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CUTOFF RESOLVENT ESTIMATES AND THE SEMILINEAR SCHRÖDINGER EQUATION

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Abstract. This paper shows how abstract resolvent estimates imply local smoothing for solutions to the Schrödinger equation. If the resolvent estimate has a loss when compared to the optimal, non-trapping estimate, there is a corresponding loss in regularity in the local smoothing estimate. As an application, we apply well-known techniques to obtain well-posedness results for the semi-linear Schrödinger equation.

1. Introduction

In this short note we show how cutoff semiclassical resolvent estimates for the Laplacian on a non-compact manifold, with spectral parameter on the real axis, lead to well-posedness results for the semilinear Schrödinger equation. Motivated by the requirements of [Chr3] and [BGT2], and the microlocal inverse estimates of [Chr1, Chr2], we first prove a general theorem for a large class of resolvents. Following the recent work of Nonnenmacher-Zworski [NoZw], we apply the general theorem in the case there is a hyperbolic fractal trapped set.

Let (M, g) be a Riemannian manifold of dimension n without boundary, with (non-negative) Laplace-Beltrami operator $-\Delta$ acting on functions. The Laplace-Beltrami operator is an unbounded, essentially self-adjoint operator on $L^2(M)$ with domain $H^2(M)$. We assume (M, g) is asymptotically Euclidean in the sense of [NoZw, §3.2] and that the classical resolvent $(-\Delta - (\lambda^2 + i\epsilon))^{-1}$ obeys a limiting absorption principle as $\epsilon \to 0+$, $\lambda \neq 0$.

Our first result is that if we have cutoff semiclassical resolvent estimates with a sufficiently small loss, then we have weighted smoothing for the Schrödinger propagator with a loss. Let ρ_s be a smooth, non-vanishing weight function satisfying

$$\rho_s(x) \equiv \langle d_g(x, x_0)\rangle^{-s}, \tag{1.1}$$

for some fixed x_0 and x outside a compact set.

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Theorem 1. *Suppose for each compactly supported function $\chi \in \mathcal{C}_c^\infty(M)$ with sufficiently small support, there is $h_0 > 0$ such that the semi-classical Laplace-Beltrami operator satisfies*

$$\|\chi(-h^2\Delta - E)^{-1}\chi u\|_{L^2(M)} \leq \frac{g(h)}{h}\|u\|_{L^2(M)}, \;\; E > 0, \tag{1.2}$$

uniformly in $0 < h \leq h_0$, where $g(h) \geq c_0 > 0$, $g(h) = o(h^{-1})$. Then for each $T > 0$ and $s > 1/2$, there is a constant $C = C_{T,s} > 0$ such that

$$\int_0^T \left\|\rho_s e^{it\Delta} u_0\right\|^2_{H^{1/2-\eta}(M)} dt \leq C\|u_0\|^2_{L^2(M)}, \tag{1.3}$$

where $\eta \geq 0$ satisfies

$$g(h)h^{2\eta} = \mathcal{O}(1), \tag{1.4}$$

and ρ_s is given by (1.1).

The assumption that (M, g) is asymptotically Euclidean is that there exists $R_0 > 0$ sufficiently large that each infinite branch of $M \backslash B(0, R_0)$ agrees with $\mathbb{R}^n$ and on each branch the semiclassical Laplacian $-h^2\Delta$ takes the form

$$-h^2\Delta|_{M\backslash B(0,R_0)} = \sum_{|\alpha|\leq 2} a_\alpha(x,h)(hD_x)^\alpha,$$

with $a_\alpha(x,h) \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and independent of h for $|\alpha| = 2$,

$$\sum_{|\alpha|=2} a_\alpha(x,h)(hD_x)^\alpha \geq C^{-1}|\xi|^2, \; 0 < C < \infty, \text{ and}$$

$$\sum_{|\alpha|\leq 2} a_\alpha(x,h)(hD_x)^\alpha \to |\xi|^2, \quad \text{as } |x| \to \infty \text{ uniformly in } h.$$

