DENSE-LINEABILITY IN SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper we provide a general method to prove that certain nonlinear families of continuous functions contain dense linear manifolds. An application is furnished.

1. INTRODUCTION

The study of a rich algebraic structure for a family of functions not being a linear space has recently attracted the attention of an increasing number of analysts. Often, it turns out that although giving a concrete example of a function with a special property can be difficult, we surprisingly obtain that there exist many such functions. This is well-known if we restrict ourselves to obtaining a topologically large set of such functions, by mainly using a Baire-category approach (see [17]). What is not so well-known is that, in many cases, such special functions can even be organized in a linear way. For this, see for instance [1], [2], [9] and the references contained in it.

In order to fix the goal of this paper, we need some recent terminology, which can be found in [12], [3], [1] and [7]. Assume that $X$ is a topological vector space and that $M$ is a subset of $X$. Then $M$ is said to be

- **lineable** if $M \cup \{0\}$ contains an infinite dimensional linear submanifold,
- **dense-lineable** or **algebraically generic** if $M \cup \{0\}$ contains a dense linear submanifold,
- **spaceable** if $M \cup \{0\}$ contains an infinite dimensional closed linear submanifold.

It is clear that “spaceable” implies “lineable” and that, if $X$ is infinite dimensional, “dense-lineable” also implies “lineable”.

For instance, Gurariy [10] (see also [11]) proved in 1966 that the family $ND$ of (continuous) nowhere differentiable functions on $[0, 1]$ is lineable in the space $C[0, 1]$ of continuous functions on $[0, 1]$. Later, Fonf, Gurariy and Kadec [8] proved that...
$ND$ is spaceable in $C[0,1]$, endowed with the supremum norm. In fact, Rodríguez-Piazza [13] showed that every separable infinite dimensional Banach space is isometrically isomorphic to a linear manifold whose nonzero elements belong to $ND$. Hencl [13] improved this result in 2000 by replacing $ND$ by the class $NH$ of nowhere Hölder functions; in particular, $NH$ is spaceable in $C[0,1]$. In 2005, Bayart [3] pp. 168–169 (see also [2]) demonstrated that, for a prefixed set $E \subset \mathbb{T}$ (:= the unit circle) of Lebesgue measure 0, the subset of functions in $C(\mathbb{T})$ whose Fourier series expansion diverges at every point of $E$ is dense-lineable.

In this paper we will prove two rather general assertions which are useful to discover dense-lineability, mainly in the context of spaces of continuous functions; see Section 2. In Section 3, an application of such assertions to three families in the space $C[0,1]$ is given. In particular, we complete the aforementioned results by Gurariy and Hencl.

2. The main results

Our setting will be the vector space $F(T, \mathbb{K})$ of $\mathbb{K}$-valued functions on a topological vector space $T$ or its subspace $C(T, \mathbb{K})$ of corresponding continuous functions. As usual, $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ (the real line) or $\mathbb{C}$ (the complex plane).

Let us fix some additional terminology. A linear manifold $A \subset F(T, \mathbb{K})$ is said to be an algebra if it is stable under multiplication. If, in addition, it satisfies that $1/f \in A$ whenever $f \in A$ and $f(t) \neq 0$ for all $t \in T$, then we say that $A$ is an algebra with division. A family $\{A(t)\}_{t \in T}$ of subsets of $F(T, \mathbb{K})$ is called locally defined if it satisfies the following property: If $t_0 \in T$, $f \in A(t_0)$, $g \in F(T, \mathbb{K})$ and $g = f$ on some neighborhood of $t_0$, then $g \in A(t_0)$. Moreover, a family $\{A(t)\}_{t \in T}$ of subsets of $F(T, \mathbb{K})$ is said to be openly defined provided that

$$A(t) = \bigcup_{U \in O(t)} \bigcap_{\tau \in U} A(\tau) \quad (t \in T),$$

where $O(t)$ is the collection of all open subsets of $T$ containing $t$.

