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MEASURABLE SENSITIVITY

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Abstract. We introduce the notions of measurable and strong measurable sensitivity, which are measure-theoretic versions of the conditions of sensitive dependence on initial conditions and strong sensitive dependence on initial conditions, respectively. Strong measurable sensitivity is a consequence of light mixing, implies that a transformation has only finitely many eigenvalues, and does not exist in the infinite measure-preserving case. Unlike the traditional notions of sensitive dependence, measurable and strong measurable sensitivity carry up to measure-theoretic isomorphism, thus ignoring the behavior of the transformation on null sets and eliminating dependence on the choice of metric.

1. Introduction

We introduce the notions of measurable sensitivity and strong measurable sensitivity, which are measure-theoretic versions of sensitive dependence and strong sensitive dependence, respectively. Sensitive dependence on initial conditions, which is widely understood to be one of the central ideas of chaos, is a topological, rather than measurable, notion. It was introduced by Guckenheimer in [8]. A transformation T on a metric space (X, d) is said to exhibit **sensitive dependence** with respect to d if there exists $\delta > 0$ such that for all $\varepsilon > 0$ and for all $x \in X$, there exists $n \in \mathbb{N}$ and $y \in B_\varepsilon(x)$ such that $d(T^n(x), T^n(y)) > \delta$. The notion of sensitive dependence has been studied extensively, and the reader is referred to [3], [7] and [2] for recent results. The relationship between measure-theoretic notions, such as weak mixing, and sensitive dependence is studied in [7], [2], [4], and [10]. A stronger notion of sensitive dependence, called strong sensitive dependence, was introduced in [1]. A transformation T on a metric space (X, d) is said to exhibit **strong sensitive dependence** with respect to d if there exists $\delta > 0$ such that for all $\varepsilon > 0$ and for all $x \in X$, there exists $N \in \mathbb{N}$ so that for all integers $n \geq N$ there exists $y \in B_\varepsilon(x)$ such that $d(T^n(x), T^n(y)) > \delta$. Both sensitive dependence and strong sensitive dependence are topological notions, depending on both the choice of metric and the behavior of the transformation on null sets.

In Section 2 we show that a doubly ergodic (a condition equivalent to weak mixing for finite measure-preserving transformations), nonsingular transformation is measurably sensitive, that a lightly mixing, nonsingular transformation (for example, a mixing, finite measure-preserving transformation) is strong measurably

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sensitive, and that strong measurable sensitivity does not imply weak mixing. In Section 3 we show that if an ergodic, nonsingular transformation T is strong measurably sensitive, then there exists an integer $n > 0$ so that T^n has n invariant subsets and the restriction of T^n to each of these subsets is weakly mixing. Section 4 shows that if an ergodic, finite measure-preserving transformation T is strong measurably sensitive, then there is an integer n so that T^n has n invariant sets of positive measure covering X a.e. and such that the restriction of T^n to each is lightly mixing. The final section shows that an ergodic, infinite measure-preserving transformation cannot be strong measurably sensitive (although it can be measurably sensitive by Section 2).

Throughout the paper, the ordered quadruple $(X, \mathcal{S}(X), \mu, T)$ denotes a standard Lebesgue space X with a nonnegative, finite or σ-finite, nonatomic measure μ, with $\mathcal{S}(X)$ the collection of μ-measurable subsets of X, and T a μ-measurable nonsingular endomorphism of X (see, e.g., [11]). In some cases we require T measure-preserving. It is a standard fact that any two such spaces $(X, \mathcal{S}(X), \mu)$ and $(X_1, \mathcal{S}(X_1), \mu_1)$ are measurably isomorphic under a nonsingular isomorphism. We define a metric d on $(X, \mathcal{S}(X), \mu)$ to be **μ-compatible** if μ assigns a positive (nonzero) measure to all nonempty, open d-balls. Given two nonempty sets A and B in a space X with metric d, we define $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Lemma 1.1. *If d is a μ-compatible metric on $(X, \mathcal{S}(X), \mu)$, then X is separable under d.*

