TURÁN TYPE INEQUALITIES
FOR HYPERGEOMETRIC FUNCTIONS

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Dedicated to the memory of Professor Alexandru Lupaş

ABSTRACT. In this note our aim is to establish a Turán type inequality for
Gaussian hypergeometric functions. This result completes the earlier result
that G. Gasper proved for Jacobi polynomials. Moreover, at the end of this
note we present some open problems.

1. INTRODUCTION

The famous result of Turán [19], established in 1950, is the following inequality:

$$(1.1) \qquad [P_{n+1}(r)]^2 > P_n(r)P_{n+2}(r) \text{ for all } r \in (-1,1) \text{ and } n = 0, 1, 2, \ldots,$$

where P_n is the Legendre polynomial, that is,

$$P_n(r) := \frac{\mathrm{d}^n}{\mathrm{d}r^n} \left[\frac{(r^2-1)^n}{n!2^n} \right] = {}_2F_1\left(-n, n+1, 1, \frac{1-r}{2} \right).$$

Here ${}_2F_1(a, b, c, r)$ denotes the Gaussian hypergeometric function [4, p. 64], which
for real numbers a, b, c and $c \neq 0, -1, -2, \ldots$ has the infinite series representation

$$(1.2) \qquad {}_2F_1(a, b, c, r) := \sum_{n \geq 0} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{r^n}{n!} \text{ for all } r \in (-1, 1),$$

where $(a)_0 = 1$ for $a \neq 0$ and $(a)_n = a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a)$
for each $n = 1, 2, \ldots$ denotes the well-known Pochhammer (or Appell) symbol.
Turán's inequality established for Legendre polynomials has generated considerable
interest. This classical result has been extended in several directions: ultraspherical,
Laguerre and Hermite polynomials [16], Jacobi polynomials [10, 11], general class
of polynomials [9], Bessel functions of the first kind [15], modified Bessel functions
of the first kind [6, 13, 18], and so forth. This inequality still attracts the attention
of mathematicians, and it is worth mentioning here that recently the inequality
(1.1) was improved by Alzer et al. [2].

Karlin and Szegő in their mammoth work [14] raised the question of determining the explicit range of parameters α and β for which the generalized Turán inequality

$$(1.3) \qquad \left[R_{n+1}^{(\alpha,\beta)}(r)\right]^2 > R_n^{(\alpha,\beta)}(r)R_{n+2}^{(\alpha,\beta)}(r)$$

holds for all $r \in (-1,1)$ and $n = 0,1,2,\ldots$, where

$$R_n^{(\alpha,\beta)}(r) = P_n^{(\alpha,\beta)}(r)/P_n^{(\alpha,\beta)}(1)$$

is the normalized Jacobi polynomial and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial, that is

$$P_n^{(\alpha,\beta)}(r) := \frac{(\alpha+1)_n}{n!}{}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1, \frac{1-r}{2}\right), \quad \alpha,\beta > -1.$$

Clearly we have $R_n^{(0,0)}(r) = P_n(r)$ and

$$(1.4) \qquad R_n^{(\alpha,\beta)}(r) := {}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1, \frac{1-r}{2}\right).$$

In 1962 Szegő [17] proved that (1.3) holds for all $\beta \geq |\alpha|$ and $\alpha > -1$. Gasper [10, 11] improved this result by showing that in fact (1.3) holds if and only if $\beta \geq \alpha > -1$. Now let us suppose that $\beta = 0$ and let us consider the notation $F_a(r) = {}_2F_1(a, c-a, c, r)$, where $r \in (0,1)$, $c = \alpha+1 \in (0,1]$, $a = -n$ and $n = 0,1,2,\ldots$. Using Gasper's result and (1.4) for $\beta = 0$, we obtain that the inequality (1.3) is equivalent to

$$(1.5) \qquad \left[F_{\frac{a_1+a_2}{2}}(r)\right]^2 > F_{a_1}(r) \cdot F_{a_2}(r),$$

where $a_1 = a$ and $a_2 = a-2$. Thus it is natural to ask whenever (1.5) or its reverse holds for other values of a. Our main motivation to write this note was this question, which we answer in the next section. It is worth mentioning here that the positive answer – in the particular case $c = 1$ – to the above question was given in [5, Theorem 2.5], which was motivated by the inequality (1.1). In fact, in Theorem 2.3 we prove a stronger statement, namely that the function $a \mapsto F_a(r)$ is strictly concave on $(0,c)$ for each fixed $r \in (0,1)$ and $c \in (0,1]$. This completes the result of Gasper in the case of $\beta = 0$. At the end of section 2 we formulate some open problems which may be of interest for further research.