In order to quote the results of [NoZw] we also need the following analyticity assumption: $\exists \varepsilon > 0$ such that the $a_\alpha(x,h)$ extend holomorphically to

$$\{r\omega : \omega \in \mathbb{C}^n, \operatorname{dist}(\omega, \mathbb{S}^n) < \varepsilon, \; r \in \mathbb{C}, \; |r| \geq R_0, \; \arg r \in (-\varepsilon, \varepsilon)\}$$

and satisfy the same estimates in this extended region. As in [NoZw], the analyticity assumption immediately implies

$$\partial_x^\beta \left(\sum_{|\alpha|\leq 2} a_\alpha(x,h)\xi^\alpha - |\xi|^2 \right) = o(|x|^{-|\beta|})\langle\xi\rangle^2, \; |x| \to \infty.$$

Recall the free Laplacian $(-\Delta_0 - \lambda^2)^{-1}$ on $\mathbb{R}^n$ has a holomorphic continuation from $\operatorname{Im}\lambda > 0$ to $\lambda \in \mathbb{C}$ for $n \geq 3$ odd, and to the logarithmic covering space for n even. This motivates the limiting absorption assumption that

$$\lim_{\epsilon\to 0+, \; \lambda\neq 0} \rho_s(-\Delta - (\lambda^2 + i\epsilon))^{-1}\rho_s$$

exists as a bounded operator

$$L^2(M, d\mathrm{vol}_g) \to L^2(M, d\mathrm{vol}_g),$$

provided $s > 1/2$. As in the free case, we allow a possible logarithmic singularity at $\lambda = 0$.

The problem of "local smoothing" estimates for the Schrödinger equation has a long history. The sharpest results to date are those of Doi [Doi] and Burq [Bur]. Doi proved if M is asymptotically Euclidean, then one has the estimate

$$\int_0^T \left\|\chi e^{it\Delta} u_0\right\|^2_{H^{1/2}(M)} dt \leq C\|u_0\|^2_{L^2(M)} \tag{1.5}$$

for $\chi \in \mathcal{C}_c^\infty(M)$ if and only if there are no trapped sets. Burq's paper showed if there is trapping due to the presence of several convex obstacles in $\mathbb{R}^n$ satisfying certain assumptions, then one has the estimate (1.5) with the $H^{1/2}$ norm replaced by $H^{1/2-\eta}$ for $\eta > 0$. In [Chr3], the author considered an arbitary, single trapped hyperbolic orbit. One of the goals of this paper is to use estimates obtained by Nonnenmacher-Zworski [NoZw] for fractal hyperbolic trapped sets to obtain similar results to [Chr3] for the semilinear Schrödinger equation. To that end we have the following corollary to Theorem 1.

Corollary 1.1. *Assume (M,g) admits a hyperbolic fractal trapped set, K_E, in the energy level $E > 0$ and that the topological pressure $P_E(1/2) < 0$. Then $-h^2\Delta - E$ satisfies* (1.2) *for some $E > 0$ with $g(h) = C\log(1/h)$, and for every $\eta > 0$, $T > 0$, and $s > 1/2$, there exists a constant $C = C_{P_E,\eta,T,s} > 0$ such that*

$$\int_0^T \left\|\rho_s e^{it\Delta} u_0\right\|^2_{H^{1/2-\eta}(M)} dt \leq C\|u_0\|^2_{L^2(M)}.$$

We remark that the assumption $P_E(1/2) < 0$ implies the trapped set K_E is filamentary or "thin" (see [NoZw] for definitions).

We consider the following semilinear Schrödinger equation problem:

$$\begin{cases} i\partial_t u + \Delta u = F(u) \text{ on } I \times M, \\ u(0,x) = u_0(x), \end{cases} \tag{1.6}$$

where $I \subset \mathbb{R}$ is an interval containing 0. Here the non-linearity F satisfies

$$F(u) = G'(|u|^2)u,$$

and $G : \mathbb{R} \to \mathbb{R}$ is at least C^3 and satisfies

$$|G^{(k)}(r)| \leq C_k \langle r\rangle^{\beta - k},$$

for some $\beta \geq \frac{1}{2}$.

In §3 we prove a family of Strichartz-type estimates which will result in the following well-posedness theorem.

