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# MAXIMAL $n$-ORTHOGONAL MODULES FOR SELFINJECTIVE ALGEBRAS

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Abstract. Let $A$ be a finite-dimensional selfinjective algebra. We show that, for any $n \geq 1$, maximal $n$-orthogonal $A$-modules (in the sense of Iyama) rarely exist. More precisely, we prove that if $A$ admits a maximal $n$-orthogonal module, then *all* $A$-modules are of complexity at most 1.

## 1. Introduction

Recently, O. Iyama introduced maximal $n$-orthogonal modules for finite-dimensional algebras, and developed an extensive theory [19], [20]. One aspect is a 'higher Auslander correspondence', generalizing the classical one-to-one correspondence between algebras of finite representation type and Auslander algebras, that is, algebras of global dimension at most 2 and dominant dimension at least 2. Maximal $n$-orthogonal modules are also called $(n+1)$-cluster tilting [23], and play a crucial role in the categorification of Fomin and Zelevinsky's cluster algebras [16], which was started in [9].

In this paper we are addressing the problem of characterizing those finite-dimensional algebras having maximal $n$-orthogonal modules. Our main result, Theorem 1 below, gives a necessary condition for this to happen, namely that all modules must have complexity at most 1.

The existence of a maximal $n$-orthogonal module of an algebra $A$ has striking consequences for the homological properties of $A$ and its modules. In particular it follows that then the representation dimension of $A$ is at most $n+2$ [20, 5.4.1]. For background on the representation dimension see [1].

Of special interest is the case $n = 1$. Maximal 1-orthogonal modules are known to exist for certain algebras of finite representation type, and also for preprojective algebras of Dynkin type [17]. If $A$ has a maximal 1-orthogonal module, then the representation dimension is at most 3. Using a result of K. Igusa and G. Todorov [18], this implies that the famous finitistic dimension conjecture holds for $A$, that is, there is a finite bound on the projective dimensions of $A$-modules of finite projective dimension.

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If there are maximal 1-orthogonal modules, then usually they are not unique. However, Iyama proved the striking result that the endomorphism rings of any two maximal 1-orthogonal modules of a fixed algebra are tilting equivalent [20, 5.3.3]. Moreover, there is an exchange rule: taking an indecomposable summand $X'$ of a maximal 1-orthogonal module, there is at most one indecomposable module $T$ not isomorphic to $X'$ which can be substituted for $X'$ giving another maximal 1-orthogonal module. For $(n+1)$-tilting objects this was proved in [21] and [22], and a prototype of this goes back to [9] and [17, 4.5].

Maximal 1-orthogonal modules are crucial for the work on cluster algebras of C. Geiß, B. Leclerc and J. Schröer [17]. Cluster algebras were introduced by Fomin and Zelevinsky in [16] to study canonical bases of quantum groups; a central feature is the introduction of an exchange graph. In the approach of [17], the exchange property of maximal 1-orthogonal modules for preprojective algebras of Dynkin type describes the exchange graph for the associated cluster algebra. For details, see [17].

Because of these results, it would be very interesting to know how common maximal 1-orthogonal modules are. However we discovered that for selfinjective algebras they are very rare, and perhaps occur only for the known cases of finite representation type and preprojective algebras of Dynkin type. The aim of this note is to give a proof of this, and also show more generally that for any $n \geq 1$ maximal $n$-orthogonal modules are rare for selfinjective algebras.

We recall the definition of a maximal $n$-orthogonal module for a finite-dimensional algebra $A$, due to Iyama [19]. For an $A$-module $X$, we denote by $\mathrm{add}(X)$ the full subcategory of the module category $\mathrm{mod}\, A$ whose objects are direct summands of direct sums of copies of $X$. A (finitely generated) $A$-module $X$ is called *maximal $n$-orthogonal* if for every $A$-module $M$ the following three conditions are equivalent:

(i) $\mathrm{Ext}^i_A(M, X) = 0$ for all $1 \leq i \leq n$.
(ii) $\mathrm{Ext}^i_A(X, M) = 0$ for all $1 \leq i \leq n$.
(iii) $M \in \mathrm{add}(X)$.

For more details and some examples illustrating this concept, we refer to Section 2.2 below.