2. A TURÁN TYPE INEQUALITY
FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

The following technical lemma improves [5, Lemma 2.2] and is one of the crucial facts in the proof of our main result.

Lemma 2.1. *Let us consider the sequence of functions $f_n(a) = (a)_n(c-a)_n$, where $n = 1,2,\ldots$ and $0 \leq a \leq c \leq 1$. For each $n = 1,2,\ldots$ the following assertions are true:*

(a) *f_n is positive, is increasing on $[0,c/2]$ and is decreasing on $[c/2,c]$;*
(b) *$g_n = f_n/a$ is strictly decreasing on $(0,c]$;*
(c) *$g_1 = f_1/a$ is concave and $g_n = f_n/a$ is strictly concave on $(0,c/2]$ for each $n \geq 2$;*
(d) *f_n is strictly concave on $(0,c)$.*

Proof. (a) Clearly the function $f_1(a) = a(c-a)$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Now suppose that for some $n \geq 2$ the function f_n has the same property. Since $(\alpha)_{n+1} = (\alpha)_n(\alpha + n)$, one has

$$(2.2) \quad f_{n+1}(a) = f_n(a)h_n(a), \text{ where } h_n(a) = (a+n)(c-a+n), \ n = 1, 2, \ldots.$$

Clearly the function h_n for each $n = 1, 2, \ldots$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Thus f_{n+1} has the same property, hence by mathematical induction the required result follows.

(b) First suppose that $a \in (0, c/2]$ and let us write $g_n(a) = u(a)v_n(a)$, where

$$u(a) = [\Gamma(a+1)\Gamma(c-a)]^{-1} \text{ and } v_n(a) = \Gamma(a+n)\Gamma(c-a+n).$$

Then clearly we have

$$[\log u(a)]' = \psi(c-a) - \psi(a+1) < 0, \quad a > 0 \geq (c-1)/2,$$
$$[\log v_n(a)]' = \psi(a+n) - \psi(c-a+n) \leq 0, \quad 0 < a \leq c/2,$$

where we have used the fact that the digamma function $x \mapsto \psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing on $(0, \infty)$, i.e. the gamma function is log-convex. Consequently the function u is strictly decreasing and v_n is decreasing, and thus the function g_n is strictly decreasing, too.

Now assume that $a \in [c/2, c]$. From part (a) f_n is decreasing and thus g_n is clearly strictly decreasing as a product of a decreasing and a strictly decreasing function.

(c) The function $g_1(a) = f_1(a)/a = c - a$ is concave on $(0, c/2]$ and $g_2(a) = f_2(a)/a$ is strictly concave on $(0, c/2]$. Now suppose that g_n is strictly concave, too, for some $n \geq 3$. From (2.2) $g_{n+1}(a) = g_n(a)h_n(a)$, and thus

$$g''_{n+1}(a) = g''_n(a)h_n(a) + 2g'_n(a)h'_n(a) + g_n(a)h''_n(a) < 0,$$

because g_n is strictly decreasing from part (b), and h_n is increasing and strictly concave on $(0, c/2]$. Mathematical induction implies the strict concavity of g_n.

(d) Since $f_n(a) = f_n(c-a)$, it is enough to show the strict concavity of f_n for $a \in (0, c/2]$. First suppose that $n = 1$. Then $f_1(a) = a(c-a)$ is clearly strictly concave on $(0, c/2]$. Now assume that $n \geq 2$. Because from part (c) g_n is strictly concave, one has for each $n = 2, 3 \ldots$ and $a \in (0, c/2]$

$$g''_n(a) = \left(\frac{f_n(a)}{a}\right)'' = \frac{1}{a^3}\left[a^2 f''_n(a) - 2af'_n(a) + 2f_n(a)\right] < 0.$$

From this we have $a^2 f''_n(a) < 2af'_n(a) - 2f_n(a)$. Finally, since g_n from part (b) is strictly decreasing on $(0, c/2]$, we obtain that $\log g_n$ is strictly decreasing, too, and consequently $(\log[f_n(a)/a])' \leq 0$. From this we obtain that $af'_n(a) \leq f_n(a)$, and hence $f''_n(a) < 0$. Thus the proof is complete.   □

Our main result of this section improves [5, Theorem 2.5].

