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NEAR-SYMMETRY IN A_∞ AND REFINED JONES FACTORIZATION

WINSTON OU

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Abstract. We use variants of the Hardy-Littlewood maximal and the Cruz-Uribe–Neugebauer minimal operators to give direct characterizations of A_1 and RH_∞ that clarify their near symmetry and yield elementary proofs of various known results, including Cruz-Uribe and Neugebauer's refinement of the Jones factorization theorem.

1. Introduction

In 1972, B. Muckenhoupt [9] demonstrated that the weights w for which the Hardy-Littlewood maximal operator M was bounded on $L^p(wdx)$ $(1 < p < \infty)$ were those which belonged to the A_p class, i.e., satisfying the condition

$$A_p(w) = \sup_Q (\frac{1}{|Q|}\int_Q w)(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}})^{p-1} < \infty.$$

The classical theory of such A_p weights (whose union $\bigcup_p A_p$ is denoted by A_∞) reached a peak in 1980 with the factorization theorem of P. Jones [8], which stated that $w \in A_p$ if and only if $w = w_0 w_1^{1-p}$ for some $w_0, w_1 \in A_1$, where the limiting class A_1 of weights for which $Mw(x) \leq cw(x)$ a.e. was in turn characterized by Coifman and Rochberg as $A_2 \cap e^{BLO}$ [2] that same year. (For the standard account of the theory, definitions, and notation, see the text of Garcia-Cuerva and Rubio de Francia [4].)

In their elegant 1995 paper [3], Cruz-Uribe and Neugebauer inverted the common view of A_∞ by focusing on the structure of the reverse Hölder classes RH_s (where $w \in RH_s$ if $RH_s(w) := \inf\{ C \,|(\frac{1}{|Q|}\int_Q w^s)^{1/s} \leq C\frac{1}{|Q|}\int_Q w$ for all cubes $Q\} < \infty)$ rather than on the A_p structure. In particular, they showed convincingly that their *minimal* operator $mf(x) = \inf_{Q\ni x}\frac{1}{|Q|}\int_Q |f|$ filled a role with respect to the RH_s structure mirroring exactly that played by the maximal operator with respect to the A_p structure: for example, the class RH_∞, defined analogously with A_1 as $\{w \mid \exists\, c > 0$ s.t. $cmw(x) \geq w(x)$ a.e.$\}$, emerged as the limiting class of the RH_s, $s > 1$; m mapped A_∞ into RH_∞, just as M mapped A_∞ into A_1 [5]; and further, RH_∞ could be characterized as e^{BUO} (and thus $A_1 = \frac{1}{RH_\infty \cap A_2}$; i.e., the limiting classes were nearly reciprocal). Their paper culminated with a symmetric

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version of the Jones factorization that encompassed both A_p and RH_s data: that $w \in A_p \cap RH_s$ if and only if $w = w_0 w_1$, where $w_0 \in A_1 \cap RH_s$ and $w_0 \in A_p \cap RH_\infty$.

In this paper, we use variations of the Hardy-Littlewood maximal and Cruz-Uribe–Neugebauer minimal operators to simplify our understanding of the fundamental, near-symmetric structure of A_∞ and its connection with BMO. We begin in section 2 by introducing the *natural minimal* operator $m^\natural$, which possesses properties identical to the natural maximal operator studied in [10]: in particular, it commutes with the logaritheorem on A_∞ and can be used to characterize BUO. In section 3, we show that these properties form the crux of both a refined version of Coifman-Rochberg's characterization of A_1 as $A_2 \cap e^{BLO}$ and, simultaneously, a direct proof of Cruz-Uribe–Neugebauer's characterization $RH_\infty = e^{BUO}$ that cleanly reveals the source of the asymmetry between the limiting weight classes. In section 4, we show how the above characterizations immediately yield various known properties of A_1 and RH_∞, including, significantly, Cruz-Uribe–Neugebauer's improvement of the Jones factorization theorem. We conclude in section 5 with some comments about the reciprocal nature of the A_p and RH_s structures themselves.

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2. Natural maximal and minimal operators

We first review and extend some previous work [10]. Let us recall the definition of the *natural* maximal operator,

$$M^\natural f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f,$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and in this paper Q will always range over cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. This variant of the Hardy-Littlewood maximal operator was introduced by Bennett [1] in 1982 to study BLO, the functions of bounded lower oscillation, i.e., functions ϕ such that over all cubes Q, $\frac{1}{|Q|}\int_Q \phi - \inf_Q \phi \le C$ (we will use $||\phi||_{BLO}$ to denote the infimum of such C; BUO, the space of functions of bounded upper oscillation, and $||\phi||_{BUO}$ are defined analogously). $M^\natural$ was later re-introduced in [10] as the heart of a simple proof of the boundedness of M on BMO, in which the following two properties were central.