The following is the main result of this note, showing that only very few selfinjective algebras can possibly admit maximal $n$-orthogonal modules.

**Theorem 1.1.**[1] *Let $A$ be a finite-dimensional selfinjective algebra, and suppose that for some $n \geq 1$, there exists a maximal $n$-orthogonal $A$-module. Then all $A$-modules have complexity at most* 1.

Recall that the complexity of a module measures the growth of its minimal projective resolution. In particular, a module $M$ has complexity at most 1 precisely when the dimensions of the modules in a minimal projective resolution of $M$ are bounded by a constant. For the precise definition of complexity see Section 4.1 below.

For selfinjective algebras, the most common modules that have complexity at most 1 are the $\Omega$-periodic modules (here $\Omega M$ is the kernel of a minimal projective cover of the module $M$, and $M$ is $\Omega$-periodic if $\Omega^k(M) \cong M$ for some $k \geq 1$).

[1]After having completed this paper, O. Iyama informed us that this result could also be obtained by combining some general facts about maximal $n$-orthogonal subcategories proved in [19].

Let us point out that the existence of maximal 1-orthogonal modules for finite-dimensional preprojective algebras is perfectly in line with our above results. In fact, for preprojective algebras of Dynkin type all modules are $\Omega$-periodic, of period at most 6. This follows from an unpublished result of C. M. Ringel and A. Schofield which shows that these algebras have a periodic bimodule resolution of period $\leq 6$; for a proof see [7, section 4.3]; see also [14], [15, 2.10].

Algebras in this paper are assumed to be finite-dimensional algebras over a field $K$. All modules are finitely generated right modules, and $\operatorname{mod} A$ denotes the category of finitely generated $A$-modules.

## 2. Background and preliminaries

2.1. **Homological algebra for selfinjective algebras.** Let $A$ be a finite-dimensional selfinjective algebra, so that projective modules and injective modules are the same. For a module $M$, we have $\Omega M$, the kernel of a minimal projective cover, and we also have $\Omega^{-1}M$, the cokernel of an injective hull. Then $\Omega$ and $\Omega^{-1}$ induce mutually inverse equivalences of the stable module category of $A$ (see for example [3, chapter IV]). Recall that the stable module category $\underline{\operatorname{mod}}\, A$ has the same objects as $\operatorname{mod} A$, and the morphisms $\underline{\operatorname{Hom}}_A(M,N)$ are equivalence classes of module homomorphisms modulo those factoring through a projective $A$-module. In particular we have for all $k \geq 1$ that

$$\operatorname{Ext}^k_A(M,N) \cong \underline{\operatorname{Hom}}_A(\Omega^k M, N) \cong \underline{\operatorname{Hom}}_A(M, \Omega^{-k} N).$$

2.2. **Maximal $n$-orthogonal modules.** For the convenience of the reader, here we restate Iyama's definition of a maximal $n$-orthogonal module for a finite-dimensional algebra, as already given in the introduction.

An $A$-module $X$ is called *maximal $n$-orthogonal* if for every $A$-module $M$ the following three conditions are equivalent:

(i) $\operatorname{Ext}^i_A(M,X) = 0$ for all $1 \leq i \leq n$.
(ii) $\operatorname{Ext}^i_A(X,M) = 0$ for all $1 \leq i \leq n$.
(iii) $M \in \operatorname{add}(X)$.

If $X$ is maximal $n$-orthogonal, then all projective indecomposable $A$-modules and all injective indecomposable $A$-modules must be summands of $X$. Moreover, $X$ does not have self-extensions, even more, $\operatorname{Ext}^i_A(X,X) = 0$ for $i = 1, \ldots, n$.

We are interested in studying such modules when the algebra $A$ is selfinjective but not semisimple. In that case, $X$ must have at least one indecomposable summand which is not projective (and injective). Namely, otherwise every indecomposable $A$-module $M$ would have to be a summand of $X$ since $\operatorname{Ext}^i_A(X,M) = 0$ for $i = 1, \ldots, n$, and then every indecomposable $A$-module would be projective and then even simple, and $A$ would be semisimple.

Furthermore, if $X$ is a maximal $n$-orthogonal module of a selfinjective algebra $A$, then so is $\Omega^t X \oplus A$ for any $t \in \mathbb{Z}$.