Proof. Assume $\mu(X) < \infty$; the extension to the σ-finite case is standard and left to the reader. Let $\{C_\alpha\}_{\alpha\in\Gamma}$ consist of all countable subsets of X. For each $n \in \mathbb{N}$, let $\tau_n = \sup_{\alpha\in\Gamma}\{\mu(\bigcup_i B^d_{1/n}(x_i)) : x_i \in C_\alpha\}$. Then for each n, there exists a sequence $\{C_{j,n}\}_{j=1}^{\infty}$ of elements of $\{C_\alpha\}_{\alpha\in\Gamma}$ such that $\limsup_{j\to\infty}\{\mu(\bigcup B^d_{1/n}(x_i)) : x_i \in C_{j,n}\} = \tau_n$. Let $D_n = \bigcup_j C_{j,n}$. D_n^c cannot contain any ball of radius $\frac{1}{n}$, as any such ball would have positive measure, contradicting the maximality of τ_n. Consequently, every point in X has distance at most $\frac{2}{n}$ to a point in D_n. Then the set $D = \bigcup_n D_n$ is a countable dense set. □

Definition 1.2. Let $(X, \mathcal{S}(X), \mu, T)$ be a nonsingular dynamical system. T is **strong measurably sensitive** if whenever a dynamical system $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ is measurably isomorphic to $(X, \mathcal{S}(X), \mu, T)$ and d is a μ_1-compatible metric on X_1, then there exists $\delta > 0$ such that for $x \in X_1$ and all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers $n \geq N$

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n x, T_1^n y) > \delta\} > 0.$$

Definition 1.3. Let $(X, \mathcal{S}(X), \mu, T)$ be a nonsingular dynamical system. T exhibits **measurable sensitivity** if whenever a dynamical system $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ is measurably isomorphic to $(X, \mathcal{S}(X), \mu, T)$ and d is a μ_1-compatible metric on X_1 then there exists $\delta > 0$ such that for $x \in X_1$ and all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0.$$

The real number δ in the above definition is referred to as a sensitivity constant.

Example 1.4. Consider a measure space X consisting of two copies of the circle S^1, labeled S_1 and S_2. Define a metric d on X as follows: for two points $x \in S_i$ and $y \in S_j$, if $i = j$ define $d(x, y)$ to be the the minimal arclength between points

x and y, and if $i \neq j$ define $d(x, y) = r$ for some fixed $r > \frac{\pi}{2}$. In each copy of S^1, pick out the orbit of a fixed point z under a fixed irrational rotation R on S^1, and denote this set by M. Define a transformation $T : X \to X$ as follows: for $x \in M$, let $T(x) = R(x)$, and for $x \in X \setminus M$, let $T(x)$ map x to $R(x)$ in the *other* copy of S^1.

Because points in M go "far away" from those in M^c, it is clear that the system exhibits sensitive dependence, but without this null set sensitive dependence would fail. Thus, the system does not exhibit either strong measurable or measurable sensitivity. In fact, $T^n(B_\varepsilon(x)) = B_\varepsilon(R^2(x))$ for all $n \equiv 0 \pmod 2$.

Proposition 1.5. *Let X be an interval of finite length in $\mathbb{R}$ and let d be the standard Euclidean metric on X. If a continuous transformation $T : X \to X$ exhibits sensitive dependence with respect to d, then T exhibits strong sensitive dependence with respect to d.*

Proof. Suppose T exhibits sensitive dependence with sensitivity constant δ. Let I_1,...,I_n be disjoint (except at endpoints) intervals with closed or open endpoints which cover X and where each has length shorter than $\frac{\delta}{2}$. Every interval of length at least δ must contain one of these intervals. Since T is sensitive, for each $1 \leq j \leq n$ there must exist a natural number m_j so that $T^{m_j}(I_j)$ has length at least δ. Consequently, for any interval I with length at least δ, and any $m \in \mathbb{N}$, $T^m(I)$ contains one of $T^k(I_i)$, where $1 \leq i \leq n$ and $1 \leq k \leq m_i$. Let δ' be one third of the minimum of the lengths of these intervals. Let $x \in X$ and $\varepsilon > 0$. Since T is sensitive, there must be some k_0 so that $T^{k_0}(B_\varepsilon(x))$ contains an interval of length δ, and thus for any $k > k_0$, $T^k(B_\varepsilon(x))$ contains an interval of length at least $3\delta'$, and hence contains a point whose distance is at least δ' from $T^k(x)$. $\square$