Lemma 2.1 (Commutation [10]). *For $w \in A_\infty$, $0 \le [\log M^\natural - M^\natural \log]w(x) \le \log A_\infty(w)$.*

Lemma 2.2 (Characterization of BLO [1, 10]). *$\phi \in BLO \iff (M^\natural - I)\phi \in L^\infty$, in which case $||\phi||_{BLO} = ||(M^\natural - I)\phi||_\infty$.*

It will be useful later to observe that if we analogously define the natural *minimal* operator as $m^\natural f(x) = \inf_{Q \ni x} \frac{1}{|Q|}\int_Q f$, then clearly $M^\natural(f)(x) = -m^\natural(-f)(x)$. Consequently, we have the following properties, which will be critical in the next section.

Lemma 2.3 (Commutation). *For all $w \in A_\infty$, $0 \le [\log m^\natural - m^\natural \log]w(x) \le \log A_\infty(w)$.*

Lemma 2.4 (Characterization of BUO). *$\phi \in BUO \Longleftrightarrow (I - m^\natural)\phi \in L^\infty$, in which case $||\phi||_{BUO} = ||(I - m^\natural)\phi||_\infty$.*

We also note, digressing slightly, that the observation further immediately implies (see [10]) the following set of results about the behavior of $m^\natural$ and m; thus the symmetry, mentioned in the introduction, between the behavior of m and M on BMO and A_∞ is nearly tautological.

Theorem 2.5. *$m^\natural$ maps BMO boundedly into BUO.*

Corollary 2.6. *m maps BMO boundedly into BUO.*

Theorem 2.7. *Boundedness of $m^\natural : BMO \to BUO$ implies $m(A_\infty) \subset RH_\infty$, where $RH_\infty(mw)$ depends only on $RH_s(w)$ and s.*

3. Asymmetric characterizations of A_1 and RH_∞

We now use the above properties of the natural maximal and minimal operators to give characterizations of A_1 and RH_∞. First we present a somewhat surprising and further, sharp, characterization of A_1, refining the result of Coifman-Rochberg [2] that $w \in A_2 \cap e^{BLO} \Longleftrightarrow w \in A_1$.

Theorem 3.1 (Characterization of A_1). *$w \in A_1 \Longleftrightarrow w \in A_\infty \cap e^{BLO}$. Precisely, $e^{||\log w||_{BLO}} \le A_1(w) \le A_\infty(w)e^{||\log w||_{BLO}}$.*

Proof. $\Rightarrow$) This direction is well-known; we follow [4, p. 157]. $w \in A_1$ implies that for every Q,

$$(\frac{1}{|Q|}\int_Q e^{\log w(x)}\,dx)(\sup_{x\in Q} e^{-\log w(x)}) \le A_1(w),$$

i.e.,

$$(\frac{1}{|Q|}\int_Q e^{\log w(x)}\,dx)(e^{-\inf_{x\in Q}\log w(x)}) \le A_1(w).$$

Then, by Jensen's inequality,

$$e^{\frac{1}{|Q|}\int_Q \log w(x)\,dx - \inf_{x\in Q}\log w(x)} \le A_1(w),$$

for every Q, i.e.,

$$e^{||\log w||_{BLO}} \le \log A_1(w).$$

$\Leftarrow$) Say that $w \in e^{BLO} \cap A_\infty$; $\log w \in BLO$. By the characterization of BLO (Lemma 2.2) above,

$$M^\natural \log w(x) \le \log w(x) + ||\log w||_{BLO} \text{ a.e.}$$

Further, since $w \in A_\infty$, the commutation lemma, Lemma 2.1, for $M^\natural$ implies

$$\log M^\natural w(x) - \log A_\infty(w) \le M^\natural \log w(x) \text{ a.e.}$$

Combining the two yields

$$\log M^\natural w(x) \le \log A_\infty(w) + \log w(x) + ||\log w||_{BLO} \text{ a.e.},$$

i.e.,

$$Mw(x) \le [A_\infty(w)e^{||\log w||_{BLO}}]w(x) \text{ a.e.}$$

Considering the case of constant weights shows the bound to be sharp. $\square$

Applying exactly the same method in the reverse Hölder setting yields a proof of the characterization of Cruz-Uribe–Neugebauer that $RH_\infty = e^{BUO}$, an approach that cleanly reveals how and why the asymmetrical relation between the limiting classes arises. First recall that $RH_\infty(w) = \inf\{c \mid cmw(x) \geq w(x) \text{ a.e.}\}$. It is easy to see that $RH_\infty \subset \bigcap_s RH_s$ (Theorem 4.1, [3]), so that $RH_\infty \subset A_\infty$.