Theorem 2. *Suppose (M,g) satisfies the assumptions of the introduction, and set*

$$\delta = \frac{4\eta}{2\eta+1} \geq 0. \tag{1.7}$$

Then for each

$$s > \frac{n}{2} - \frac{2}{\max\{2\beta - 2, 2\}} + \delta \tag{1.8}$$

and each $u_0 \in H^s(M)$ there exists $p > \max\{2\beta-2, 2\}$ and $0 < T \leq 1$ such that (1.6) *has a unique solution*

$$u \in C([-T,T]; H^s(M)) \cap L^p([-T,T]; L^\infty(M)). \tag{1.9}$$

Moreover, the map $u_0(x) \mapsto u(t,x) \in C([-T,T]; H^s(M))$ is Lipschitz continuous on bounded sets of $H^s(M)$, and if $\|u_0\|_{H^s}$ is bounded, T is bounded from below.

If, in addition, (M, g) *satisfies the assumptions of Corollary* 1.1*,* $n \leq 3$*,* $\beta < 3$*, and* $G(r) \to +\infty$ *as* $r \to +\infty$*, then* u *in* (1.9) *extends to a solution*

$$u \in C((-\infty, \infty); H^1(M)) \cap L^p((-\infty, \infty); L^\infty(M)).$$

Remark 1.2. In particular, the cubic defocusing non-linear Schrödinger equation is globally H^1-well-posed in three dimensions with a fractal trapped hyperbolic set which is sufficiently filamentary. Of course other non-linearities can be considered, but for simplicity we consider only those in this work.

2. Proof of Theorem 1

Since we are assuming $(-\Delta - z)^{-1}$ obeys a limiting absorption principle, we have

$$\|\rho_s(-\Delta - (\tau - i\epsilon))^{-1}\rho_s\|_{L^2 \to L^2} \leq C_\epsilon$$

for $0 < \epsilon_0 \leq |\tau| \leq C$. For $|\sigma| \geq C$ for some $C > 0$, $\sigma \in \mathbb{C}$ in a neighbourhood of the real axis, write

$$\begin{aligned} -\Delta - \sigma &= -\Delta - \frac{z}{h^2} \\ &= h^{-2}(-h^2\Delta - z) \end{aligned}$$

for

$$z \in [E - \alpha, E + \alpha] + i[-c_0 h, c_0 h].$$

Now

$$(-h^2\Delta - z)$$

is a Fredholm operator for z in the specified range, and hence the "gluing" techniques from [Vod] and [Chr3, §2] can be used to conclude for $s > 1/2$,

$$\rho_s(-h^2\Delta - z)^{-1}\rho_s$$

has a holomorphic extension to a slightly smaller neighbourhood in z, and in particular,

$$\|\rho_s(-h^2\Delta - E)^{-1}\rho_s\|_{L^2 \to L^2} \leq C\frac{g(h)}{h}.$$

Rescaling, we have

$$\left\|\rho_s(-\Delta - \tau)^{-1}\rho_s\right\|_{L^2 \to L^2} \leq C\frac{g(\langle\tau\rangle^{1/2})}{\langle\tau\rangle^{1/2}}, \ \ \tau \in \mathcal{C}_{\pm\epsilon}, \tag{2.1}$$

where (see Figure 1)

$$\mathcal{C}_{\pm\epsilon} = \{\tau \in \mathbb{R} : |\tau| \geq \epsilon\} \cup \{\tau \in \mathbb{C} : |\tau| = \epsilon, \pm \operatorname{Im}\tau \geq 0\}.$$

As in [Chr3] and [Bur], the following lemma follows from integration by parts and interpolation, together with the condition on η, (1.4).

Lemma 2.1. *With the notation and assumptions above, we have*

$$\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to H^1} \leq Cg(\langle\tau\rangle^{1/2}), \ \ \tau \in \mathcal{C}_{\pm\epsilon},$$

and for every $r \in [-1, 1]$*,*

$$\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{H^r \to H^{1+r-\eta/2}} \leq C, \ \ \tau \in \mathcal{C}_{\pm\epsilon}.$$

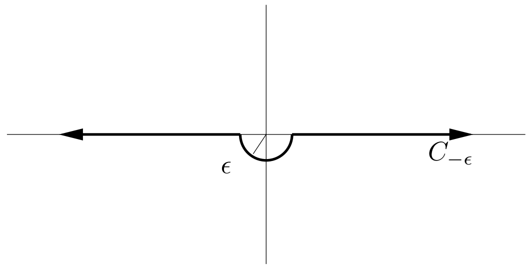

FIGURE 1. The curve $\mathcal{C}_{-\epsilon}$ in the complex plane.

