PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 9, September 2008, Pages 3171–3176
S 0002-9939(08)09460-4
Article electronically published on April 30, 2008

# A NOTE ON GENERATING FUNCTIONS FOR HAUSDORFF MOMENT SEQUENCES

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(Communicated by Andreas Seeger)

Abstract. For functions $f$ whose Taylor coefficients at the origin form a Hausdorff moment sequence we study the behaviour of $w(y) := |f(\gamma + iy)|$ for $y > 0$ ($\gamma \leq 1$ fixed).

## 1. Introduction and statement of the results

A sequence $\{a_k\}_{k\geq 0}$ of non-negative real numbers, $a_0 = 1$, is called a Hausdorff moment sequence if there is a probability measure[1] $\mu$ on $[0,1]$ such that

$$a_k = \int_0^1 t^k \, d\mu(t), \quad k \geq 0,$$

or, equivalently,

$$F(z) = \sum_{k=0}^{\infty} a_k \, z^k \; = \; \int_0^1 \frac{d\mu(t)}{1 - tz},$$

and $F$ is its generating function.

It is well known (Hausdorff [2]) that a sequence $\{a_k\}_{k\geq 0}$ with $a_0 = 1$ is a Hausdorff moment sequence if and only if it is *completely monotone*, i.e.

$$\Delta^n a_k \quad := \quad \Delta^{n-1} a_k - \Delta^{n-1} a_{k+1} \; \geq \; 0, \quad k \geq 0, \; n \geq 1,$$

where $\Delta^0$ is the identity operator: $\Delta^0 a = a$.

Let $\mathcal{T}$ denote the set of such generating functions $F$. They are analytic in the slit domain $\Lambda := \mathbb{C} \setminus [1, \infty)$ and also belong to the set of Pick functions $P(-\infty, 1)$ (see Donoghue [1] for more information on Pick functions).

Wirths [5] has shown that $f \in \mathcal{T}$ implies that the function $zf(z)$ is univalent in the half-plane $\operatorname{Re} z < 1$, and recently the theory of universally prestarlike mappings

Received by the editors July 5, 2007.

2000 *Mathematics Subject Classification.* Primary 30E05, 26A48.

*Key words and phrases.* Hausdorff moment sequences, completely monotone sequences, Pick functions, convolution, polylogarithms.

The first and second authors acknowledge partial support from the German-Israeli Foundation (grant G-809-234.6/2003). The second and third authors received partial support from FONDECYT (grants 1070269 and 7070131) and DGIP-UTFSM (grant 240721).

[1] Here and in the sequel we always assume that the measures are Borel.

has been developed, showing a close link to $\mathcal{T}$; see [4]. Many classical functions belong to $\mathcal{T}$ or are closely related to it. We mention only the polylogarithms

$$Li_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha}, \quad \alpha \geq 0,$$

where $Li_\alpha(z)/z \in \mathcal{T}$ and which we are going to study somewhat closer in the sequel.

The main result in this paper is

**Theorem 1.1.** *For $f \in \mathcal{T}$ we have*

$$\operatorname{Re} \frac{f(\gamma + iy_1)}{f(\gamma + iy_2)} \geq 1, \quad \gamma \in (-\infty, 1], \ 0 < y_1 \leq y_2. \tag{1.1}$$

*This relation does not hold, in general, for $\gamma > 1$.*

Theorem 1.1 has the following immediate consequence.

**Corollary 1.2.** *For $f \in \mathcal{T}$ and $\gamma \in (-\infty, 1]$ fixed, the function $|f(\gamma + iy)|$ is monotonically decreasing with $y > 0$ increasing.*

In the case $\gamma = 0$ Theorem 1.1 admits a slight generalization. It is well-known and easy to verify that $\mathcal{T}$ is invariant under the Hadamard product: if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \in \mathcal{T}, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathcal{T},$$

then also

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k \in \mathcal{T}.$$

**Theorem 1.3.** *For $f, g \in \mathcal{T}$ we have*

$$\operatorname{Re} \frac{(f * g)(iy)}{f(iy)} \geq 1, \quad y > 0.$$

Therefore, under the same assumption,

$$|f(iy)| \leq |(f * g)(iy)|, \quad y > 0. \tag{1.2}$$

For the polylogarithms and $0 < \alpha \leq \beta$ it is clear that $Li_\beta = Li_\alpha * Li_{\beta-\alpha}$, so that we get

**Corollary 1.4.** *For $0 \leq \alpha < \beta$*

$$|Li_\alpha(iy)| \leq |Li_\beta(iy)|, \quad y > 0.$$

This result can also be obtained and even strengthened using Corollary 1.2 and the deeper relation

$$\frac{Li_\alpha}{Li_\beta} \in \mathcal{T}, \quad 0 \leq \alpha \leq \beta,$$

recently established in [4].