Theorem 3.2 (Characterization of RH_∞)**.** *$w \in RH_\infty \iff w \in e^{BUO}$. Precisely, $RH_\infty(w) \leq e^{||\log w||_{BUO}} \leq A_\infty(w) RH_\infty(w)$.*

Proof. $\Leftarrow$) Say that $w \in e^{BUO}$; $\log w \in BUO$. By the characterization of BUO (Lemma 2.4),

$$m^\natural \log w(x) \geq \log w(x) - ||\log w||_{BUO} \text{ a.e.}$$

Now, although we have not assumed $w \in A_\infty$, we do not need the full commutation lemma, Lemma 2.3, but only that part based on Jensen's inequality, i.e.,

$$\log m^\natural w(x) \geq m^\natural \log w(x) \text{ a.e.}$$

Thus

$$\log m^\natural w(x) \geq \log w(x) - ||\log w||_{BUO} \text{ a.e.,}$$

i.e.,

$$mw(x) \geq e^{-||\log w||_{BUO}} w(x) \text{ a.e.,}$$

as desired. (Notice that this demonstrates $e^{BUO} \subset A_\infty$, as $RH_\infty \subset \bigcap_s RH_s$.)

$\Rightarrow$) Say that $w \in A_\infty \cap RH_\infty$. By the reverse Jensen inequality and the definition of RH_∞, we see that

$$A_\infty(w) e^{\frac{1}{Q}\int_Q \log w} \geq \frac{1}{|Q|}\int_Q w \geq \frac{1}{RH_\infty(w)} w(x);$$

i.e. $\log A_\infty(w) + \log RH_\infty(w) \geq \log w(x) - \frac{1}{Q}\int_Q \log w$ for all $x \in Q$. Taking the supremum over $x \in Q$ shows the BUO norm bounded by $\log A_\infty(w) + \log RH_\infty(w)$.

As in the previous theorem, considering the case of constant weights shows the bounds to be sharp. □

Remark. The characterization of A_1 (Theorem 3.1) is only "somewhat surprising" in that it can be realized as a simple consequence of $A_1 = A_2 \cap e^{BLO}$ and Proposition 4.1 below. Given any $w \in A_\infty \cap e^{BLO}$, by the John-Nirenberg inequality w^ϵ (for any $0 < \epsilon < \frac{1}{2^{n+1}e||\log w||_*}$) will lie in A_2 (see [4], p. 409); thus $w^\epsilon \in A_2 \cap e^{BLO} = A_1$. By Proposition 4.1 (which can be proven independently of Theorem 3.1; see [6]) below, $w \in A_1$. However, this approach, though simple, does not reveal the heart of the asymmetry.

4. Consequences of $A_1 = A_\infty \cap e^{BLO}$ and $RH_\infty = e^{BUO}$

With the above characterizations in hand, various important properties of A_1 and RH_∞ now become transparent. For example, the following two propositions were used in [6] by Johnson and Neugebauer to characterize homeomorphisms preserving A_1 (i.e., h such that $w \circ h \cdot h'^\alpha \in A_1$ for all $w \in A_1$, $0 < \alpha \leq 1$):

Proposition 4.1. *If $w \in A_\infty$ and $w^s \in A_1$ for any $s > 0$, then $w \in A_1$.*

Proof. $w^s \in A_1$ implies $\log w^s = s \log w \in BLO$; since BLO is closed under multiplication by positive scalars, $\log w \in BLO$ also. □

Proposition 4.2. *If $w, w^{-1} \in A_1$, then $w \approx 1$; i.e., w is bounded below away from zero and above.*

Proof. $w, w^{-1} \in A_1$ implies that $\log w$ and $-\log w$ are in BLO; thus $\log w \in L^\infty$. $\square$

On the reverse Hölder side, one has elementary proofs of various properties (first given in [3]) of RH_∞ analogous to those of A_1, the first of which was originally used to demonstrate the characterization $RH_\infty = e^{BUO}$.