Theorem 1 now follows from the standard "TT^*" argument, letting $\epsilon \to 0$ in (2.1) (see [BGT2], the references cited therein, and [Chr3]). □

The following corollary uses interpolation with an H^2 estimate to replace the $H^{1/2-\eta}$ norm on the left hand side of (1.3) with $H^{1/2}$, and will be of use in §3. See [Chr3] for the details of the proof.

Corollary 2.2. *Suppose* (M, g) *satisfies the assumptions of Theorem* 1. *For each* $T > 0$ *and* $s > 1/2$, *there is a constant* $C > 0$ *such that*

$$\int_0^T \left\| \rho_s e^{it\Delta} u_0 \right\|^2_{H^{1/2}(M)} dt \leq C \|u_0\|^2_{H^\delta(M)}, \tag{2.2}$$

where $\delta \geq 0$ *is given by* (1.7).

In particular, if (M, g) *satisfies the assumptions of Corollary* 1.1, *then for any* $\delta > 0$, *there is* $C = C_\delta > 0$ *such that* (2.2) *holds.*

3. STRICHARTZ-TYPE INEQUALITIES

In this section we give several families of Strichartz-type inequalities and prove Theorem 2. The statements and proofs are mostly adaptations of similar inequalities in [BGT2], so we leave out the proofs of these in the interest of space.

If we view $M \setminus U$, where U is a neighbourhood of K_E, as a manifold with non-trapping geometry, we may apply the results of [HTW] or [BoTz] to a solution of the Schrödinger equation away from the trapping region, resulting in perfect Strichartz estimates. For this section we need (1.3) only with a compact cutoff χ instead of with the more general weight ρ_s.

Proposition 3.1. *For every* $0 < T \leq 1$ *and each* $\chi \in \mathcal{C}_c^\infty(M)$ *satisfying* $\chi \equiv 1$ *near* U, *there is a constant* $C > 0$ *such that*

$$\|(1-\chi)u\|_{L^p([0,T])W^{s,q}(M)} \leq C \|u_0\|_{H^s(M)}, \tag{3.1}$$

where $u = e^{it\Delta} u_0$, $s \in [0, 1]$, *and* (p, q), $p > 2$ *satisfy*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Remark 3.2. In the sequel, wherever unambiguous, we will write

$$L^p_T W^{s,q} := L^p([0,T])W^{s,q}(M)$$

and

$$H^s := H^s(M).$$

Proposition 3.3. *Suppose (M,g) satisfies the assumptions of the Introduction, $u = e^{it\Delta}u_0$, and*

$$v = \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau.$$

Then for each $0 < T \leq 1$ and $\delta \geq 0$ satisfying (1.7), we have the estimates

$$\|u\|_{L^p_T W^{s-\delta,q}} \leq C\|u_0\|_{H^s} \tag{3.2}$$

and

$$\|v\|_{L^p_T W^{s-\delta,q}} \leq C\|f\|_{L^1_T H^s}, \tag{3.3}$$

where $s \in [0,1]$ and (p,q), $p > 2$ satisfy the Euclidean scaling

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{3.4}$$

The proof uses a local WKB expansion also localized in time to the scale of inverse frequency, followed by summing over frequency bands (see [Chr3] and [BGT1]). The only difference here is the explicit dependence of δ on η, which is related to the growth of the function $g(h)$.