Theorem 2.3. *For $0 < a < c \leq 1$, $r \in (0,1)$ the function $a \mapsto F_a(r) = {}_2F_1(a, c-a, c, r)$ is strictly sub-additive and strictly concave, consequently is strictly log-concave. In particular, for all $a_1, a_2 \in (0, c)$ and $r \in (0, 1)$, we have*

$$\sqrt{F_{a_1}(r)F_{a_2}(r)} \leq \frac{F_{a_1}(r) + F_{a_2}(r)}{2} \leq F_{\frac{a_1+a_2}{2}}(r) \leq F_{\frac{a_1}{2}}(r) + F_{\frac{a_2}{2}}(r),$$

and in each of the above inequalities equality holds if and only if $a_1 = a_2$.

Proof. Using (1.2) clearly we have

$$F_a(r) = \sum_{n \geq 0} \frac{f_n(a)}{(c)_n n!} r^n,$$

where $f_n(a) = (a)_n (c-a)_n$. From part (b) of Lemma 2.1 the function $a \mapsto f_n(a)/a$ is strictly decreasing on $(0, c)$ for each $n = 1, 2, \ldots$, thus clearly f_n is strictly sub-additive. From this we have that for all $a_3, a_4 \in (0, c)$, $a_3 \neq a_4$ and $r \in (0, 1)$

$$F_{a_3 + a_4}(r) = \sum_{n \geq 0} \frac{f_n(a_3 + a_4)}{(c)_n n!} r^n < \sum_{n \geq 0} \frac{f_n(a_3) + f_n(a_4)}{(c)_n n!} r^n = F_{a_3}(r) + F_{a_4}(r),$$

i.e. the function $a \mapsto F_a(r)$ is strictly sub-additive. Now from part (d) of Lemma 2.1 we know that $a \mapsto f_n(a)$ is strictly concave, thus for all $a_1, a_2 \in (0, c)$, $a_1 \neq a_2$, $r \in (0, 1)$ and $\lambda \in (0, 1)$ we have

$$\begin{aligned}
F_{\lambda a_1 + (1-\lambda) a_2}(r) &= \sum_{n \geq 0} \frac{f_n(\lambda a_1 + (1-\lambda) a_2)}{(c)_n n!} r^n \\
&> \sum_{n \geq 0} \frac{\lambda f_n(a_1) + (1-\lambda) f_n(a_2)}{(c)_n n!} r^n = \lambda F_{a_1}(r) + (1-\lambda) F_{a_2}(r),
\end{aligned}$$

i.e. the function $a \mapsto F_a(r)$ is strictly concave, too. Finally, since the concavity is stronger than the log-concavity, the proof is complete. $\qquad\square$

We note that in fact the first and second inequalities in Theorem 2.3 may be confined as inequalities between geometric and arithmetic means. Namely, the first inequality is actually the arithmetic-geometric inequality between the values $F_{a_1}(r)$ and $F_{a_2}(r)$. These suggest the following interesting open problem:

Open problem. If m_1 and m_2 are two-variable means, i.e. for $i = 1, 2$ and for all $x, y, \alpha > 0$ we have $m_i(x, y) = m_i(y, x)$, $m_i(x, x) = x$, $m_i(\alpha x, \alpha y) = \alpha m_i(x, y)$ and $x < m_i(x, y) < y$ whenever $x < y$, then *find conditions on $a_1, a_2 > 0$ and $c > 0$ for which the inequality*

$$m_1(F_{a_1}(r), F_{a_2}(r)) \leq (\geq) F_{m_2(a_1, a_2)}(r)$$

holds true for all $r \in (0, 1)$.

The decreasing homeomorphism $\mu_a : (0, 1) \to (0, \infty)$, defined by

$$\mu_a(r) := \frac{\pi}{2 \sin \pi a} \frac{{}_2F_1(a, 1-a, 1, 1-r^2)}{{}_2F_1(a, 1-a, 1, r^2)},$$

where $a \in (0, 1)$, is the so-called generalized Grötzsch ring function, which appears in Ramanujan's generalized modular equations (see for example [3]). This function and its particular form $\mu_{1/2} = \mu$ play an important role in various fields of mathematics, for example they appear in quasiconformal theory, number theory and in estimates of the hyperbolic distance (see for instance [5, Theorem 4.1]).