For instance, if $T = [0,1]$ and we denote $A(t) = \{\text{functions } f : [0,1] \to \mathbb{K} \text{ that are differentiable at } t\}$ and $\tilde{A}(t) = \{\text{functions } f : [0,1] \to \mathbb{K} \text{ that are analytic at } t\}$, then $A(t)$, $\tilde{A}(t)$ are algebras with division for all $t \in [0,1]$, and both families $\{A(t)\}_{t \in T}$, $\{\tilde{A}(t)\}_{t \in T}$ are locally defined. Moreover $\{\tilde{A}(t)\}_{t \in T}$ is openly defined (recall that a function is analytic at a point $t_0$ if and only if it is analytic at a neighborhood of $t_0$). But $\{A(t)\}_{t \in T}$ is not openly defined; indeed, according to the well-known Weierstrass example (see e.g. [20] pp. 48–49), the function

$$f(x) := x^2 \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi(8\pi+3)^n x)$$

is (continuous on $[0,1]$ but) differentiable only at the origin.

Finally, we establish the two promised assertions about algebraic genericity. The first one is abstract, and its main usefulness is to deduce dense-lineability from the mere lineability. The second one deals with families of continuous functions.

**Theorem 2.1.** Assume that $X$ is a metrizable separable topological vector space. Suppose that $\Gamma$ is a family of linear submanifolds of $X$ such that $\bigcap_{S \in \Gamma} S$ is dense in $X$ and $\bigcap_{S \in \Gamma} (X \setminus S)$ is lineable. Then $\bigcap_{S \in \Gamma} (X \setminus S)$ is dense-lineable.
From the hypothesis, we can choose a dense countable set \( \{ z_n \}_{n \geq 1} \) in \( X \) as well as a translation-invariant distance \( d \) defining the topology of \( X \). By denseness, we also can take, for each \( N := \{1, 2, \ldots \} \), a vector \( y_n \in \bigcap_{S \in \Gamma} S \) such that
\[
d(y_n, z_n) < 1/n.
\]
Since \( \bigcap_{S \in \Gamma} (X \setminus S) \) is lineable, there is an infinite dimensional linear manifold \( L \) with
\[
L \setminus \{0\} \subset \bigcap_{S \in \Gamma} (X \setminus S).
\]
Therefore we can select a linearly independent sequence of vectors \( \{v_n\}_{n \geq 1} \subset L \).
Since scalar multiplication is continuous on \( X \), we derive the existence of a sequence \( \{\varepsilon_n\}_{n \geq 1} \subset (0, +\infty) \) such that
\[
d(\varepsilon_nv_n, 0) < 1/n \quad (n \in \mathbb{N}).
\]
Now, we define \( x_n := y_n + \varepsilon_nv_n \) \( (n \geq 1) \) and
\[
D := \text{span} \{x_n : n \geq 1\}.
\]

Thanks to (1), (2), the triangle inequality and the translation-invariance of \( d \), we get
\[
d(x_n, z_n) \leq d(y_n + \varepsilon_nv_n, y_n) + d(y_n, z_n)
= d(\varepsilon_nv_n, 0) + d(y_n, z_n) < 2/n \quad (n \to \infty).
\]
From (3) and the denseness of \( \{z_n\}_{n \geq 1} \), we derive that \( \{x_n\}_{n \geq 1} \) is dense. Consequently, \( D \) is a dense linear submanifold of \( X \).

It remains to prove that \( D \setminus \{0\} \subset \bigcap_{S \in \Gamma} (X \setminus S) \). To this end, fix a vector \( x \in D \setminus \{0\} \). Then there exist \( N \in \mathbb{N} \) and scalars \( c_1, \ldots, c_N \) with \( c_N \neq 0 \) satisfying \( x = c_1x_1 + \cdots + c_Nx_N \); that is,
\[
x = c_1y_1 + \cdots + c_Ny_N + c_1\varepsilon_1v_1 + \cdots + c_N\varepsilon_nv_N.
\]
Assume, by way of contradiction, that \( x \notin \bigcap_{S \in \Gamma} (X \setminus S) \). Then there would be some \( S_0 \in \Gamma \) for which \( x \in S_0 \). But \( y_1, \ldots, y_N \in \bigcap_{S \in \Gamma} S \subset S_0 \) and \( S_0 \) is a linear manifold, so
\[
x - (c_1y_1 + \cdots + c_Ny_N) \in S_0.
\]
Since \( c_N\varepsilon_N \neq 0 \) and the vectors \( v_n \) are linearly independent, we deduce that \( c_1\varepsilon_1v_1 + \cdots + c_N\varepsilon_nv_N \in L \setminus \{0\} \subset \bigcap_{S \in \Gamma} (X \setminus S) \), which contradicts (5) because of (4).