For a certain subset of $\mathcal{T}$ we can go one step beyond Corollary 1.2 as far as the behaviour of $|f(iy)|$ for $y > 0$ is concerned.

**Theorem 1.5.** *Let*

$$f(z) = \int_0^1 \frac{\sigma(t)dt}{1-tz}, \tag{1.3}$$

*where* $\sigma \in \mathcal{C}^1((0,1))$ *is positive and with* $t\sigma'(t)/\sigma(t)$ *decreasing. Then, for* $w(y) := |f(iy)|$, *the function* $yw'(y)/w(y)$ *decreases with* $y > 0$ *increasing.*

Fundamental for the proof of Theorem 1.5 is the following result, which is based on a general theorem in [4].

**Theorem 1.6.** *Let* $f$ *be as in Theorem* 1.5. *Then, for* $x \in [0,1]$,

$$\frac{f(z)}{f(xz)} \in \mathcal{T}.$$

One can show that the conclusion of Theorem 1.6 is not generally valid for $f \in \mathcal{T}$. However, for the functions $g_\alpha(z) := \frac{1}{z} Li_\alpha(z),\ \alpha > 0$, we have

$$g_\alpha(z) = \frac{1}{\Gamma(a)} \int_0^1 \frac{\log^{\alpha-1}(1/t)}{1-tz} dt,$$

for which the assumptions of Theorem 1.5 are fulfilled. Thus both Theorem 1.5 and Theorem 1.6 apply to $g_\alpha$.

## 2. Proofs

We first note that the convex set $\mathcal{T}$ satisfies the condition of the main theorem in [3], which for the present case can be stated as follows:

**Lemma 2.1.** *Let* $\lambda_1, \lambda_2$ *be two continuous linear functionals on* $\mathcal{T}$ *and assume that* $0 \notin \lambda_2(\mathcal{T})$. *Then the range of the functional*

$$\lambda(f) := \frac{\lambda_1(f)}{\lambda_2(f)}$$

*over* $\mathcal{T}$ *equals the set*

$$\left\{ \lambda\left( \frac{\rho}{1-t_1 z} + \frac{1-\rho}{1-t_2 z} \right) \ : \ \rho, t_1, t_2 \in [0,1] \right\}.$$

*Proof of Theorem* 1.1. First we note that it is enough to prove (1.1) for $\gamma = 1$ only. This is because $f \in \mathcal{T}$ implies $f(z-\delta)/f(-\delta) \in \mathcal{T}$ for all $\delta > 0$. In Lemma 2.1 we choose $\lambda_j(f) := f(1+iy_j),\ j = 1,2$. Since $\operatorname{Im} f(z) > 0$ for $f \in \mathcal{T}$ and $\operatorname{Im} z > 0$, it is clear that $0 \notin \lambda_2(\mathcal{T})$. Lemma 2.1 now implies that for the proof of Theorem 1.1 we only need to show that the expression

$$\frac{\dfrac{\rho}{1-t_1-it_1y_1} + \dfrac{1-\rho}{1-t_2-it_2y_1}}{\dfrac{\rho}{1-t_1-it_1y_2} + \dfrac{1-\rho}{1-t_2-it_2y_2}}, \quad \rho, t_1, t_2 \in [0,1],$$

is located in the half-plane $\{w \ : \ \operatorname{Re} w \geq 1\}$. To simplify this expression we set $\kappa := (1-\rho)/\rho,\ \tau := y_1/y_2$. Then our claim is

$$\operatorname{Re} q(\kappa, y, \tau, t_1, t_2) \geq 1, \quad \kappa \geq 0,\ y > 0,\ t_1, t_2, \tau \in [0,1],$$

where

$$q(\kappa, y, \tau, t_1, t_2) = \frac{\dfrac{1}{1-t_1-i\tau y t_1} + \dfrac{\kappa}{1-t_2-i\tau y t_2}}{\dfrac{1}{1-t_1-iyt_1} + \dfrac{\kappa}{1-t_2-iyt_2}}.$$

Note that by symmetry we may assume that $t_1 \le t_2$. For fixed $y, \tau, t_1, t_2$ the values of $w(\kappa) := q(\kappa, y, \tau, t_1, t_2)$, $\kappa \ge 0$, form a circular arc connecting the points $w(0) = v(t_1)$ and $w(\infty) = v(t_2)$, where

$$v(t) = \frac{1-t-iyt}{1-t-i\tau yt}.$$

It is easily checked that under our assumptions for $y$ and $\tau$ the function $\operatorname{Re} v(t)$ increases with $t \in [0,1]$ and, in particular, $\operatorname{Re} v(t) \ge \operatorname{Re} v(0) = 1$. This implies that

$$1 \le \operatorname{Re} w(0) \le \operatorname{Re} w(\infty).$$

We will prove that $\operatorname{Re} w'(0) \ge 0$. Once this is done a simple geometric consideration shows that under these circumstances the circular arc $w(\kappa)$, $\kappa \ge 0$, cannot leave the half-plane $\{w : \operatorname{Re} w \ge 1\}$, which then completes the proof of (1.1).