Proposition 4.3. *The following are equivalent:*

(1) $w \in RH_\infty$,
(2) $w^{s_0} \in RH_\infty$ *for some* $s_0 > 0$,
(3) $w^s \in RH_\infty$ *for all* $s > 0$.

Proof. (2) $\Longrightarrow$ (3): $w^{s_0} \in RH_\infty = e^{BUO}$ implies $\log w^{s_0} = s_0 \log w \in BUO$. Since BUO is closed under multiplication by positive scalars, $s \log w \in BUO$ also for all $s > 0$. $\square$

Proposition 4.4. *If $w, w^{-1} \in RH_\infty$, then $w \approx 1$; i.e., w is bounded below away from zero and above.*

Proof. $w, w^{-1} \in RH_\infty$ implies $\log w$ and $-\log w$ are in BUO; thus $\log w \in L^\infty$. $\square$

The characterization also cleanly yields Cruz-Uribe–Neugebauer's characterization ([3]) of the multipliers of RH_∞, i.e., those functions ϕ such that $\phi w \in RH_\infty$ for all $w \in RH_\infty$.

Theorem 4.5. *ϕ is a multiplier of $RH_\infty \Longleftrightarrow \phi \in RH_\infty$.*

Proof. $\Rightarrow$) Again obvious, since $w \equiv 1$ is in RH_∞.

$\Leftarrow$) Suppose $\phi = e^f$ and $w = e^g$ are both in $RH_\infty = e^{BUO}$. Since BUO is closed under vector addition, $\phi w = e^{f+g} \in e^{BUO}$. $\square$

The characterization of the multipliers of A_1 (by Johnson and Neugebauer [6]) as $\frac{1}{\bigcap_{p>1} A_p} \cap e^{BLO}$, for its part, was simplified in [3] using the following result, which now becomes a consequence of the duality $w \in A_p \Longleftrightarrow w^{-\frac{1}{p-1}} \in A_{p'}$.

Theorem 4.6 (Cruz-Uribe–Neugebauer)**.** $(p > 1)$. $w \in A_1 \Longleftrightarrow w^{1-p} \in A_p \cap RH_\infty$.

Proof. $\Rightarrow$) $w \in A_1 \Longrightarrow w \in A_{p'} \Longrightarrow w^{1-p} \in A_p$; further, since $A_1 \subset e^{BLO}$, $w^{1-p} \in e^{BUO} = RH_\infty$, so $w^{1-p} \in A_p \cap RH_\infty$.

$\Leftarrow$) $w^{1-p} \in A_p \Longrightarrow w^{(1-p)(1-p')} = w \in A_{p'}$, and $w^{1-p} \in RH_\infty = e^{BUO}$ (see below) implies $w^{(1-p)(1-p')} \in e^{BLO}$; thus $w \in A_\infty \cap e^{BLO} = A_1$. $\square$

In fact, the above is the now much-simplified crux of Cruz-Uribe and Neugebauer's proof of their refined Jones factorization theorem, which we include briefly for completeness. We will need the following fact, a consequence of the Hölder and reverse Hölder inequalities; for convenience, we single out the case $p = 1$ as a corollary.

Theorem 4.7 ([7])**.** $w \in A_p \cap RH_s \Longleftrightarrow w^s \in A_{s(p-1)+1}$.

Corollary 4.8. $w \in A_1 \Longleftrightarrow w^{1/s} \in A_1 \cap RH_s$; $s > 1$.

Theorem 4.9 ([3]). *$w \in A_p \cap RH_s$ if and only if $w = w_0 w_1$ for some $w_0 \in A_1 \cap RH_s$, $w_1 \in A_p \cap RH_\infty$.*

Proof. Let $p > 1$ and $s < \infty$ (the case $p = 1$ or $s = \infty$ is immediate; see [3]). By Theorem 4.7, $w \in A_p \cap RH_s$ if and only if $w^s \in A_{s(p-1)+1}$. In turn, by the (original) Jones factorization, this fact is equivalent to $w^s = v_0 v_1^{-s(p-1)}$ for some $v_0, v_1 \in A_1$; i.e., $w = v_0^{1/s} v_1^{1-p}$ for some $v_0, v_1 \in A_1$. By Corollary 4.8 and Theorem 4.6, respectively, this fact in turn is true if and only if $w = v_0^{1/s} v_1^{1-p}$, where $v_0^{1/s} \in A_1 \cap RH_s$ and $v_1^{1-p} \in A_p \cap RH_\infty$. Take $w_0 = v_0^{1/s}$ and $w_1 = v_1^{1-p}$. $\square$