2. Mixing notions and strong measurable sensitivity

We begin by proving that double ergodicity implies measurable sensitivity. A nonsingular transformation is said to be **doubly ergodic** if for all sets A and B of positive measure there exists an integer $n > 0$ such that $\mu(T^{-n}(A) \cap A) > 0$ and $\mu(T^{-n}(A) \cap B) > 0$. Double ergodicity is equivalent to weak mixing for measure-preserving transformations on finite measure spaces [6], but strictly stronger than weak mixing in the infinite measure-preserving case [5].

Proposition 2.1. *If $(X, \mathcal{S}(X), \mu, T)$ is a nonsingular, doubly ergodic dynamical system, then T exhibits measurable sensitivity. In particular, weakly mixing, finite measure-preserving transformations exhibit measurable sensitivity.*

Proof. Let $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ be measurably isomorphic to $(X, \mathcal{S}(X), \mu, T)$, and let d be a μ_1-compatible metric on X_1. Then there exist sets $A, C \subset X_1$ of positive measure such that $d(A, C) > 0$. Pick $\delta < d(A, C)/2$. Let $x \in A$ and $\varepsilon > 0$. As $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ is doubly ergodic, there exists $n \in \mathbb{N}$ such that

$$\mu_1(T_1^{-n}C \cap B_\varepsilon(x)) > 0 \quad \text{and} \quad \mu_1(T_1^{-n}A \cap B_\varepsilon(x)) > 0.$$

Therefore, $\mu_1\{y \in B_\varepsilon(x) : T_1^n(y) \in A\} > 0$ and $\mu_1\{y \in B_\varepsilon(x) : T_1^n(y) \in C\} > 0$. Since $T_1^n(x)$ cannot be within δ of both A and C, $\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0$. $\square$

We now turn our attention to strong measurable sensitivity and light mixing. Recall that a system $(X, \mathcal{S}(X), \mu, T)$ on a finite measure space is said to be **lightly mixing** if $\liminf_{n\to\infty} \mu(T^{-n}(A) \cap B) > 0$ for all sets A and B of positive measure.

Proposition 2.2. *If $(X, \mathcal{S}(X), \mu, T)$ is a nonsingular, lightly mixing dynamical system, then T is strong measurably sensitive.*

Proof. Let $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ be measurably isomorphic to $(X, \mathcal{S}(X), \mu, T)$, and let d be a μ_1-compatible metric on X_1. Then there exist sets $A, C \subset X_1$ of positive measure with $d(A, C) > 0$. Pick $\delta < \frac{d(A,C)}{2}$. Let $x \in X_1$ and $\varepsilon > 0$. By light mixing, there exists $N \in \mathbb{N}$ such that for any integer $n \geq N$, $\mu_1(T_1^{-n}(C) \cap B_\varepsilon(x)) > 0$ and $\mu_1(T_1^{-n}(A) \cap B_\varepsilon(x)) > 0$. Thus, $n \geq N$ implies $\mu_1\{y \in B_\varepsilon(x) : T_1^n(y) \in A\} > 0$ and $\mu_1\{y \in B_\varepsilon(x) : T_1^n(y) \in C\} > 0$. Since $d(A, C) > 2\delta$, $T_1^n(x)$ cannot be within δ of both A and C. As $T_1^n(B_\varepsilon(x))$ intersects both A and C in sets of positive measure, $\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0$, so T is strong measurably sensitive. $\square$

In Proposition 4.4, we prove that a finite measure-preserving, weakly mixing transformation that is not lightly mixing is not strong measurably sensitive (but is measurably sensitive).