**Theorem 2.2.** Let \( T \) be a topological space and let \( X \) be a topological vector space of \( \mathbb{K} \)-valued continuous functions on \( T \) that is an algebra. Assume that \( \{A(t)\}_{t \in T} \) is a family of subsets of \( F(T, \mathbb{K}) \), and denote \( A := \bigcap_{t \in T} (A(t) \cap X), B := \bigcap_{t \in T} (X \setminus A(t)) \).

(A) Suppose that the following conditions are satisfied:
   (a) For any \( t \in T \), the set \( A(t) \) is an algebra with division.
   (b) The family \( \{A(t)\}_{t \in T} \) is locally defined and openly defined.
   (c) \( B \) is not empty.
   (d) There exists a function \( \varphi \in A \) such that the image \( \varphi(U) \) of every nonempty open subset \( U \subset T \) is an infinite set.
   Then the set \( B \) is lineable.

(B) If, in addition, \( X \) is metrizable and separable and \( A \) is dense in \( X \), then \( B \) is dense-lineable in \( X \).
Proof. Part (B) follows from (A) and Theorem 2.1 as applied to \( \Gamma := \{ A(t) \cap X \}_{t \in T} \).

In order to demonstrate (A), let us assume that conditions (a) to (d) hold. Consider the function \( \varphi \) furnished by (d). By using (c), we can define the set

\[
D := \{(P \circ \varphi)F : P \in \Lambda \},
\]

where \( \Lambda \) is the family of all polynomials \( P : \mathbb{K} \to \mathbb{K} \) and \( F \in B \). Then \( D \) is, evidently, a linear submanifold of \( X \). In order to see that \( \dim(D) = \infty \), it is enough to show that, for each \( N \in \mathbb{N} \), the functions \( F, \varphi F, \ldots, \varphi^N F \) (which are in \( D \)) are linearly independent. This is true because, otherwise, there would be a nonzero polynomial \( P \) such that \( (P \circ \varphi)F = 0 \). By (a), the set \( A \) is an algebra, so \( P \circ \varphi \in A \). Taking \( U = T \) in (d), we have that \( \varphi(T) \) is infinite, so we can select a point \( t_0 \in T \) such that \( P(\varphi(t_0)) \neq 0 \); indeed, \( P \) has only finitely many zeros. By continuity, \( P(\varphi(t)) \neq 0 \) for all \( t \) belonging to some neighborhood of \( t_0 \). Hence \( F(t) = 0 \) on such a neighborhood, which by (b) implies that \( F \in A(t_0) \), a contradiction. Thus, \( \dim(D) = \infty \).

It remains to prove that \( D \setminus \{0\} \subset B \). For this, fix a function \( f \in D \setminus \{0\} \). Then there is a nonzero polynomial \( P \) with \( f = (P \circ \varphi)F \). Assume, by way of contradiction, that \( f \notin B \). Then there exists a point \( t_0 \in T \) such that \( f \in A(t_0) \).

Since \( \{ A(t) \}_{t \in T} \) is openly defined, we can find an open set \( U \subset T \) with \( t_0 \in U \) such that \( f \notin A(t) \) for all \( t \in U \). Due to (d), the set \( \varphi(U) \) is infinite, so one can choose a point \( t_1 \in U \) such that \( P(\varphi(t_1)) \neq 0 \). Again by continuity, there is an open set \( U_1 \) with \( t_1 \in U_1 \subset U \) satisfying \( P(\varphi(t)) \neq 0 \) for all \( t \in U_1 \).

