ON A PROBLEM OF BERNARD CHEVREAU
CONCERNING THE $\rho$-CONTRACTIONS

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Abstract. We prove new results for the operators of class $C_\rho$ ($\rho > 0$) on
Hilbert spaces defined by B. Sz.-Nagy and C. Foiaș. The main result is an
answer to a problem posed in 2006 by B. Chevreau: Let $p \geq 2$ be a natural
number and $T \in L(\mathcal{H})$; if there exists $\rho_0 > 0$ such that $T^p \in C_{\rho_0}$, then
necessarily is $T \in \bigcup_{\rho > 0} C_\rho$?

1. Introduction

We denote by $\mathcal{H}$ a Hilbert space over field $\mathbb{C}$ of all complex numbers. Let $L(\mathcal{H})$
denote the algebra of all bounded operators on $\mathcal{H}$. $I_\mathcal{H}$ is the identity operator on
$\mathcal{H}$. For $\rho > 0$, we denote by $C_\rho(\mathcal{H})$ the class of operators which admits a unitary $\rho$-
dilation: an operator $T \in L(\mathcal{H})$ admits a unitary $\rho$-dilation if there exists a Hilbert
space $K \supset \mathcal{H}$ and $U \in L(K)$ unitary such that

$$T^n = \rho PU^n|_{\mathcal{H}}, \ n \geq 1,$$

where $P$ is the orthogonal projection of $K$ on $\mathcal{H}$.

If $T \in C_\rho(\mathcal{H})$, then $T$ is also called a $\rho$-contraction. The classes $C_\rho$ were
introduced by B. Sz.-Nagy and C. Foiaș in [15] (see also [16]). The class $C_1(\mathcal{H})$ is
precisely the class of all contractions on $\mathcal{H}$. The class $C_2(\mathcal{H})$ is precisely the class
of all operators $T \in L(\mathcal{H})$ which have numerical radius equal to at most one ([1]).

We remember that for an operator $T \in L(\mathcal{H})$, the numerical radius is defined by

$$w(T) = \sup \{|\langle Th, h \rangle|, h \in \mathcal{H}, \|h\| = 1\}.$$  

It is known ([16]) that $C_\rho(\mathcal{H}) \subset C_{\rho'}(\mathcal{H})$ if $0 < \rho < \rho' < \infty$ and $C_\rho(\mathcal{H}) \neq C_{\rho'}(\mathcal{H})$
if $\rho \neq \rho'$ and $\text{dim}(\mathcal{H}) \geq 2$.

We denote $C_\infty(\mathcal{H}) = \bigcup_{\rho > 0} C_\rho(\mathcal{H})$.

In February 2006, Professor Bernard Chevreau [3] posed the following:

Problem. Let $p \geq 2$ be a natural number and $T \in L(\mathcal{H})$. If there exists $\rho_0 > 0$
such that $T^p \in C_{\rho_0}$, then necessarily is $T \in C_\infty(\mathcal{H})$?

The answer is important for recent development in the theory of operators of
class $C_\rho$ ([2], [4], [5]).

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We give an answer to this question. In fact, we prove that there exists $T \in L(C^p)$ such that $T^p = I_{C^p}$ and $T \notin C^\omega(C^p)$.

We use the following result given by G. Eckstein [6]:

**Theorem 1.1.** Let $0 < \rho < \infty$ and $T \in C_\rho(\mathcal{H})$. Then, for any $h \in \mathcal{H}$, the sequence $\{||T^nh||\}_{n=0}^\infty$ converges.

A short proof of this theorem was given in [11]. An another proof was given in [9].

2. The results

We proved before the following result:

**Theorem 2.1.** Let $p \geq 2$ be a natural number and $0 < \rho < \infty$. If $T \in C_\rho(\mathcal{H})$ and $T^p$ is a unitary operator, then $T$ is unitary.

**Proof.** We denote $V := T^p$. Let $h \in \mathcal{H}$ and $n$ a natural number. We have

$$||T^nph|| = ||V^n h|| = ||h||$$

and

$$||T^{n+1}h|| = ||T^nTh|| = ||V^nTh|| = ||Th||.$$

Since $T \in C_\rho(\mathcal{H})$, it follows by Theorem 1.1 that the sequence $\{||T^nh||\}_{n=0}^\infty$ converges. It follows that the sequences

$\{||T^nh||\}_{n=0}^\infty$ and $\{||T^{n+1}h||\}_{n=0}^\infty$

converge to the same limit. It results in $||h|| = ||Th||$. Hence $T$ is an isometry.

We also have $V^* = T^{*p}$. But $T^* \in C_\rho(\mathcal{H})$ since we can write

$$T^*n h = \rho PU^*n h, \quad n \geq 1, \ h \in \mathcal{H}.$$

By the previous argument it follows that $T^*$ is also an isometry. Hence $T$ is a unitary operator.