Proof of Theorem 2. The proof of Theorem 2 is a slight modification of the proof of Proposition 3.1 in [BGT1], but we include it here in the interest of completeness. Fix s satisfying (1.8) and choose $p > \max\{2\beta - 2, 2\}$ satisfying

$$s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{1}{\max\{2\beta - 2, 2\}}$$

where $\delta \geq 0$ satisfies (1.7). Set $\sigma = s - \delta$ and

$$Y_T = C([-T,T]; H^s(M)) \cap L^p([-T,T]; W^{\sigma,q}(M))$$

for

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

equipped with the norm

$$\|u\|_{Y_T} = \max_{|t| \leq T} \|u(t)\|_{H^s(M)} + \|u\|_{L^p_T W^{\sigma,q}}.$$

Let Φ be the non-linear functional

$$\Phi(u) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.$$

If we can show that $\varphi : Y_T \to Y_T$ and is a contraction on a ball in Y_T centered at 0 for sufficiently small $T > 0$, this will prove the first assertion of the proposition, along with the Sobolev embedding

$$W^{\sigma,q}(M) \subset L^\infty(M), \tag{3.5}$$

since $\sigma > n/q$. From Proposition 3.3, we bound the W^σ part of the Y_T norm by the H^s norm, giving

$$\begin{aligned} \|\Phi(u)\|_{Y_T} &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^T \|F(u(\tau))\|_{H^s} d\tau\right) \\ &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^T \|(1 + |u(\tau)|)\|_{L^\infty}^{2\beta-2})\|u(\tau)\|_{H^s} d\tau\right), \end{aligned}$$

where the last inequality follows by our assumptions on the structure of F. Applying Hölder's inequality in time with $\tilde{p} = p/(2\beta - 2)$ and $\tilde{q}$ satisfying

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} = 1$$

gives

$$\|\varphi(u)\|_{Y_T} \leq C\left(\|u_0\|_{H^s} + T^\gamma \|u\|_{L_T^\infty H^s} \|(1+|u|)\|_{L_T^p L^\infty}^{2\beta-2}\right)$$

where $\gamma = 1/\tilde{q} > 0$. Thus

$$\|\varphi(u)\|_{Y_T} \leq C\left(\|u_0\|_{H^s} + T^\gamma(\|u\|_{Y_T} + \|u\|_{Y_T}^{2\beta})\right).$$

Similarly, we have for $u, v \in Y_T$,

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_{Y_T} && (3.6)\\
&\quad\leq CT^\gamma \|u - v\|_{L_T^\infty H^s} \|(1+|u|)\|_{L_T^p L^\infty}^{2\beta-2} + \|(1+|v|)\|_{L_T^p L^\infty}^{2\beta-2}) && (3.7)\\
&\quad\leq CT^\gamma \|u - v\|_{Y_T} \|(1+|u|)\|_{Y_T}^{2\beta-2} + \|(1+|v|)\|_{Y_T}^{2\beta-2}),
\end{aligned}$$

which is a contraction for sufficiently small T. This concludes the proof of the first assertion in the proposition.

To get the second assertion, we observe from (3.6) and the definition of Y_T, if u and v are two solutions to (1.6) with initial data u_0 and u_1 respectively, so

$$\widetilde{\Phi}(v) = e^{it\Delta}u_1 - i\int_0^t e^{i(t-\tau)\Delta} F(v(\tau))d\tau,$$

we have

$$\begin{aligned}
&\max_{|t|\leq T} \|u(t) - v(t)\|_{H^s}\\
&\quad= \max_{|t|\leq T} \|\Phi(u)(t) - \widetilde{\Phi}(v)(t)\|_{H^s}\\
&\quad\leq C\Bigg(\|u_0 - u_1\|_{H^s}\\
&\qquad + T^\gamma \max_{|t|\leq T} \|u(t) - v(t)\|_{H^s} \|(1+|u|)\|_{L_T^p L^\infty}^{2\beta-2} + \|(1+|v|)\|_{L_T^p L^\infty}^{2\beta-2})\Bigg),
\end{aligned}$$

which, for $T > 0$ sufficiently small gives the Lipschitz continuity.

If (M, g) satisfies the assumptions of Corollary 1.1, $n \leq 3$, $\beta < 3$, and $G(r) \to +\infty$ as $r \to +\infty$, we can take s and p satisfying $p > \max\{2\beta - 2, 2\}$ and

$$s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{2}{\max\{2\beta - 2, 2\}}$$

for any $\delta > 0$. Then $\sigma = s - \delta > q/n$ and the preceding argument holds. Finally, the proof of the global well-posedness now follows from the standard global well-posedness arguments from, for example, [Caz, Chapter 6]. □

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