Lemma 2.3. *If T_1 is a finite measure-preserving, lightly mixing transformation on $(X, \mathcal{S}(X), \mu)$ and T_2 is a rotation on two points with counting measure λ, then $T_1 \times T_2$ with product measure is strong measurably sensitive and ergodic but not weakly mixing.*

Proof. It is well known that all powers of T_1 are lightly mixing. We claim that $T_1 \times T_2$ is strong measurably sensitive. $(T_1 \times T_2)^2$ acts as a lightly mixing transformation on $X \times \{1\}$ and $X \times \{2\}$. Let $(Y, \mathcal{S}(Y), \nu, S)$ be isomorphic to $(X \times \{1, 2\}, \mu \times \lambda, T_1 \times T_2)$ and let $g : X \times \{1, 2\} \to Y$ be the corresponding isomorphism of measure spaces. Under a ν-compatible metric on Y, there exist sets A_i and B_i in $g(X \times \{i\})$ with a positive distance between A_i and B_i. Any ball $B_\epsilon(y)$ will intersect at least one of $g(X \times \{1\})$ and $g(X \times \{2\})$ with positive measure. Then, as a consequence of light mixing, for n sufficiently large, either both $S^{-n}(A_1)$ and $S^{-n}(B_1)$ or both $S^{-n}(A_2)$ and $S^{-n}(B_2)$ intersect $B_\epsilon(x)$ in sets of positive measure. Consequently, S exhibits strong measurable sensitivity with strong measurable sensitivity constant $\delta = \min\{d(A_1, B_1\}, d(A_2, B_2)\}$. Hence, S is strong measurably sensitive. As T_2 is ergodic and T_1 is weakly mixing, $T_1 \times T_2$ is ergodic. Finally, $T_1 \times T_2$ is not weakly mixing since $-1 \in e(T_1 \times T_2)$, the eigenvalue group of $T_1 \times T_2$. $\square$

3. Strong measurable sensitivity and eigenvalues

We show that if an ergodic, nonsingular transformation is strong measurably sensitive, then it can have only finitely many eigenvalues. (Recall that λ is an (L^∞) eigenvalue of T if there is a nonzero a.e. $f \in L^\infty$ such that $f \circ T = \lambda f$ a.e. Also, if T is ergodic and finite measure-preserving, its L^2 eigenfunctions are in L^∞.) This enables a further characterization of strong measurably sensitive transformations. All (L^∞) eigenvalues of ergodic transformations lie on the unit circle. We refer to an eigenvalue as rational if it is of finite order and irrational if it is not.

Lemma 3.1. *Let X be a Lebesgue space with measure μ. Suppose a nonsingular transformation $T : X \to X$ has an eigenfunction f with an eigenvalue that is of the form $\exp(2\pi i q)$ with q irrational. Then for any open interval $I \subset (0, 1)$, the forward orbit of the set $f^{-1}(\exp(2\pi i I))$ equals $X \pmod{\mu}$.*

Proof. Choose f so that $|f| = 1$ a.e. Let h be a translation by q on $\mathbb{R}/\mathbb{Z}$. Then $\bigcup_{n=0}^{\infty} h^n(I) = \mathbb{R}/\mathbb{Z}$, and consequently

$$\bigcup_{n=0}^{\infty} T^n(f^{-1}(\exp(iI))) = \bigcup_{n=0}^{\infty} f^{-1}(h^n(\exp(iI))) = f^{-1}(\exp(\mathbb{R}/\mathbb{Z})) = X,$$

where the final equality is (mod μ). $\square$

Lemma 3.2. *Suppose an ergodic nonsingular transformation $T : X \to X$ has an eigenfunction f, $|f| = 1$, with an eigenvalue that is of the form $\exp(2\pi iq)$ with q irrational. Then T is not strong measurably sensitive.*