Let us consider the function \( \psi : X \to \mathbb{K} \) given by

\[
\psi(t) = \begin{cases} 
\frac{f(t)}{P(\varphi(t))} & \text{if } t \in U_1, \\
1 & \text{if } t \in X \setminus U_1.
\end{cases}
\]

Since \( A(t_1) \) is a locally defined algebra with division, we obtain (observe that \( f \) and \( P \circ \varphi \) are in \( A(t_1) \)) that \( \psi \in A(t_1) \). But \( F = f/(P \circ \varphi) \) on \( U_1 \). Thus, once more by local definition, our function \( F \) belongs to \( A(t_1) \), which is absurd. \( \square \)

Remark 2.3. In view of the last theorem, one is tempted to say that if there are sufficiently many “good” functions (the set \( A \)) enjoying an adequate structure, then the set of “pathological” functions (the set \( B \)) is either empty or algebraically huge.

3. An application

In this section, we apply the statements obtained in Section 2 to three families within the space of continuous functions on \([0,1] \). Let us recall some standard notation and facts. We set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( f^{(0)} := f \) for any function \( f \).

If \( p \in \mathbb{N}_0 \), then the algebra \( C^p[0,1] \) of \( C^p \)-functions \( f : [0,1] \to \mathbb{K} \) becomes a Banach space when endowed with the norm \( \|f\| := \sum_{j=0}^p \sup_{[0,1]} |f^{(j)}| \) (see [14]). This norm generates the topology of uniform convergence of functions and their derivatives up to the order \( p \). Moreover, the algebra \( C^\infty[0,1] \) of infinitely many times differentiable functions on \([0,1] \) can be endowed with a distance \( d \) generating the topology of uniform convergence of derivatives of all orders. For instance, we can take \( d(f,g) = \sum_{j=0}^\infty 2^{-j} \min(1,\sup_{[0,1]} |f^{(j)}|) \). So all spaces \( C^p[0,1] \) \((p \in \mathbb{N}_0 \cup \{\infty\})\) are metrizable and complete. They are also separable: the set of polynomials is dense in each of them.

By using a Baire category approach, it has been proved that the family of continuous functions on \([0,1] \) that are nowhere differentiable is not also nonempty but
is even residual in $C[0,1]$ [17] pp. 45–46] (recall that a residual set $A$ in a Baire space is a subset whose complement is of the first category; in some sense, $A$ is topologically large). Also the family of $C^\infty$-functions on $[0,1]$ that are nowhere analytic is residual in $C^\infty[0,1]$ (see [10], [11] and [15]). By applying the results of the last section, we will be able to prove that both families are also algebraically large. Note that Proposition 3.1 below improves Gurariu’s result [10].

**Proposition 3.1.** Let $p \in \mathbb{N}_0$. Then the class of functions $f \in C^p[0,1]$ such that $f^{(p)}$ is nowhere differentiable on $[0,1]$ is dense-lineable in $C^p[0,1]$.

**Proof.** Let us denote by $M(p)$ the class of the statement of the theorem. For the case $p = 0$, we know by [10] that $M(0)$ is lineable. Thus, it is enough to apply Theorem 2.1 to $X = C[0,1]$ and $\Gamma = \{S(t)\}_{t \in [0,1]}$, where $S(t) = \{f \in X : f^{(p)}$ is differentiable at $t\}$. Finally, recall that the polynomials are differentiable everywhere and form a dense set.

Assume now that $p \in \mathbb{N}$. As expected, this time we apply Theorem 2.1 to $X = C^p[0,1]$ and $\Gamma = \{S(t)\}_{t \in [0,1]}$, where $S(t) = \{f \in X : f^{(p)}$ is differentiable at $t\}$. Again, recall that the polynomials are $p + 1$ times differentiable everywhere and form a dense subset of $C^p[0,1]$. So it must be proved that the class $M(p)$ is lineable. To see this, consider a linearly independent sequence $\{f_n\}_{n \geq 1}$ whose linear span is included in $M(0) \cup \{0\}$ and, for each $n \in \mathbb{N}$, denote by $F_n$ the unique antiderivative of order $p$ of $f_n$ such that $F_n^{(k)}(0) = 0$ for all $k = 0, \ldots, p - 1$. It is clear that $\text{span}(\{F_n\}_{n \geq 1}) \subset M(p) \cup \{0\}$. Now if $c_1, \ldots, c_N$ are scalars satisfying $c_1F_1 + \cdots + c_NF_N = 0$ on $[0,1]$, then after $p$ derivations we are driven to $c_1f_1 + \cdots + c_Nf_N = 0$ on $[0,1]$, which implies $c_1 = \cdots = c_N = 0$. Therefore the sequence $\{F_n\}_{n \geq 1}$ is linearly independent and $M(p)$ is lineable. □

**Proposition 3.2.** The class of functions $f \in C^\infty[0,1]$ which are nowhere analytic on $[0,1]$ is dense-lineable in $C^\infty[0,1]$.

