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A NEW PROOF OF THE RIGIDITY PROBLEM

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Abstract. In this short note we give a new proof of the boundary rigidity problem in a Euclidean setting proved by Croke. Our method is based on the differentiability of Busemann functions and the characteristic of Euclidean metric on Riemannian manifolds without conjugate points.

1. Introduction

A compact Riemannian manifold with boundary $(M, \partial M, g)$ is said to be *boundary rigid* if for any Riemannian manifold with the same boundary, say $(M', \partial M, h)$, an isometry between the two induced chord length metrics $\mathrm{dist}_g, \mathrm{dist}_h \colon \partial M \times \partial M \to \mathbb{R}$ implies that there exists a diffeomorphism $\psi \colon M' \to M$ with $\psi^* g = h$. Recently Pestov and Uhlmann [8] showed boundary rigidity for the two-dimensional compact simply connected conjugate point free Riemannian manifolds with strictly convex boundary; that is, the second fundamental form of the boundary is positive definite in every boundary point.

It was known that compact subdomains of Euclidean space and of Euclidean hemisphere are boundary rigid (see [5]). We give a simple proof for Euclidean space:

Theorem. *Any compact subdomains of Euclidean metric on $\mathbb{R}^n$ are boundary rigid.*

In [1], the above theorem was extended to Lorentzian surfaces. However in [2], Arcostanzo has a negative answer in the Finsler case for boundary rigidity, and hence the rigidity problem in Finsler geometry requires more scrutiny (cf. [3]). The author and Yim [7] proved boundary rigidity for compact subdomains with vanishing mean tangent curvature on Minkowski space.

Our proof does not use the Jacobi and curvature tensors, which are of totally different form [5, Theorem 6.4]. Among other things, we use the differentiability of Busemann functions and the characteristic of Euclidean metric on manifolds without conjugate points instead; see §3 below. Although our proof is longer than Croke's, most of our arguments are elementary and self-contained.

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2. Preliminaries

Let SM be the unit tangent bundle of a Riemannian manifold M with natural projection $\pi : SM \to M$. The geodesic flow on SM is denoted by ϕ_t and is given by

$$\phi_t(v) = \gamma_v'(t),$$

where $\gamma_v(t)$ denotes the geodesic with initial point $\pi(v) = \gamma_v(0)$ and initial velocity vector $v = \gamma_v'(0)$. Let $d\mu$ be the Liouville measure on SM. Then we have from the Liouville theorem that the geodesic flow ϕ_t preserves the volume form $d\mu$ on SM; that is, $\frac{d}{dt}\{(\phi_t)^*(d\mu)\} = 0$.

We are given a Riemannian manifold M and a relatively compact subdomain Ω in M with a smooth boundary. For any $v \in S\Omega$ we set

$$\tau(v) = \sup\{\tau > 0 \mid \gamma_v(t) \in \Omega, t \in (o, \tau)\},$$

that is, when $\tau(v)$ is finite, then $\gamma_v(\tau(v))$ will be the first point on the geodesic to hit the boundary $\partial\Omega$ of Ω. Let η denote the inward unit normal vector field along $\partial\Omega$, and let $S^+\partial\Omega$ denote the collection of inward pointing unit vectors along $\partial\Omega$; that is

$$S^+\partial\Omega := \{v \in S\Omega \mid \pi(v) \in \partial\Omega, g(v, \eta) > 0\},$$

with measure $d\sigma(v) = dx_{\pi(v)}(v)dA\big(\pi(v)\big)$, where dA denotes the $(n-1)$-measure on $\partial\Omega$. The following theorem is very useful for estimating the volume and eigenvalue of compact Riemannian manifolds with boundary and can be found in [9].

Theorem 2.1 (Santaló's formula)**.** *For all integrable functions F on $S\Omega$ we have*

$$\int_{S\Omega} F\ d\mu = \int_{S^+\partial\Omega} \left\{ \int_0^{\tau(v)} F\big(\phi_t(v)\big)\ dt \right\} g(v, \eta)\ d\sigma(v).$$

For all points $x, y \in M$, call $\mathrm{dist}(x, y)$ the distance function between x and y. For each unit vector $v \in S\Omega$ and each $t \geq 0$, define the function

$$b_{vt}(x) := \{\mathrm{dist}\left(\gamma_v(t), x\right) - t\}, \quad x \in M.$$

These functions are smooth except at $\gamma_v(t)$ and, by the triangle inequality, decreasing with t and absolutely bounded by $\mathrm{dist}\left(\gamma_v(t), x\right)$. So the function

$$b_v(x) := \lim_{t\to\infty} \{\mathrm{dist}\left(\gamma_v(t), x\right) - t\}$$

is defined everywhere on M and called the *Busemann function* of v. The following proposition proved by Eschenburg [6] will play a crucial role in this note.

Proposition 2.2. *Let M be a complete simply connected Riemannian manifold without conjugate points. Then the Busemann function b_v is $\mathcal{C}^1$-differentiable with gradient*

$$\nabla b_v = \lim_{t\to\infty} \nabla b_{vt}$$

pointwise convergent for each unit vector $v \in SM$. In particular $g(\nabla b_v, \nabla b_v) = 1$.