2.2.1. *Some examples.* Here are some easy explicit examples to illustrate the concept.

(1) First, let $A = K[T]/(T^t)$, a truncated polynomial ring. Then $A$ has $t$ indecomposable modules (up to isomorphism), of dimensions $1, 2, \ldots, t$. But all non-projective indecomposable $A$-modules have selfextensions. Hence, there is no maximal $n$-orthogonal $A$-module for any $n \geq 1$.

(2) Let $Q$ be the following quiver:

$$1 \bullet \rightleftarrows \bullet\, 2$$

Set $A = KQ/\operatorname{rad}^2(KQ)$, a four-dimensional selfinjective algebra. This algebra has precisely four indecomposable modules, namely two simple modules $S_1$ and $S_2$, and two indecomposable projectives $P_1$ and $P_2$ with $P_i$ the projective cover of $S_i$. Let $X := P_1 \oplus P_2 \oplus S_1$; then $X$ is a maximal 1-orthogonal $A$-module. In fact, $\operatorname{Ext}^1_A(X, S_2) \cong \operatorname{Ext}^1_A(S_1, S_2) \neq 0$ and $\operatorname{Ext}^1_A(S_2, X) \cong \operatorname{Ext}^1_A(S_2, S_1) \neq 0$.

Then $X' := P_1 \oplus P_2 \oplus S_2$ also is a maximal 1-orthogonal $A$-module, since $X' = \Omega X \oplus A$.

(3) We consider the quiver $Q$ as in (2), and now let $A = KQ/\operatorname{rad}^3(KQ)$. Then the selfinjective algebra $A$ has six indecomposable modules, namely the simple modules $S_1, S_2$, their projective covers $P_1$, $P_2$, and furthermore a 2-dimensional module $U_{1,2}$ with top $S_1$, and a 2-dimensional module $U_{2,1}$ with top $S_2$. Note that $U_{1,2} \cong \Omega S_2$ and $U_{2,1} \cong \Omega S_1$. Suppose we have a maximal 1-orthogonal $A$-module $X$; then $X$ must have at least one non-projective indecomposable summand. We may assume that it has a simple summand (otherwise we replace $X$ by $\Omega^{-1}X \oplus A$). At most one of the simples can be a summand of $X$ (since $S_1$ and $S_2$ have a non-split extension). Suppose, say, $S_1$ is a summand of $X$. Now, for any indecomposable non-projective module $M \neq S_1$, we have $\operatorname{Ext}^1_A(M, S_1) \neq 0$ or $\operatorname{Ext}^1_A(S_1, M) \neq 0$. So $X$ can have no further non-projective summands. On the other hand,

$$\operatorname{Ext}^1_A(X, U_{1,2}) \cong \operatorname{Ext}^1_A(S_1, U_{1,2}) \cong \underline{\operatorname{Hom}}_A(U_{2,1}, U_{1,2}) = 0,$$

a contradiction, since $X$ is assumed to be maximal 1-orthogonal. Hence, $A$ does not have a maximal 1-orthogonal module.

However, it can be seen by similar calculations as above that for every non-projective indecomposable $A$-module $M$, the module $A \oplus M$ is a maximal 2-orthogonal $A$-module. We leave the details to the reader.

(4) For any natural number $n$ there exists a selfinjective algebra $A$ with an $n$-orthogonal module, as the following example shows.

For any $n \geq 1$, let $Q$ be the cyclic (oriented) quiver with $m := 2n + 2$ vertices. Then consider the selfinjective algebra $A = KQ/\operatorname{rad}^2(KQ)$. Note that the indecomposable $A$-modules are the projectives $P_0, P_1, \ldots, P_{m-1}$ and the simple modules $S_0, S_1, \ldots, S_{m-1}$. We label the simple modules so that $P_i$ has socle $S_{i+1}$, with indices taken modulo $m$. Then it is straightforward to check that the module

$$X := P_0 \oplus P_1 \oplus \ldots \oplus P_{m-1} \oplus S_0 \oplus S_{n+1}$$

is a maximal $n$-orthogonal $A$-module.