5. Closing remarks

Given the near-reciprocity between $A_1 = e^{BLO} \cap A_\infty$ and $RH_\infty = e^{BUO}$, one might hope for a similar relation between the *non*-limiting A_p and RH_s classes. We give in closing an extension of Cruz-Uribe–Neugebauer's result that $A_1 = \frac{1}{RH_\infty \cap A_2}$, using the following version of the useful lemma of Strömberg and Wheeden [11].

Lemma 5.1. *$w^s \in A_\infty \iff w \in RH_s$, $s > 1$. Precisely, $[RH_s(w)]^s \le A_\infty(w^s) \le [A_\infty(w) RH_s(w)]^s$.*

Proof. $w^s \in A_\infty$, so $\frac{1}{|Q|}\int_Q w^s \le A_\infty(w^s) e^{\frac{1}{|Q|}\int_Q s \log w}$. By Jensen's inequality, $(\frac{1}{|Q|}\int_Q w^s)^{1/s} \le [A_\infty(w^s)]^{1/s} \frac{1}{|Q|}\int_Q w$; thus $RH_s(w) \le A_\infty(w^s)^{1/s}$.

Conversely, if $w \in RH_s$, then $(\frac{1}{|Q|}\int_Q w^s)^{1/s} \le RH_s(w) \frac{1}{|Q|}\int_Q w$; by the reverse Jensen inequality $(\frac{1}{|Q|}\int_Q w^s)^{1/s} \le RH_s(w) A_\infty(w) e^{\frac{1}{|Q|}\int_Q \log w}$. Thus $(\frac{1}{|Q|}\int_Q w^s) \le [RH_s(w) A_\infty(w)]^s e^{\frac{1}{|Q|}\int_Q \log w^s}$, i.e., $A_\infty(w^s) \le [RH_s(w) A_\infty(w)]^s$. $\square$

Theorem 5.2. *$A_{1+\frac{1}{s}} = \frac{1}{RH_s \cap A_2}$ for all $s > 1$. Precisely, $RH_s(\frac{1}{w}) \le A_{1+\frac{1}{s}}(w) \le A_\infty(w) A_\infty(\frac{1}{w}) RH_s(\frac{1}{w})$.*

Proof. Suppose $w \in A_{1+\frac{1}{s}}$, i.e., $(\frac{1}{|Q|}\int_Q w)(\frac{1}{|Q|}\int_Q w^{-s})^{1/s} \le A_{1+\frac{1}{s}}(w)$. Thus

$$(\frac{1}{|Q|}\int_Q (\frac{1}{w})^s)^{1/s} \le A_{1+\frac{1}{s}}(w) (\frac{1}{|Q|}\int_Q (\frac{1}{w})^{-1})^{-1} \le A_{1+\frac{1}{s}}(w) \frac{1}{|Q|}\int_Q \frac{1}{w}$$

by negative Hölder's inequality; therefore $\frac{1}{w} \in RH_s$ with $RH_s(\frac{1}{w}) \le A_{1+\frac{1}{s}}(w)$.

Conversely, say $\frac{1}{w} \in RH_s \cap A_2$. By Lemma 5.1, $\frac{1}{w} \in RH_s \iff (\frac{1}{w})^s \in A_\infty$, with

$$A_\infty((\frac{1}{w})^s) \le [A_\infty(\frac{1}{w}) RH_s(\frac{1}{w})]^s.$$

So $w, w^{-s} \in A_\infty$; thus ([4], p. 408) $w \in A_{1+\frac{1}{s}}$, with

$$A_{1+\frac{1}{s}}(w) \le A_\infty(w) A_\infty(\frac{1}{w}) RH_s(\frac{1}{w}). \qquad \square$$

We note in passing that if we consider *fractional* reverse Hölder classes for $0 < s < 1$, defined as $w = e^\phi \in RH_s$ if $\sup_Q(\frac{1}{|Q|}\int_Q e^{s[\phi(x) - \phi_Q]} dx)^{1/s} = RH_s(w) < \infty$, it is not difficult to extend the above statement to the full range $0 < s \le \infty$.

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Department of Mathematics, Scripps College, Claremont, California 91711
E-mail address: wcwou@scrippscollege.edu