**Proof.** The existence of nowhere analytic $C^\infty$-functions is known at least from du Bois-Reymond [6]. A specially elegant easy example is that of Lerch [15]:

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(3^n t)}{n!}.$$ 

Then it is sufficient to apply Theorem 2.2 with $T = [0,1]$, $X = C^\infty[0,1]$, $A(t) = \{f \in F([0,1],\mathbb{K}) : f$ is analytic at $t\}$ and $\varphi(t) = t$ ($t \in [0,1]$). Take into account that the polynomials are in the corresponding set $A$. □

Finally, let us denote by $CBV[0,1]$ the algebra of all continuous functions on $[0,1]$ with bounded variation. It is well-known that it becomes a Banach space under the norm $\|f\| := \sup_{[0,1]}|f| + V(f)$, where $V(f)$ is the variation of $f$ on $[0,1]$; that is, $V(f) := \sup\{\sum_{j=1}^N|f(t_j) - f(t_{j-1})| : 0 = t_0 < t_1 < t_2 < \cdots < t_N = 1, N \in \mathbb{N}\}$. Then $CBV[0,1]$ is a metrizable space. Moreover, it is separable, because the set of polynomials is also dense in this space; indeed, $C^1[0,1]$ contains the polynomials and is continuously contained in $CBV[0,1]$. Our final assertion has to do with functions in this space which are not differentiable at any interval.

**Proposition 3.3.** The class of functions $f \in CBV[0,1]$ such that $f$ is differentiable on no interval in $[0,1]$ is dense-lineable in $CBV[0,1]$. 

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Proof. Just apply Theorem 2.2 to $X = CBV[0, 1]$, $A(t) = \{ f \in F(0, 1, \mathbb{K}) : f$ is differentiable on some neighborhood of $t \}$ and $\varphi(t) = t \ (t \in [0, 1])$. Again, the corresponding set $A$ contains the polynomials, so it is dense. It remains to show that the class of the statement is not empty. This is a consequence of a more general result; namely, given a set $E \subset [0, 1]$ with Lebesgue measure 0, there is an increasing (so of bounded variation) continuous function on $[0, 1]$ which is differentiable at no point of $E$ (see [20], pp. 25–26). Now it is enough to choose $E = \{ \text{the rational points in } [0, 1] \}$. \hfill \square

Remarks 3.4. 1. Assume that $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous strictly increasing function with $\varphi(0) = 0$. Then a function $f : [0, 1] \to \mathbb{K}$ is said to be $\varphi$-Hölder (or $\varphi$-Lipschitz) at a point $x_0 \in [0, 1]$ whenever there are a neighborhood $U \subset [0, 1]$ of $x_0$ and a constant $M = M(f, x_0) \in (0, +\infty)$ such that $|f(x) - f(x_0)| \leq M \varphi(|x - x_0|)$ for all $x \in U$.

If, in addition, $f \in C[0, 1]$, then one can take $U = [0, 1]$. By Hencl’s result [13], the family of functions $f \in C[0, 1]$ which are nowhere $\varphi$-Hölder is spaceable, and so is lineable. Hence, with a very similar proof, Proposition 3.1 can be reinforced by changing “nowhere differentiable” to “nowhere $\varphi$-Hölder”. This complements Fonf-Gurariy-Kadec-Rodríguez-Hencl’s statements about spaceability.

2. With a different proof, we have obtained in [11] a statement that is stronger than Proposition 3.2. Namely, the class of functions $f \in C^\infty[0, 1]$ such that the associated Taylor series has null radius of convergence at each point is dense-lineable.

3. In Proposition 3.3 one cannot replace the condition “differentiable on no interval” by “nowhere differentiable”. Indeed, every function of bounded variation is differentiable almost everywhere.

References


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