Proof. Construct a nonsingular isomorphism from (X, μ) to $[0, 1)$ as follows. By Lemma 3.1 and countable subadditivity, each of the sets $f^{-1}(\exp(i(2^{1-n}\pi, 2^{2-n}\pi]))$ has positive measure. Then for $n \geq 1$, there exist nonsingular isomorphisms g_n from $f^{-1}(\exp(i(2^{1-n}\pi, 2^{2-n}\pi]))$ to $(2^{-n}, 2^{-n+1})$ with Lebesgue measure. Let g be the point map associated with g_n on $f^{-1}(\exp(i(2^{1-n}\pi, 2^{2-n}\pi]))$ for each positive n. We now define a new metric d on X given by $d(x, y) = |g(x) - g(y)|$. As each isomorphism is nonsingular and open sets have positive measure under Lebesgue measure, d is a μ-compatible metric. Let $x \in X$ be a point such that $|g(x) - 2^{-n}| < 2^{-n-1}$ and let $\varepsilon < 2^{-n-1}$. Then $B^d_\varepsilon(x) \subset f^{-1}(\exp(i(0, 2^{1-n}\pi]))$. Let h be a translation by q on $\mathbb{R}/\mathbb{Z}$. Then there is a sequence $\{m_k\}_{k=1}^{\infty}$ for which

$$h^{m_k}((0, 2^{1-n}]) \subset (0, 2^{2-n}].$$

Consequently, $T^{m_k}(B_\varepsilon(x))$ is contained in $f^{-1}(\exp(i(0, 2^{2-n}\pi]))$, and so for almost every point $y \in T^{m_k}(B_\varepsilon(x))$, $d(T^{m_k}(y), T^{m_k}(x)) < 2^{2-n}$. As $n \to \infty$, $2^{2-n} \to 0$, so there is no possible strong measurable sensitivity constant. $\square$

Lemma 3.3. *If an ergodic nonsingular transformation T on a Lebesgue space X has infinitely many rational eigenvalues, then T is not strong measurably sensitive.*

Proof. Since the eigenvalues of a transformation form a multiplicative group, there exists an increasing sequence of positive integers a_i so that $e^{\frac{2\pi i}{a_i}}$ is an eigenvalue and $a_i | a_{i+1}$. These eigenvalues indicate the existence of a_i invariant sets under T^{a_i}. There are $\frac{a_{i+1}}{a_i}$ invariant sets for T^{a_i+1} contained in each invariant set for T^{a_i}. The invariant sets can be enumerated so that the invariant sets for T^{a_i} are represented by integers $A_{i,0}, ..., A_{i,a_i-1}$ and the $T^{a_{i+1}}$-invariant sets $A_{i+1,m}$ contained in a given T^{a_i}-invariant set $A_{i,n}$ have $m \equiv 2 \pmod{a_i}$. Then any sequence of integers $\{b_i\}_{i=1}^{\infty}$, for which $0 \leq b_i < \frac{a_{i+1}}{a_i}$, corresponds to a sequence of sets C_i such that $C_{i+1} \subset C_i$, where $C_i = A_{i, \sum_{j=1}^{i} b_j a_j}$. For any such sequence, the set $\bigcap_{i=1}^{\infty} C_i$ cannot intersect any analogous set corresponding to another sequence. There are uncountably many possible sequences of integers, so there must be a sequence of integers $\{b_i\}_{i=1}^{\infty}$ so that $\bigcap_{i=1}^{\infty} C_i$ has measure 0 for the corresponding sequence of sets $\{C_i\}$. Choose such a sequence. Then letting $C_0 = X$, the space may be expressed as $X = \bigsqcup_{i=0}^{\infty} C_i \backslash C_{i+1}$ (mod μ).

Let g_i be a nonsingular isomorphism from $C_i \backslash C_{i+1}$ to $(2^{-i-1}, 2^{-i})$ with Lebesgue measure. Let N be the backwards orbit of $\bigcap_{i=1}^{\infty} C_i$, where none of the maps g_i are defined. N must have measure zero. Then let $g : X \backslash N \to X \backslash N$ be the union of the maps g_i. Let T' be the restriction of T to $X \backslash N$. Let d be a metric on $X \backslash N$ defined by $d(x, y) = |g(x) - g(y)|$. Since each map g_i is nonsingular, d is a μ-compatible metric. Choose a point $x \in C_i \backslash N$ and let ε be small enough so that $B^d_\varepsilon(x) \subset C_i$.

