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SMITH EQUIVALENT Aut(A_6)-REPRESENTATIONS ARE ISOMORPHIC

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Dedicated to Professor Katsuhiro Komiya on his 60th birthday

ABSTRACT. Many authors, e.g. M. Atiyah, R. Bott, J. Milnor, G. Bredon, S. Cappell, J. Shaneson, C. Sanchez, T. Petrie, E. Laitinen, K. Pawałowski, R. Solomon and so on, studied Smith equivalent representations for finite groups. Observing their results, E. Laitinen conjectured that nonisomorphic Smith equivalent real G-modules exist if a_G, the number of real conjugacy classes of elements not of prime power order in G, is greater than or equal to 2. This paper shows that in the case $G = \mathrm{Aut}(A_6)$, $a_G = 2$ any two Smith equivalent real G-modules are isomorphic.

1. INTRODUCTION

Let G be a finite group. If G smoothly acts on a smooth manifold M and x is a G-fixed point, then the tangent space $T_x(M)$ at x is a real G-module. In 1960, P. A. Smith [30] posed the problem: If G smoothly acts on a sphere S with exactly two G-fixed points a and b, then is it true that $T_a(S)$ and $T_b(S)$ are isomorphic as real G-modules? It is easy to show that the answer is affirmative if G is a group such that each element has order 1, 2 or 4. Atiyah-Bott [1] and Milnor [16, Theorem 12.11] affirmatively solved the problem under the hypothesis that the G-action on $S \smallsetminus \{a, b\}$ is free, and Sanchez [28] also affirmatively solved the problem under the hypothesis that the order of G is a power of an odd prime. On the other hand, Petrie [21]–[25] and Cappell-Shaneson [3]–[5] obtained negative answers to Smith's problem above in the cases $G = C_{pqr} \times C_{pqr}$ and $G = C_n$ such that $n = 4m$ with $m \geq 2$, where p, q and r are distinct odd primes and C_n denotes the cyclic group of order n.

Let us restate the problem in terms of Smith equivalence. Two real G-modules U and V are called *Smith equivalent* if there exists a smooth G-action on a homotopy sphere Σ such that $\Sigma^G = \{a,\ b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real G-modules. This Σ will be referred to as a *Smith sphere* for U and V. Petrie and his collaborators found various nonisomorphic, Smith equivalent real G-modules U and V; see e.g. [22]–[25], [26], [27], [6], [9], [10], [31], [15], [7], [12], [11].

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Let $\mathrm{RO}(G)$ denote the real representation ring of G. Define the subset $\mathrm{Sm}(G)$ of $\mathrm{RO}(G)$ by

$$\mathrm{Sm}(G) = \{[U] - [V] \in \mathrm{RO}(G) \mid U \text{ and } V \text{ are Smith equivalent}\}.$$

Let d_G denote the homomorphism $\mathrm{RO}(G) \to \mathbb{Z}$ defined by

$$d_G([U] - [V]) = \dim U^G - \dim V^G.$$

Immediately, $\mathrm{Sm}(G) \subseteq \mathrm{Ker}(d_G)$ follows. Let $\mathcal{P}(G)$ denote the set of all subgroups P of G of prime power order. Let $\mathcal{P}^*(G)$ denote the subset of $\mathcal{P}(G)$ consisting of P such that $|P| \leq 4$ if 2 divides the order $|P|$ of P. It is clear that

$$\mathrm{Sm}(G) \subseteq \bigcap_{P \in \mathcal{P}^*(G)} \mathrm{Ker}[\mathrm{res} : \mathrm{RO}(G) \to \mathrm{RO}(P)]. \tag{1.1}$$

For $g \in G$, let (g) denote the conjugacy class of g in G. The union $(g)^{\pm} = (g) \cup (g^{-1})$ is called the *real conjugacy class* of g in G. Let a_G denote the number of real conjugacy classes $(g)^{\pm}$ in G such that the order of g is not a prime power. A finite group G is called an *Oliver group* if G admits a smooth action on a disk without G-fixed points. In other words, G is an Oliver group if G never admits a normal series $P \triangleleft H \triangleleft G$ such that P and G/H both have prime power orders and H/P is cyclic.