Now, for $0 < a < c$ consider the decreasing homeomorphism $\mu_{a,c} : (0, 1) \to (0, \infty)$, defined by

$$\mu_{a,c}(r) := \frac{\Gamma(a)\Gamma(c-a)}{2\Gamma(c)} \frac{{}_2F_1(a, c-a, c, 1-r^2)}{{}_2F_1(a, c-a, c, r^2)},$$

which is a natural extension of μ_a and is called the generalized modulus [12]. In [5, Corollary 2.13] we proved that the function $a \mapsto \mu_a(r)$ is strictly log-convex on $(0, 1)$. Since $\mu_{a,1} = \mu_a$, the next result improves [5, Corollary 2.13].

Corollary 2.4. *For $c \in (0, 1]$ and $r \in (0, 1)$ fixed the function $a \mapsto \mu_{a,c}(r)$ is strictly log-convex on $(0, c)$. In particular, $\mu_{a,c}$ satisfies the reversed Turán type inequality, that is, for all $a_1, a_2 \in (0, c)$ and $r \in (0, 1)$ we have*

$$\left[\mu_{\frac{a_1+a_2}{2},c}(r)\right]^2 \leq \mu_{a_1,c}(r)\mu_{a_2,c}(r),$$

where equality holds if and only if $a_1 = a_2$.

Proof. In [5, Theorem 2.14] we proved that the generalized complete elliptic integral [12]

$$a \mapsto \mathcal{K}_{a,c}(r) := \frac{\Gamma(a)\Gamma(c-a)}{2\Gamma(c)} {}_2F_1(a, c-a, c, 1-r^2)$$

is log-convex, whenever $0 < a < \min\{c, 1\}$ and $r \in (0, 1)$. On the other hand, from Theorem 2.3 we know that the function $a \mapsto {}_2F_1(a, c-a, c, r^2)$ is strictly log-concave on $(0, c)$. Thus, the function $a \mapsto \mu_{a,c}(r)$ is strictly log-convex as a product of a strictly log-convex and log-convex functions.   □

Open Problems.

(a) Recall that from [5, Theorem 2.14] it is known that in fact the function $c \mapsto \mathcal{K}_{a,c}(r)$ is log-convex, too, whenever $0 < a < \min\{c, 1\}$ and $r \in (0, 1)$. This suggest the following question: is the generalized modulus $\mu_{a,c}$ strictly log-convex with respect to the parameter c? Our numerical experiments suggest the validity of the following conjecture: *for each $n \geq 1$ and $0 < a < c$ the function $c \mapsto (c-a)_n/(c)_n$ is strictly concave.* If this result were true, then this would imply that the function $c \mapsto {}_2F_1(a, c-a, c, r^2)$ is strictly concave on (a, ∞) and this in turn would imply that $c \mapsto \mu_{a,c}(r)$ is strictly log-convex on (a, ∞).

(b) It is known that [5, Theorem 2.14] the generalized elliptic integral $\mathcal{K}_{a,c}$ is log-convex as a function of two variables with respect to variables (a, c). This naturally suggest the following question: *is it true that for each $n \geq 1$ and $c > a > 0$ the function $f_n(a, c) = (a)_n(c-a)_n/(c)_n$ is strictly concave as a function of two variable?* If this were true, then it would imply that ${}_2F_1(a, c-a, c, r^2)$ is strictly concave and $\mu_{a,c}(r)$ is strictly log-convex as functions of variables (a, c).

(c) A function f with domain $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $f^{(m)}(x)$ for each $m = 0, 1, 2, \ldots$ and if $(-1)^m f^{(m)}(x) \geq 0$ for all $x > 0$. For more information on complete monotonicity the interested reader is referred to the papers [1], [20] and to the references therein. Non-negative functions with a completely monotone derivative appear in literature as Bernstein functions [7]. It is known [8] that the function

$$x \mapsto \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}$$

is completely monotonic on $(0, \infty)$ for each $a, b > 0$. From this we have that the function

$$c \mapsto \frac{(c-a)_n}{(c)_n} = \frac{\Gamma(c)\Gamma(c-a+n)}{\Gamma(c-a)\Gamma(c+n)}$$

is completely monotonic, too, on $(0, \infty)$ for each $n = 1, 2, \ldots$ and $a < 0$. On the other hand, it is easy to verify that the function $c \mapsto (c-a)_n/(c)_n$ is increasing on (a, ∞) for all $n = 1, 2, \ldots$ and $a > 0$. Thus, in view of part (a) of these open problems we may ask the following: *is it true that the function*

$$x \mapsto \frac{\Gamma(x)\Gamma(x-a+b)}{\Gamma(x-a)\Gamma(x+b)}$$

is a Bernstein function on (a, ∞) for each $a, b > 0$?

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