In general, it is not at all easy to decide whether or not maximal $n$-orthogonal modules exist. In [20], O. Iyama discusses the case $n = 1$ for selfinjective algebras of finite representation type, and gives a combinatorial reformulation in terms of certain triangulations of regular $m$-gons. This is related to work of Caldero, Chapoton and Schiffler for the case of $A_n$; see [10]. For preprojective algebras of Dynkin type, the existence of maximal 1-orthogonal modules is proved in [17].

2.3. **Auslander-Reiten formula ([2]).** Let $A$ be any finite-dimensional algebra, and let $\tau = D\,Tr$ be the Auslander-Reiten translation. When $A$ is selfinjective, we have $\tau \cong \Omega^2\nu$ where $\nu$ is a Nakayama automorphism of $A$; see [3, Chapter IV, 3.7]. Then for any $A$-modules $M, N$ we have

$$D\,\mathrm{Ext}^1_A(M,N) \cong \underline{\mathrm{Hom}}_A(\tau^{-1}N, M).$$

We will freely use that $\Omega$ and $\nu$ commute for a selfinjective algebra $A$ (see [3, IV.3.5 and IV.3.7]).

## 3. Periodicity of $n$-orthogonal modules

The following theorem is the first crucial step in proving our main result.

**Theorem 3.1.** *Let $A$ be a selfinjective algebra, and, for some $n \geq 1$, let $X$ be a maximal $n$-orthogonal $A$-module. If $Y$ is a direct summand of $X$, then so is $\Omega^{n+2}\nu Y$. Hence every non-projective indecomposable summand of $X$ is $\Omega^{n+2}\nu$-periodic.*

Before embarking on the proof, we make an easy but useful observation.

**Lemma 3.2.** *Let $A$ be a selfinjective algebra. For any $A$-module $M$ and any $i \geq 1$ we have an isomorphism of vector spaces*

$$\mathrm{Ext}^i_A(M,N) \cong \mathrm{Ext}^1_A(N, \Omega^{i+2}\nu M).$$

*Proof.* Using the Auslander-Reiten formula of Section 2.3 and the formula of Section 2.1 we get the following isomorphisms of vector spaces:

$$\begin{aligned}
\mathrm{Ext}^i_A(M,N) &\cong \underline{\mathrm{Hom}}_A(\Omega^i M, N) \cong \underline{\mathrm{Hom}}_A(\Omega M, \Omega^{-i+1}N)\\
&\cong \mathrm{Ext}^1_A(M, \Omega^{-i+1}N) \cong \underline{\mathrm{Hom}}_A(\tau^{-1}\Omega^{-i+1}N, M)\\
&\cong \underline{\mathrm{Hom}}_A(\Omega N, \tau\Omega^i M) \cong \mathrm{Ext}^1_A(N, \tau\Omega^i M)\\
&\cong \mathrm{Ext}^1_A(N, \Omega^{i+2}\nu M)
\end{aligned}$$

where for the last isomorphism we use the fact that, for $A$ a selfinjective algebra, one has $\tau = \Omega^2\nu$. □

Now we are in the position to complete the proof of Theorem 3.1.

*Proof of Theorem* 3.1. Let $X$ be a maximal $n$-orthogonal module for the selfinjective algebra $A$. We consider the $A$-module $\Omega^{n+2}\nu X$. For any $i$ such that $0 < i \leq n$ we obtain

$$\begin{aligned}
\mathrm{Ext}^i_A(X, \Omega^{n+2}\nu X) &\cong \underline{\mathrm{Hom}}_A(\Omega^i X, \Omega^{n+2}\nu X) \cong \underline{\mathrm{Hom}}_A(\Omega X, \Omega^{-i+n+3}\nu X)\\
&\cong \mathrm{Ext}^1_A(X, \Omega^{-i+n+3}\nu X) \cong \mathrm{Ext}^{-i+n+1}_A(X,X)
\end{aligned}$$

where the last isomorphism comes from Lemma 3.2. Note that the superscripts $-i+n+1$ run through the set $\{1, 2, \ldots, n\}$. Since $X$ is maximal $n$-orthogonal, we conclude that $\mathrm{Ext}^{-i+n+1}_A(X,X) = 0$ for all $i = 1, \ldots, n$, which means that $\mathrm{Ext}^i_A(X, \Omega^{n+2}\nu X) = 0$ for $i = 1, \ldots, n$. Using again that $X$ is maximal $n$-orthogonal we deduce that $\Omega^{n+2}\nu X \in \mathrm{add}(X)$.

