CONCORDANCE CROSSCAP NUMBERS OF KNOTS
AND THE ALEXANDER POLYNOMIAL

CHARLES LIVINGSTON

(Communicated by Daniel Ruberman)

Abstract. For a knot $K$ the concordance crosscap number, $c(K)$, is the minimum crosscap number among all knots concordant to $K$. Building on work of G. Zhang, which studied the determinants of knots with $c(K) < 2$, we apply the Alexander polynomial to construct new algebraic obstructions to $c(K) < 2$.

With the exception of low crossing number knots previously known to have $c(K) < 2$, the obstruction applies to all but four prime knots of 11 or fewer crossings.

Every knot $K \subset S^3$ bounds an embedded surface $F \subset S^3$ with $F \sim n \mathbb{P}^2 - B_2$ for some $n \geq 0$, where $\mathbb{P}^2$ denotes the real projective plane. The crosscap number of $K$, $\gamma(K)$, is defined to be the minimum such $n$. The careful study of this invariant began with the work of Clark in [Cl]; other references include [HT, MY1]. The study of the 4-dimensional crosscap number, $\gamma_4(K)$, defined similarly but in terms of $F \subset B^4$, appears in such articles as [MY2, Vi, Ya].

Gengyu Zhang [Zh] recently introduced a new knot invariant, the concordance crosscap number, $\gamma_c(K)$. This is defined to be the minimum crosscap number of any knot concordant to $K$. This invariant is the nonorientable version of the concordance genus, originally studied by Nakanishi [Na] and Casson [Ca], and later investigated in [Li].

In [Zh], Zhang presented an obstruction to $\gamma_c(K) \leq 1$ based on the homology of the 2-fold branched cover of the knot, or equivalently, $\det(K)$. Inspired by her work, in this note we will observe that the obstruction found in [Zh] extends to one based on the Alexander polynomial of $K$, $\Delta_K(t)$, and the signature of $K$, $\sigma(K)$.

**Theorem 1.** Suppose $\gamma_c(K) = 1$ and set $q = |\sigma(K)| + 1$. For all odd prime power divisors $p$ of $q$, the $2p$-cyclotomic polynomial $\phi_{2p}(t)$ has odd exponent in $\Delta_K(t)$. Furthermore, every other symmetric irreducible polynomial $\delta(t)$ with odd exponent in $\Delta_K(t)$ satisfies $\delta(-1) = \pm 1$.

**Proof.** Any knot $K'$ with $\gamma(K') = 1$ bounds a Mobius band and is thus a $(2, r)$–cable of some knot $J$ for some odd $r$. If $K$ is concordant to $K'$, then $\sigma(K) = \sigma(K') = \pm |r| - 1$; the signature $\sigma(K')$ is given by a formula of Shinohara [Sh] for the signature of 2-stranded cable knots. It follows that $|\sigma(K)| = |r| - 1$, so $|r| = |\sigma(K)| + 1 = q$.

According to a result of Seifert [Se], the Alexander polynomial of $K'$ is given by $\Delta_2,q(t)\Delta_J(t^2)$, where $\Delta_{2,q}(t)$ is the Alexander polynomial of the $(2, q)$–torus knot.

Received by the editors May 11, 2007.

2000 Mathematics Subject Classification. Primary 57M25.
A standard result states that $\Delta_{2,q}(t) = \frac{(2^q-1)(t-1)}{(t-1)(t^{2^q}-1)} = \frac{t^q+1}{t+1}$. This can be written as the product of cyclotomic polynomials,

$$\Delta_{2,q}(t) = \prod_{p|q, \ p > 1} \Phi_{2p}(t).$$

Since $K$ is concordant to $K'$, $K \# - K'$ is slice, and thus has Alexander polynomial of the form $g(t)g(t^{-1})$. That is, with $q = |\sigma(K)| + 1$,

$$\Delta_K(t)\Delta_J(t^2)\Delta_{2,q}(t) = g(t)g(t^{-1}).$$

We now make two observations: (1) Any symmetric irreducible polynomial has even exponent in $g(t)g(t^{-1})$, and thus even exponent in $\Delta_K(t)\Delta_J(t^2)\Delta_{2,q}(t)$; (2) since $\Delta_J(t)$ is an Alexander polynomial, $\Delta_J(1) = \pm 1$, and thus $\Delta_J(t^2)|_{t=-1} = \pm 1$.