We call real G-modules U and V 2-*properly Smith equivalent* if there exists a Smith sphere Σ for U and V such that $\dim \Sigma^P \geq 1$ (equivalently, Σ^P is connected) for every cyclic 2-subgroup P of G with $|P| \geq 8$. This is the same as the notion of *Laitinen-Smith equivalent* used in [20]. Laitinen and Pawałowski were interested in the subset $\mathrm{Sm}_2(G)$ of $\mathrm{Sm}(G)$ defined by

$$\mathrm{Sm}_2(G) = \{[U] - [V] \in \mathrm{Sm}(G) \mid U \text{ and } V \text{ are 2-properly Smith equivalent}\},$$

and a conjecture occurred.

Laitinen's Conjecture. Let G be an Oliver group. Then $\mathrm{Sm}_2(G)$ is nontrivial if and only if $a_G \geq 2$.

Laitinen-Pawałowski [14] affirmatively proved the conjecture for an arbitrary perfect group G, and Pawałowski-Solomon [20] affirmatively proved it in the following cases:

(1) G is an Oliver group with a cyclic quotient of order pq for distinct odd primes p and q.
(2) G is an Oliver group of odd order.
(3) G is a nonsolvable gap group not isomorphic to $P\Sigma L(2, 27)$, where $P\Sigma L(2, 27)$ is the splitting extension of $PSL(2, 27)$ by the group $\mathrm{Aut}(\mathbb{F}_{27})$.

For the sake of the reader's convenience, we recall the definition of a gap group. A finite group G is called a *gap group* if there exists a (finite dimensional) real G-module W satisfying

(1) $W^K = 0$ for all normal subgroups K of G such that $|G/K|$ is a power of a prime (possibly $|G/K| = 1$), and
(2) $\dim W^P > 2 \dim W^K$ for all subgroups $P \in \mathcal{P}(G)$ and $K \subseteq G$ such that $K \supsetneq P$.

Let A_6 denote the alternating group of degree 6 and $\mathrm{Aut}(A_6)$ the group of all automorphisms on A_6. Pawałowski-Solomon [20, Theorem A.3 (3)] pointed out that $G = \mathrm{Aut}(A_6)$ is a nonsolvable Oliver group such that $a_G = 2$.

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We however obtain the next result contrary to Laitinen's Conjecture.

Theorem 1.2. *If* $G = \mathrm{Aut}(A_6)$*, then* $\mathrm{Sm}(G) = 0$.

The purpose of the present paper is to prove this result.

2. Proof of Theorem 1.2

We begin the section with a key lemma.

Lemma 2.1. *Let* G *be a group of order* 2 *and* X *a connected closed smooth manifold with smooth* G*-action. If* $\dim X \geq 1$ *and* $X^G \neq \varnothing$*, then* $|X^G| \geq 2$.

Although the lemma follows from Theorem 25.1 of [8], we give here a proof of the lemma because of its simplicity.

Proof. Contrarily assume that $X^G = \{x\}$. Then G acts freely on $X \smallsetminus \{x\}$. Let D be a G-invariant disk neighborhood of x. Pinching the subset

$$X \smallsetminus \mathrm{Interior}(D)$$

of X, we obtain the sphere $S(\mathbb{R} \oplus W)$ with linear G-action, where $W = T_x(X)$. There exists a smooth G-map $f : X \to S(\mathbb{R} \oplus W)$ such that $f(x) = (1, 0)$, $f(X \smallsetminus \{x\}) = S(\mathbb{R} \oplus W) \smallsetminus \{(1, 0)\}$, and f is transverse regular to the point $(1, 0)$ in $S(\mathbb{R} \oplus W)$, where 1 is the element in $\mathbb{R}$ and 0 is the element in W. Then the mod 2 mapping degree of f is 1 in $\mathbb{Z}/2\mathbb{Z}$. Deforming f by a G-homotopy, we may assume that f is transverse regular to the point $(-1, 0)$ in $S(\mathbb{R} \oplus W)$. Since $f^{-1}((-1, 0))$ is a free G-set, $f^{-1}((-1, 0))$ consists of an even number of points. This contradicts the fact that the mod 2 mapping degree of f is 1. We have established $|X^G| \neq 1$. □

For an upper closed, conjugation-invariant family $\mathcal{F}$ consisting of subgroups of G, a real G-module W is called *$\mathcal{F}$-free* if $W^H = 0$ for all $H \in \mathcal{F}$.