The Busemann function, b_v, is said to be *skew symmetric* if, whenever γ_v is a ray and $\gamma_v(0)$ is the midpoint of p and q, we have

$$b_v(p) + b_v(q) = 0.$$

The skew symmetry for the Busemann functions is satisfied in Euclidean space, since the central reflection in $\gamma_v(0)$ takes the horosphere $b_v^{-1}(0)$ into itself as $b_v^{-1}(0)$ is flat. The following theorem is used in the proof of the main theorem and the

characteristic of Euclidean metric on manifolds without conjugate points is shown by Busemann and Phadke [4, Theorem III.6.17].

Theorem 2.3. *A simply connected noncompact Riemannian manifold without conjugate points is Euclidean if and only if the Busemann functions are skew symmetric.*

3. Proof of main theorem

In this section we prove our main theorem.

Theorem. *Any compact subdomains of Euclidean metric g_0 on $\mathbb{R}^n$ are boundary rigid.*

Proof. Let $(K, \partial K, g)$ be a compact subdomain of the n-dimensional Euclidean space. We may assume that the compact subdomain K is contained in the compact set Ω in $(\mathbb{R}^n, g)$ such that $(\mathbb{R}^n - \Omega, g)$ is isometric to $(\mathbb{R}^n - B(r), g_0)$ for some $r > 0$, where $B(r)$ is the metric ball of radius r. Since $\mathrm{dist}_g = \mathrm{dist}_{g_0}$ on $\partial\Omega \times \partial\Omega$ in the Euclidean setting, i.e., $\tau(v) = \tau_0(v)$ for all $v \in S^+\partial\Omega$, by Santaló's formula 2.1,

$$\begin{aligned}\mathrm{vol}(S\Omega) &= \int_{S\Omega} 1 \; d\mu = \int_{S^+\partial\Omega} \left\{ \int_0^{\tau(v)} 1 \; dt \right\} g(v, \eta) \; d\sigma(v) \\ &= \int_{S^+\partial\Omega} \left\{ \int_0^{\tau_0(v)} 1 \; dt \right\} g_0(v, \eta) \; d\sigma_0(v) = \mathrm{vol}_0(S\Omega).\end{aligned}$$

Given $v \in S\Omega$ and $(t, w) \in [0, \tau(w)] \times S^+\partial\Omega$, we define $f_v(t, w)$ by

$$f_v(t, w) := g\Big(\nabla b_v\big(\gamma_w(t)\big), \gamma_w'(t)\Big).$$

Since $(\mathbb{R}^n, g)$ has no conjugate points, by Proposition 2.2, this function is well defined. Using the chain rule, we obtain

$$f_v(t, w) = \frac{d}{dt}\Big\{ b_v\big(\gamma_w(t)\big) \Big\}.$$

Then we have

$$\begin{aligned}\left\{ \int_0^{\tau(w)} f_v(t, w)^2 \; dt \right\} \cdot \left\{ \int_0^{\tau(w)} 1^2 \; dt \right\} &\geq \left\{ \int_0^{\tau(w)} f_v(t, w) \cdot 1 \; dt \right\}^2 \\ &= \left\{ \int_0^{\tau(w)} \frac{d}{dt}\Big\{ b_v\big(\gamma_w(t)\big) \Big\} \; dt \right\}^2 \\ &= \Big\{ b_v\big(\gamma_w(\tau(w))\big) - b_v\big(\gamma_w(0)\big) \Big\}^2 \\ &= \{ g_0(v, w)\tau_0(w) \}^2.\end{aligned} \tag{3.1}$$

We note that the first line is obtained from Schwarz's inequality and the second line is obtained from the definition of $f_v(t, w)$. Thus we obtain

$$\int_0^{\tau(w)} f_v(t, w)^2 \; dt \geq \tau_0(w) \{ g_0(v, w) \}^2.$$

Integrating on $w \in S^+\partial\Omega$ and using Santaló's formula 2.1, we have

$$\int_{S\Omega} g\big(\nabla b_v, w\big)^2 \; d\mu(w) \geq \int_{S\Omega} g_0(v, w)^2 \; d\mu_0(w). \tag{3.2}$$

Obviously the latter integral does not depend on $v \in S\Omega$. Adding its value over $v_1, v_2, \ldots, v_n$ where $\{v_i\}_{i=1}^n$ forms an orthonormal basis in $S_{\pi(v)}\Omega$, we obtain

$$\begin{aligned}\mathrm{vol}(S_{\pi(v)}\Omega) &= \int_{S_{\pi(v)}\Omega} 1 \; dx_{\pi(v)} \\ &= \int_{S_{\pi(v)}\Omega} \sum_{i=1}^n g_0(v_i, w)^2 \; dx_{\pi(v)} \\ &= n \cdot \int_{S_{\pi(v)}\Omega} g_0(v, w)^2 \; dx_{\pi(v)},\end{aligned}$$

and hence

$$\int_{S_{\pi(v)}\Omega} g_0(v, w)^2 \; dx_{\pi(v)} = \frac{\mathrm{vol}(S_{\pi(v)}\Omega)}{n}. \tag{3.3}$$

From (3.3), we gather that the left-hand side and the right-hand side of (3.2) are the same and equal to $\mathrm{vol}(S\Omega)/n$ and hence by the equality case of Schwarz's inequality in (3.1), $f_v(t, w)/1$ is constant. Since $f_v(t, w) = \frac{d}{dt}\{b_v\big(\gamma_w(t)\big)\}$ is constant, we have $b_v\big(\gamma_w(-t)\big) + b_v\big(\gamma_w(t)\big) = 0$ for all $t \in \mathbb{R}$, $v, w \in S\Omega$ and the main theorem is proved by Theorem 2.3. □

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