By Lemma 2 $\Phi_{2p}(-1) = p$ if $p$ is an odd prime power, and $\Phi_{2p}(-1) = \pm 1$ if $p$ is an odd composite. Thus, for $p$ an odd prime power divisor of $q$, $\Phi_{2p}(t)$ has odd exponent in $\Delta_{2,q}(t)$ and does not divide $\Delta_J(t^2)$, so has odd exponent in $\Delta_K(t)$. Any other irreducible factor of $\Delta_K(t)$ with odd exponent is either a factor $\delta(t)$ of $\Delta_{2,q}(t)$, and thus of the form $\Phi_{2p}(t)$ with $p$ an odd composite (and so $\delta(-1) = \pm 1$), or else is not a factor of $\Delta_{2,q}(t)$ and so has odd exponent in $\Delta_J(t^2)$, and again must satisfy $\delta(-1) = \pm 1$. This completes the argument. □

**Lemma 2.** The cyclotomic polynomial $\Phi_{2p}(t)$ satisfies $\Phi_{2p}(-1) = p$ if $p$ is an odd prime power and $\Phi_{2p}(-1) = \pm 1$ if $p$ is an odd composite.

**Proof.** For an odd $r$, $h_r(t) = \frac{t^r+1}{t+1}$ satisfies $h_r(-1) = r$ by l'Hôpital’s rule. We have that $h_r(t)$ is the product

$$h_r(t) = \prod_{p|r, \ p > 1} \Phi_{2p}(t).$$

For $p$ a prime power, $s^n, \ \Phi_{2p}(t) = \frac{t^{n+1}}{t^n+1}$, and so, again by l'Hôpital’s rule, $\Phi_{2s^n}(-1) = s$. Thus, the product

$$\prod_{p|r, \ p > 1, \ p \text{ a prime power}} \Phi_{2p}(-1) = r.$$  

It follows that all the other terms in the product expansion of $h_r(t)$ must equal $\pm 1$ when evaluated at $t = -1$, as desired. □

**Example.** Theorem [1] is quite effective in ruling out $\gamma_c(K) = 1$. For instance, there are 801 prime knots with 11 or fewer crossings. Of these, 51 are known to be topologically slice, and 23 are known to be concordant to a $(2,q)$–torus knot for some $q$ and thus have $\gamma_c = 1$. Of the remaining 727 knots, all but four can be shown to have $\gamma_c \geq 2$. These four are $11n_{45}$ and $11n_{115}$, both of which are possibly slice, and $9_{40}$ and $11n_{46}$, both of which are possibly concordant to the trefoil. Of the collection of 727 knots, Yasuhara’s result [Ya] applies to show that 207 of them have 4–ball crosscap number $\gamma_4(K) \geq 2$. The 4–ball crosscap numbers of the rest are unknown.

As a second set of examples, consider knots $K$ with $\Delta_K(t)$ of degree 2. It follows immediately from Theorem [1] that there are only two possibilities: either $\sigma(K) = 0$ and $\Delta_K$ is reducible (an irreducible symmetric quadratic $f(t)$ cannot satisfy $f(1) = \pm 1$ and $f(-1) = \pm 1$) or $\sigma(K) = \pm 2$ and $\Delta_K(t) = t^2 - t + 1$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We conclude with the further special case consisting of \((p, q, r)\)-pretzel knots, \(P(p, q, r)\), with \(p, q,\) and \(r\) odd; some of these were studied in \([Zh]\). If we let \(D = D(p, q, r) = pq + qr + rp\), then

\[
\Delta_{P(p,q,r)}(t) = \frac{D + 1}{4} t^2 - \frac{D - 1}{2} t + \frac{D + 1}{4},
\]

which has discriminant \(-D\). Thus, by the previous argument we have:

**Corollary 3.** If \(\gamma_c(P(p, q, r)) = 1\), then either \(\sigma(P(p, q, r)) = 0\) and \(D(p, q, r) = -l^2\) for some integer \(l\) or \(\sigma(P(p, q, r)) = \pm 2\) and \(D(p, q, r) = 3\).

These pretzel knots include some shown by Zhang \([Zh]\) to have 4–dimensional crosscap number \(\gamma_4(K) = 1\).

**References**