In particular, if $Y$ is an indecomposable direct summand of $X$, then $\Omega^{n+2}\nu Y$ is also a direct summand of $X$.

This means that $\Omega^{n+2}\nu$ permutes the indecomposable non-projective summands of $X$ (recall that $\Omega^{n+2}\nu$ induces a permutation on the set of non-projective indecomposable $A$-modules). But $X$ has by definition only finitely many indecomposable summands. Hence some power of $\Omega^{n+2}\nu$ is the identity permutation on the non-projective summands of $X$, that is, $X$ is $\Omega^{n+2}\nu$-periodic. □

## 4. Complexity at most 1

4.1. **Complexity.** Let $A$ be a finite-dimensional algebra. For any $A$-module $M$, let

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution. The complexity of $M$ measures the rate of growth of the terms of such a resolution. More precisely, the *complexity of* $M$ is defined as

$$\mathrm{cx}(M) := \inf\{b \in \mathbb{N}_0 \mid \exists\, c > 0 : \dim P_n \leq c\, n^{b-1} \text{ for all } n\}$$

if it exists, otherwise $\mathrm{cx}(M) = \infty$. Note that $\mathrm{cx}(M) = 0$ precisely for modules $M$ having finite projective dimension. Moreover, we have $\mathrm{cx}(M) \leq 1$ if and only if the dimensions of the $P_n$'s are bounded. Clearly, if $M$ is $\Omega$-periodic, then $\mathrm{cx}(M) = 1$. The converse is not true in general; see [24] for a counterexample.

The following well-known result will be useful later.

**Lemma 4.1.** *Let $A$ be a finite-dimensional algebra.*

(a) *Suppose*

$$0 \to L \to M \to N \to 0$$

*is a short exact sequence of $A$-modules. Then, if two of the modules in the sequence have complexity $\leq 1$, then so does the third.*

(b) *Let $\alpha$ be an automorphism of the algebra $A$, and for any $A$-module $M$ let $M_\alpha$ denote the $A$-module with twisted action $m \cdot a := m\alpha(a)$. Then $M$ and $M_\alpha$ have the same complexity.*

To prove Theorem 1.1, we will use the following standard construction which we recall here for the convenience of the reader.

**Lemma 4.2** (Universal extension)**.** *Let $A$ be a finite-dimensional algebra, and let $X$ be an $A$-module with $\mathrm{Ext}^1_A(X,X) = 0$. Moreover, let $V$ be an $A$-module such that $n := \dim \mathrm{Ext}^1_A(X,V) > 0$. Then there exists a short exact sequence*

$$0 \to V \to U \to X^n \to 0$$

*for which $\mathrm{Ext}^1_A(X,U) = 0$.*

*Proof.* ([6], Lemma 2.1). We choose a basis of $\mathrm{Ext}^1_A(X,V)$, say $e_1, \ldots, e_n$. Then we construct the short exact sequence

$$0 \to V \to U \to X^n \to 0$$

such that the pullback under the $i^{th}$ canonical injection $X \to X^n$ is $e_i$. Upon applying $\mathrm{Hom}_A(X, -)$ we get a long exact sequence

$$\ldots \to \mathrm{Hom}_A(X, X^n) \overset{\delta}{\twoheadrightarrow} \mathrm{Ext}^1_A(X,V) \to \mathrm{Ext}^1_A(X,U) \to \underbrace{\mathrm{Ext}^1_A(X,X^n)}_{0} \to \ldots .$$

Note that the map $\delta$ is surjective by construction. Hence, it follows that $\mathrm{Ext}^1_A(X,U) = 0$, as desired. $\square$

*Proof of Theorem 1.1.* Any module which is $\Omega^{n+2}\nu$-periodic has complexity $\leq 1$. So we assume now that $A$ has an indecomposable module $V$ which is not $\Omega^{n+2}\nu$-periodic.

Let $V = U_0$. We inductively construct modules $U_1, U_2, \dots, U_n$ and short exact sequences

$$(\zeta_i) \qquad 0 \to U_{i-1} \to U_i \to (\Omega^{n-i}X)^{r_i} \to 0$$

for $1 \le i \le n$, such that $\mathrm{Ext}^1_A(\Omega^j X, U_i) = 0$ for $n - i \le j \le n-1$.