Proposition 2.2. *Let* $\mathcal{F}_2$ *denote the set of all subgroups of* G *with index* 1 *or* 2*. If* U *and* V *are Smith equivalent real* G*-modules, then there exists a real* G*-module* W *such that complementary* G*-modules* U_0 *and* V_0 *are both* $\mathcal{F}_2$*-free, where* U_0 *and* V_0 *are* G*-modules satisfying* $U = U_0 \oplus W$ *and* $V = V_0 \oplus W$*, respectively.*

Proof. Let Σ be a homotopy sphere such that $\Sigma^G = \{a,\ b\}$, $U = T_a(\Sigma)$ and $V = T_b(\Sigma)$. Let K be an arbitrary subgroup of G with index 2. If $U^K \neq 0$ or $V^K \neq 0$, then by Lemma 2.1 the points a and b are contained in a same connected component of Σ^K. It follows that $\dim U^K = \dim V^K$, and hence $U^K \cong V^K$ as real G-modules. Thus, the real G-module $W = \bigoplus_K U^K$, where K runs over the set of all subgroups of G with index 2, satisfies the property required in the proposition above. □

Given a finite group G and a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup H of G such that $|G/H|$ is a power of p.

Now let $G = \mathrm{Aut}(A_6)$. Then $[G, G] \cong A_6$ and $G/[G, G] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $G^{\{2\}} = [G, G]$ and $G^{\{p\}} = G$ for any odd prime p. Let U and V be Smith equivalent real G-modules and Σ a Smith sphere for U and V. By Proposition 2.2, there exists a real G-module W such that $U_0^{[G,G]} = 0$ and $V_0^{[G,G]} = 0$, where $U = U_0 \oplus W$ and $V = V_0 \oplus W$. The (complex) character table of G is as in Table 2.3 (by means of GAP [13]).

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TABLE 2.3

	$1a$	$4a$	$2a$	$8a$	$8b$	$4b$	$2b$	$4c$	$10a$	$5a$	$2c$	$6a$	$3a$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	1	-1	-1	1	-1	-1	-1	1	1	1	1
$X.3$	1	1	1	1	-1	-1	-1	1	-1	1	-1	-1	1
$X.4$	1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1
$X.5$	9	1	1	-1	-1	1	1	1	1	-1	-3	0	0
$X.6$	9	1	1	-1	1	-1	-1	1	-1	-1	3	0	0
$X.7$	9	1	1	1	-1	-1	1	-1	1	-1	3	0	0
$X.8$	9	1	1	1	1	1	-1	-1	-1	-1	-3	0	0
$X.9$	10	-2	2	0	0	-2	0	0	0	0	-2	1	1
$X.10$	10	-2	2	0	0	2	0	0	0	0	2	-1	1
$X.11$	16	0	0	0	0	0	-4	0	1	1	0	0	-2
$X.12$	16	0	0	0	0	0	4	0	-1	1	0	0	-2
$X.13$	20	0	-4	0	0	0	0	0	0	0	0	0	2

For each $k = 5, \ldots, 13$, let V_k denote an irreducible complex G-module such that the character of V_k is $X.k$ in the table. By (1.1),

$$[\mathbb{C} \otimes_{\mathbb{R}} U_0] - [\mathbb{C} \otimes_{\mathbb{R}} V_0] = 2\alpha[V_5] + 2\alpha[V_6] - 2\alpha[V_7] - 2\alpha[V_8] - \alpha[V_{11}] + \alpha[V_{12}]$$

for some integer α. We are going to show $\alpha = 0$.

So, contrarily suppose $\alpha \neq 0$. Then either $\mathbb{C} \otimes_{\mathbb{R}} U_0$ or $\mathbb{C} \otimes_{\mathbb{R}} V_0$ has an irreducible summand isomorphic to V_7. Let x, y, and $z \in G$ be representatives of the conjugacy classes $4a$, $2a$, and $8a$, respectively. Then $(z^3) = (z)$, $(z^2) = (x)$, and $(x^2) = (y)$, where (x) denotes the conjugacy class of x in G. Since $\dim V_7^{\langle z \rangle} \neq 0$ and $\Sigma^{\langle z \rangle}$ is a $\mathbb{Z}/2\mathbb{Z}$-coefficient-homology sphere, $\Sigma^{\langle z \rangle}$ is connected and hence $\mathrm{res}^G_{\langle z \rangle} U_0 \cong \mathrm{res}^G_{\langle z \rangle} V_0$. This implies $\alpha = 0$, which is a contradiction.

Thus we obtain $[U] - [V] = 0$ and have completed the proof of Theorem 1.2.

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