inducement for us to finally publish. The authors wish to thank the SFB 343 at Bielefeld University and especially Friedhelm Waldhausen for providing the atmosphere that enabled us to accomplish this research.

References

[*] *Nil Phenomena in Topology*, Workshop at Vanderbilt University, Nashville, Tennessee, April 14-15, 2007.

[B] Bass, H: Algebraic K-Theory, Benjamin, New York, 1968. MR0249491 (40:2736)

[F] Farrell, T.: Private Communication, 2006.

[H_+] Hüttemann, T., Klein, J.R., Vogell, W., Waldhausen, F., Williams, B.: The "fundamental theorem" for the algebraic K-theory of spaces. I. *J. Pure Appl. Algebra* **160**, 21–52 (2001). MR1829311 (2002a:19003)

[H_+2] Hüttemann, T., Klein, J.R., Vogell, W., Waldhausen, F., Williams, B.: The "fundamental theorem" for the algebraic K-theory of spaces. II. *J. Pure Appl. Algebra* **167**, 53–82 (2002). MR1868117 (2002i:19003)

[G] Grayson, D.: Higher algebraic K-theory II (after Daniel Quillen), *Algebraic K-theory*, Lecture Notes in Math., vol. 551, Springer, Berlin, 1976, pp. 217–240. MR0574096 (58:28137)

[GKM] Grunewald, J., Klein, J.R., Macko, T.: Operations on the A-theoretic nil-terms, submitted to *Jour. of Topology*, http://arxiv.org/pdf/math/0702580

[VS] Schwänzl, R., Vogt, R.M.: The categories of A_∞- and E_∞-monoids and ring spaces as closed simplicial and topological model categories. *Arch. Math. (Basel)* **56**, 405–411 (1991). MR1094430 (92b:18006)

[W] Waldhausen, F.: Algebraic K-theory of spaces. *Algebraic and Geometric Topology*, Proceedings Rutgers, 1983, Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR802796 (86m:18011)

Department of Mathematics, Wayne State University, Detroit, Michigan 48202
E-mail address: klein@math.wayne.edu

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
E-mail address: williams.4@nd.edu