(a) We first construct $U_1$. If $\mathrm{Ext}^1_A(\Omega^{n-1}X, V) = 0$, then take $U_1 = V \oplus \Omega^{n-1}X$ and $r_1 = 1$. Otherwise, construct the universal extension (see Section 4.2)

$$0 \to V \to U_1 \to (\Omega^{n-1}X)^{r_1} \to 0.$$

Then $\mathrm{Ext}^1_A(\Omega^{n-1}X, U_1) = 0$.

(b) For the inductive step, suppose $U_1, \dots, U_{i-1}$ have been constructed.

If $\mathrm{Ext}^1_A(\Omega^{n-i}X, U_{i-1}) = 0$, then take $U_i = U_{i-1} \oplus \Omega^{n-i}X$ and $r_i = 1$. Otherwise, we construct the universal extension

$$0 \to U_{i-1} \to U_i \to (\Omega^{n-i}X)^{r_i} \to 0.$$

Then by construction we have $\mathrm{Ext}^1_A(\Omega^{n-i}X, U_i) = 0$. Furthermore, for $n - i < j$ we have by the inductive hypothesis that $\mathrm{Ext}^1_A(\Omega^j X, U_{i-1}) = 0$.

Since $X$ is maximal $n$-orthogonal we have for all $k = 0, 1, \dots, n-1$ that

$$\mathrm{Ext}^1_A(\Omega^k X, X) = \mathrm{Ext}^{k+1}_A(X, X) = 0.$$

In particular, in our situation we then know that

$$\mathrm{Ext}^1_A(\Omega^j X, \Omega^{n-i}X) = \mathrm{Ext}^1_A(\Omega^{j-(n-i)}X, X) = 0.$$

By considering the long exact sequence to the previous universal extension, one concludes that $\mathrm{Ext}^1_A(\Omega^j X, U_i) = 0$, thus completing the inductive step.

The module $U_n$ satisfies

$$\mathrm{Ext}^{j+1}_A(X, U_n) = \mathrm{Ext}^1_A(\Omega^j X, U_n) = 0$$

for $0 \le j \le n-1$. Since $X$ is maximal $n$-orthogonal it follows that $U_n$ belongs to $\mathrm{add}(X)$. In particular, $U_n$ is $\Omega^{n+2}\nu$-periodic, by Theorem 3.1, and then has complexity $\le 1$, by Lemma 4.1. Now we use downward induction. In the extension $(\zeta_n)$, the last two terms have complexity $\le 1$ and hence so does the first term, that is, $U_{n-1}$, by Lemma 4.1. For the inductive step, suppose $U_j$ has complexity $\le 1$; then the last two terms in the sequence $(\zeta_j)$ have complexity $\le 1$ and hence so does $U_{j-1}$, again by Section 4.1. The last step shows that $V$ has complexity $\le 1$. $\square$

## 5. Concluding remarks and open questions

5.1. We have proved that if $A$ is selfinjective and $A$ has a maximal $n$-orthogonal module, then all $A$-modules have complexity $\le 1$. For algebras of finite representation type, this is no restriction. On the other hand, it is a very strong restriction in general. One would like to know which algebras of infinite representation type have this property, and whether any such algebra has a maximal $n$-orthogonal module.

Algebras for which every module has $\Omega^3\nu$-period $\le 2$ were classified in [8]. The list consists of the preprojective algebras of Dynkin type, then one series of algebras denoted by $P(\mathbb{L}_n)$ (where $n \ge 2$) which have precisely one simple module with self-extensions, and otherwise certain deformations of these algebras. By [17], preprojective algebras of Dynkin type do have maximal 1-orthogonal modules. On the other hand, they do not have maximal $n$-orthogonal modules for $n \ge 2$ since $\mathrm{Ext}^2(M, M) \ne 0$ for all non-projective modules. It is also easy to see that $P(\mathbb{L}_2)$ (which is of finite type) does not have a maximal 1-orthogonal module. We do not know whether or not $P(\mathbb{L}_n)$ for $n \geq 3$, or the deformations of preprojective algebras of Dynkin type have maximal 1-orthogonal modules.

