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# A CANCELLATION CONJECTURE FOR FREE ASSOCIATIVE ALGEBRAS

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ABSTRACT. We develop a new method to deal with the Cancellation Conjecture of Zariski in different environments. We prove the conjecture for free associative algebras of rank two. We also produce a new proof of the conjecture for polynomial algebras of rank two over fields of zero characteristic.

## 1. Introduction and main results

There is a famous

**Conjecture 1.1** (Cancellation Conjecture of Zariski)**.** *Let $R$ be an algebra over a field $K$. If $R[z]$ is $K$-isomorphic to $K[x_1,\dots,x_n]$, then $R$ is isomorphic to $K[x_1,\dots,x_{n-1}]$.*

Conjecture 1.1 was proved for $n=2$ by Abhyankar, Eakin and Heizer [1], and Miyanishi [10]. For $n=3$, the conjecture was proved by Fujita [5], and Miyanishi and Sugie [11] for zero characteristic, and by Russell [12] for arbitrary fields $K$. For $n\geq 4$, the conjecture remains open to the best of our knowledge. See [4, 6, 7, 8, 9, 14] for Zariski's conjecture and related topics.

Denote by $A*B$ the free product of two $K$-algebras $A$ and $B$. In view of Conjecture 1.1, it is natural and interesting to raise

**Conjecture 1.2** (Cancellation Conjecture for Free Associative Algebras)**.** *Let $R$ be an algebra over a field $K$. If $R*K[z]$ is $K$-isomorphic to $K\langle x_1,\dots,x_n\rangle$, then $R$ is $K$-isomorphic to $K\langle x_1,\dots,x_{n-1}\rangle$.*

In this paper we develop a new method based on the conditions of algebraic dependence, which can be used in different environments. In particular, by this method we prove Conjecture 1.2 for $n=2$:

**Theorem 1.3.** *Let $R$ be an algebra over an arbitrary field $K$. If $R*K[z]$ is $K$-isomorphic to $K\langle x,y\rangle$, then $R$ is $K$-isomorphic to $K[x]$.*

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We also produce a new and simple proof for Conjecture 1.1 for $n = 2$ in the zero characteristic case [1]:

**Proposition 1.4.** *Let $R$ be an algebra over a field $K$ of zero characteristic. If $R[t]$ is $K$-isomorphic to $K[x, y]$, then $R$ is isomorphic to $K[x]$.*

## 2. Preliminaries

Call a set of elements of an associative $K$-algebra *algebraically dependent* over $K$ if the $K$-subalgebra generated by the elements is not free on that generating set. To prove the main results, we need well-known necessary and sufficient conditions for algebraic dependence.

**Lemma 2.1.** *Let $K$ be an arbitrary field, $f$, $g \in K\langle x_1, \ldots, x_n\rangle$. Then $f$ and $g$ are algebraically dependent over $K$ if and only if $[f, g] = 0$, where $[f, g] = fg - gf$ is the commutator of $f$ and $g$.*

See Corollary 6.7.4, p. 338, Cohn [3].

**Lemma 2.2.** *Let $K$ be a field of zero characteristic, $f$, $g \in K[x_1, \ldots, x_n]$. Then $f$ and $g$ are algebraically dependent over $K$ if and only if $J_{x_i,x_j}(f, g) = 0$ for all $1 \le i < j \le n$, where $J_{x_i,x_j}(f, g)$ is the Jacobian determinant of $f$ and $g$ with respect to $x_i$ and $x_j$.*

See, for instance, Jie-Tai Yu [15], for a proof.

We also need a description of the subset of all elements of a polynomial or a free associative algebra which are algebraically dependent on a fixed element. The following result is due to Bergman [2]. See also Cohn [3].

**Lemma 2.3.** *Let $K$ be an arbitrary field, $f \in K\langle x_1, \ldots, x_n\rangle - K$, $\mathcal{C}(f)$ the set of all $g \in K\langle x_1, \ldots, x_n\rangle$ such that $[f, g] = 0$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K\langle x_1, \ldots, x_n\rangle$.*

For polynomial algebras, the analogue of the above result has been obtained by Shestakov and Umirbaev [13]:

**Lemma 2.4.** *Let $K$ be a field of zero characteristic, $f \in K[x_1, \ldots, x_n] - K$, $\mathcal{C}(f)$ the set of all $g \in K[x_1, \ldots, x_n]$ such that $J_{x_i,x_j}(f, g) = 0$ for all $1 \le i < j \le n$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K[x_1, \ldots, x_n]$.*

## 3. Proofs of the main results

*Proof of Theorem* 1.3. Let $R * K[z] \cong K\langle x, y\rangle$. The endomorphism of $R * K[z]$ taking $z$ to 0 and acting as the identity on $R$ is not one-to-one. Hence the images $v$ and $w$ of the generators $x$, $y$ under that endomorphisms are algebraically dependent over $K$. Obviously $R$ is generated by $v$, $w$. By Lemma 2.1, it is easy to deduce that any element $f = f(v, w) \in R$ and $v$ are algebraically dependent over $K$. By Lemma 2.1 and Lemma 2.3, $R \subset K[u]$ for some $u \in R * K[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$, $u_1$ contains only monomials occurring in $u$ with $z$-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore, $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore, $R$ is $K$-isomorphic to $K[x]$. $\square$

*Proof of Proposition* 1.4. As $R[z]$ is $K$-isomorphic to $K[x, y]$, it is easy to deduce that $R$ has a transcendence degree 1 over $K$. Therefore, there exists a $g \in R - K$ such that for all $f \in R$, $f$ and $g$ are algebraically dependent over $K$. By Lemma 2.2 and Lemma 2.4, $R \subset K[u]$ for some $u \in R[t]$. Write $u = u_0 + u_1$, where $u_0 \in R$, $u_1$ contains only monomials occurring in $u$ with $z$-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore, $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore, $R$ is $K$-isomorphic to $K[x]$. □

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