We give a generalization of Theorem 2.1 but in the proof we use another technique.

Let $T_1, T_2 \in L(\mathcal{H})$. We say that $T_1$ and $T_2$ are *double commuting* if $T_1$ commutes with $T_2$ and with $T_2^*$.

We use the following result given by B. Fuglede [7]:

**Theorem 2.2.** Let $T_1, T_2$ be bounded operators on $\mathcal{H}$, with $T_2$ being normal. If $T_1$ and $T_2$ commute, then $T_1$ and $T_2$ are double commuting.

Our result is the following:

**Theorem 2.3.** Let $p \geq 2$ be a natural number and $0 < \rho < \infty$. If $T \in C_\rho(\mathcal{H})$ is such that $N := T^p$ is normal and norm increasing (i.e. $N^*N \geq I_\mathcal{H}$), then $T$ is unitary.

**Proof.** Since $N$ is norm increasing and normal, we have

$$||Nh|| \geq ||h|| \quad \text{and} \quad ||N^*h|| = ||Nh|| \geq ||h||, \ h \in \mathcal{H}.$$ 

It follows that $N$ is invertible and $||N^{-1}h|| \leq ||h||, \ h \in \mathcal{H}.$

We have $TN = T^{p+1}$ and $NT = T^{p+1}$; hence $N^{-1}T = TN^{-1}$. We obtain

$$T(T^{p-1}N^{-1}) = I_\mathcal{H} \quad \text{and} \quad (T^{-1}N^{-1})T = I_\mathcal{H}.$$ 


It follows that $T$ is invertible and $T^{-1} = T^{p-1}N^{-1}$. Since $T \in C_p(H)$, it follows by [16] that $T^{p-1} \in C_p(H)$.

Since $T^{p-1}$ and $N^{-1}$ commute and $N^{-1}$ is normal, by Theorem 2.2 it results that $T^{p-1}$ and $N^{-1}$ are double commuting. We use the following known result (see [5], [10], [14]):

If $T_1 \in C_{\rho_1}(H)$, $T_2 \in C_{\rho_2}(H)$ are double commuting, then $T_1T_2 \in C_{\rho_1\rho_2}(H)$ ($0 < \rho_1, \rho_2 < \infty$).

We apply this result for $T_1 = T^{p-1} \in C_{\rho}(H)$ and $T_2 = N^{-1} \in C_{\rho}(H)$. It follows that $T^{-1} = T^{p-1}N^{-1} \in C_{\rho}(H)$. Since $T, T^{-1} \in C_{\rho}(H)$, by a well-known result of J. G. Stampfli [13] (see also [12]), it follows that $T$ is unitary. □

We now prove the main result of this paper.

**Theorem 2.4.** Let $p \geq 2$ be a natural number. Then there exists $T \in L(C^p)$ such that $T^p = I_{C^p}$ and $T \notin C_{\infty}(C^p)$.

**Proof.** We consider $T$ the operator on $C^p$ which has the matrix

$$
[T] = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & a_1 \\
a_2 & 0 & 0 & \ldots & 0 & 0 \\
0 & a_3 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_p & 0
\end{pmatrix}
$$

where $a_1 \cdot a_2 \ldots \cdot a_p = 1$ and there exists $j \in \{1, 2, \ldots, p\}$ such that $a_j \neq 1$. We have

$$
det[T] = (-1)^{p+1}a_1 \cdot a_2 \cdots \cdot a_p \neq 0;
$$

hence $T$ is diagonalizable.

We denote by $\lambda_1, \lambda_2, \ldots, \lambda_p$ the roots of the equation

$$
det([T] - \lambda I_p) = 0,
$$

where $I_p$ is the identity matrix of order $p$. Then we can we write

$$
[T] = S^{-1} \Lambda S,
$$

where

$$
[\Lambda] = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)
$$

and $S$ is an invertible matrix.

The equation (2.1) is equivalent to

$$
\begin{vmatrix}
-\lambda & 0 & 0 & \ldots & 0 & a_1 \\
a_2 & -\lambda & 0 & \ldots & 0 & 0 \\
0 & a_3 & -\lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_p & -\lambda
\end{vmatrix} = 0;
$$

that is,

$$
(-\lambda)^p + (-1)^{p+1}a_1a_2 \ldots a_p = 0
$$

or

$$
\lambda^p = a_1 \ldots a_p.
$$

We have

$$
$$
Hence $T^p = I_{C^p}$.

But $T \notin C^\infty (C^p)$. In a contrary case, by Theorem 2.1, it follows that $T$ is unitary. \hfill \Box

References

3. B. Chevreau, Talk given at West University of Timișoara, February 2006.

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