Tame selfinjective algebras for which all modules are periodic can be found in [4], [5]. Moreover, there are as well the algebras of quaternion type in [11].

5.2. For algebras $A$ of quaternion type, we can show that they cannot have a maximal $n$-orthogonal module, for any $n \neq 2$. Recall that by definition $A$ is symmetric (hence $\tau \cong \Omega^2$ and $\nu \cong \mathrm{id}$) and for any $A$-module $M$ one has $\Omega^4 M \cong M$ (see [11]). First, for any non-projective indecomposable $A$-module $M$ we get from Lemma 3.2

$$\mathrm{Ext}^3_A(M, M) \cong \mathrm{Ext}^1_A(M, \Omega^5 M) \cong \mathrm{Ext}^1_A(M, \Omega M) \cong \underline{\mathrm{Hom}}_A(\Omega M, \Omega M) \neq 0.$$

In particular, $A$ cannot have a maximal $n$-orthogonal module where $n \geq 3$.

Secondly, suppose $X$ is a maximal 1-orthogonal $A$-module. Then by Lemma 3.2 we have

$$\mathrm{Ext}^1_A(X, \Omega^{-1} X) \cong \mathrm{Ext}^1_A(X, X) = 0.$$

Hence, for any non-projective indecomposable summand $Y$ of $X$, we also have $\Omega^{-1} Y \in \mathrm{add}(X)$. But on the other hand,

$$\mathrm{Ext}^1_A(\Omega^{-1} X, X) \cong \underline{\mathrm{Hom}}_A(X, X) \neq 0,$$

which contradicts the maximal 1-orthogonality of $X$.

5.3. Suppose $A$ is the group algebra of a finite group $G$ over a field of prime characteristic $p > 0$. Then $A$ is symmetric, hence selfinjective. If $A$ has a maximal $n$-orthogonal module, then by Theorem 1, all $A$-modules must have complexity $\leq 1$. It is well known that this holds if and only if a Sylow $p$-subgroup of $G$ is cyclic, or generalized quaternion (and then $\mathrm{char} K = 2$).

Suppose a Sylow $p$-subgroup of $G$ is cyclic; then $A$ has finite representation type, and each block of $A$ is stably equivalent to a Nakayama algebra. (Note that the property being maximal $n$-orthogonal is invariant under stable equivalence.) Such $A$ may or may not have maximal $n$-orthogonal modules. For example, let $A$ be the group algebra of the symmetric group $\mathcal{S}_3$ where $p = 3$. Then $A$ is isomorphic to the algebra in Subsection 2.2.1, part (3). So it does not have maximal 1-orthogonal modules but it has maximal 2-orthogonal modules.

Now assume that $K$ has characteristic 2 and a Sylow 2-subgroup of $G$ is quaternion. Then any block of $A$ which is not stably equivalent to a Nakayama algebra is an algebra of quaternion type. In Section 5.2 we have seen that such a block does not have maximal $n$-orthogonal modules for $n \neq 2$. We do not know whether maximal 2-orthogonal modules exist in this case.

5.4. We do not know any algebra of infinite type for which all modules have complexity $\leq 1$ but which has modules which are not $\Omega^{n+2}\nu$-periodic. Such an algebra, if it exists, would have very unusual homological properties. For example, if for such algebra the Nakayama automorphism $\nu$ has finite order, then the finite generation properties Fg1 and Fg2 in [12] must fail, since those imply that $\Omega$-periodicity is the same as having complexity one.

5.5. Maximal $n$-orthogonal modules for an algebra $A$ have all projective indecomposable and all injective indecomposable $A$-modules as direct summands. Since $\mathrm{Ext}^i_A(X,X) = 0$ for $i = 1, \ldots, n$ for any maximal $n$-orthogonal module, there cannot be any non-split extensions between injective and projective $A$-modules. Therefore, when searching for maximal $n$-orthogonal modules it seems very natural to consider selfinjective algebras, as we did in this paper. However, there are non-selfinjective algebras having maximal $n$-orthogonal modules, but the examples we know are all of finite type. It would be interesting to know whether there can exist maximal $n$-orthogonal modules for non-selfinjective algebras $A$ for which not all $A$-modules are of complexity at most 1.

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