Calculation yields

$$\operatorname{Re} w'(0) = (1-\tau)(t_2-t_1)y^2 \frac{Z}{N},$$

where

$$\begin{aligned} Z &= t_1^* t_2^* (t_2 - t_1) + (t_1 t_2^* + t_2 t_1^*) t_1^* t_2^* \tau + t_1 t_2 y^2 \tau \left( t_1 t_2^* + t_2 t_1^* - \tau (t_2 - t_1) \right), \\ N &= \left( (1-t_1)^2 + (t_1 y \tau)^2 \right) \left( (1-t_2)^2 + (t_2 y \tau)^2 \right) \left( (1-t_2)^2 + (t_2 y)^2 \right), \end{aligned}$$

and $t_j^* := 1 - t_j$. Here all terms are non-negative (note that

$$s(\tau) := t_1 t_2^* + t_2 t_1^* - \tau(t_2 - t_1)$$

decreases with $\tau$ and is therefore not smaller than $s(1) = 2t_1 t_2^* \ge 0$).

It remains to show that (1.1) does not hold, in general, for $\gamma > 1$. Let $\gamma = 1 + \varepsilon$, $\varepsilon > 0$, and choose

$$f(z) := \frac{1}{1+2\varepsilon} + \frac{2\varepsilon}{1+2\varepsilon}\frac{1}{1-z} \in \mathcal{T}.$$

Then, using $y_1 = \varepsilon$, $y_2 = 1$,

$$\operatorname{Re} \frac{f(\gamma + i\varepsilon)}{f(\gamma + i)} = \frac{2\varepsilon}{1+\varepsilon^2} < 1. \qquad \square$$

*Proof of Theorem* 1.3. If

$$g(z) = \int_0^1 \frac{d\mu(t)}{1-tz},$$

then

$$\frac{(f * g)(iy)}{f(iy)} = \int_0^1 \frac{f(ity)}{f(iy)} d\mu(t),$$

which is a convex combination of the values of $f(ity)/f(iy)$. By Theorem 1.1 these are all in the half-plane $\{w : \operatorname{Re} w \ge 1\}$. $\square$

For the proof of Theorem 1.6 we need the following result from [4].

**Lemma 2.2.** *Let $f, g \in \mathcal{T}$ be represented by*

$$f(z) = \int_0^1 \frac{\varphi(t)dt}{1-tz}, \quad g(z) = \int_0^1 \frac{\psi(t)dt}{1-tz}$$

*with non-negative Borel functions $\varphi, \psi$ on $(0,1)$. If $\varphi(t)\psi(s) \geq \varphi(s)\psi(t)$ holds for all $0 < s < t < 1$, then $f/g \in \mathcal{T}$.*

*Proof of Theorem* 1.6. We have

$$f(xz) = \int_0^1 \frac{\sigma(t)dt}{1-txz} = \int_0^1 \frac{\sigma^*(t)dt}{1-tz},$$

with

$$\sigma^*(t) := \begin{cases} \frac{1}{x}\sigma(t/x), & 0 < t \leq x, \\ 0, & x < t < 1. \end{cases}$$

The condition

$$\sigma(t)\sigma^*(s) \geq \sigma(s)\sigma^*(t), \quad 0 < s < t < 1, \tag{2.1}$$

is immediately fulfilled if $t > x$. Otherwise we are left with

$$\sigma(t)\sigma(s/x) \geq \sigma(s)\sigma(t/x), \quad 0 < s < t \leq x.$$

This requires that $\sigma(t)/\sigma(t/x)$ increases with $t$. Taking logarithms and differentiating w.r.t. the variable $t$, we find as a necessary and sufficient condition for (2.1) that $t\sigma'(t)/\sigma(t)$ decreases for $t$ increasing. The result now follows from Lemma 2.2. □

*Proof of Theorem* 1.5. We apply Theorem 1.1 to the function $F$ of Theorem 1.6. Then, for $x, \tau \in (0,1)$, we get

$$\left|\frac{f(iy\tau)f(iyx)}{f(iyx\tau)f(iy)}\right| \geq 1, \quad y > 0.$$

Taking logarithms we obtain

$$(\log w(y) - \log w(xy)) - (\log w(\tau y) - \log w(x\tau y)) \leq 0.$$

Dividing by $1-x$ and letting $x \to 1-0$ yield

$$\frac{yw'(y)}{w(y)} \leq \frac{\tau y w'(\tau y)}{w(\tau y)},$$

which implies the assertion. □

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