Then for each positive integer k, $(T')^{ka_i}(B^d_\varepsilon(x)) \subset C_i \backslash N$. Any two points in $C_i \backslash N$ can have a distance of at most 2^{-i} between them and so any strong measurable sensitivity constant δ for T' would have to be at most 2^{-i}. Consequently, there can be no positive strong measurable sensitivity constant, and T is not strong measurably sensitive. $\square$

Corollary 3.4. *Any ergodic, nonsingular, strong measurably sensitive transformation has finitely many eigenvalues.*

Proposition 3.5. *If an ergodic, nonsingular transformation $T : X \to X$ is strong measurably sensitive, then for some $n \in \mathbb{N}$, T^n has n invariant subsets and the restriction of T^n to each of these subsets is weakly mixing.*

Proof. By Corollary 3.4, T must have finitely many eigenvalues. The eigenvalues form a cyclic group of finite order n. Let f be an eigenfunction whose eigenvalue has order n. Then the sets $A_k = f^{-1}\left(\exp\left(i\left[\frac{2\pi k}{n}, \frac{2\pi(k+1)}{n}\right)\right)\right)$ for $0 \le k \le n-1$ are invariant under T^n. First, we show that the restriction of T^n to each of these sets is ergodic. Suppose there exists a T^n-invariant set C of positive measure. Notice that since $T^{-n}(C) = C$, then $\bigcup_{i=0}^{n-1} T^{-i}(C)$ is T-invariant, and thus equals $X (\text{mod } \mu)$. Each of $T^{-i}(C)$ for $0 \le i \le n-1$ must be contained in A_k for a different k and so $C \cap A_k = A_k \ (\text{mod } \mu)$ for all k. Hence, the restriction of T^n to A_k is ergodic for $0 \le k \le n-1$.

Next, we prove that the restriction of T^n to any of these sets admits no eigenvalues other than 1. Suppose T^n restricted to A_0 admits an eigenvalue $\lambda \ne 1$. For $0 \le k \le n-1$, let $h(T^{-k}(x)) = \lambda^{\frac{-k}{n}} f(x)$. Then h will be an eigenfunction of T which will have order greater than n. Thus the restriction of T^n to A_i for each $0 \le i \le n-1$ is ergodic and admits no eigenvalues other than 1, so it must be weakly mixing. $\square$

Corollary 3.6. *A totally ergodic, strong measurably sensitive transformation is weakly mixing.*

4. Strong measurable sensitivity for finite measure-preserving transformations

In this section, we consider strong measurable sensitivity for measure-preserving transformations on finite measure spaces. Considering only such spaces, Proposition 3.5 is strengthened to include a requirement of light mixing.

Lemma 4.1. *Suppose $T : X \to X$, $\mu(X) = 1$, is measure-preserving and not lightly mixing. Then there exist sets C_1 and D_1 of positive measure and an infinite sequence $\{n_k\}_{k=1}^\infty$ such that $T^{n_k}(C_1) \cap D_1 = \emptyset$ for all $k \in \mathbb{N}$.*

Proof. From the definition of lightly mixing, we may assume that there exist sets C and D of positive measure so that $\liminf_{n\to\infty} \mu(C \cap T^{-n}(D)) = 0$. Choose an increasing sequence of distinct natural numbers $\{n_k\}_{k=1}^\infty$ so that $\mu(C \cap T^{-n_k}(D)) \le 2^{-k-1}\mu(C)$. Let $C_1 = C \backslash \bigcup_{k=1}^\infty (C \cap T^{-n_k}(D))$ and $D_1 = D$. Then $\mu(C_1) > \frac{1}{2}\mu(C) > 0$ and so $T^{n_k}(C_1) \cap D_1 = \emptyset$ for every k. $\square$

The proof of the following lemma is standard and is omitted.

Lemma 4.2. *Let $f : (0,1) \to (0,1)$ be a continuous function with $f(x) > x$ for every $x \in (0,1)$. Then for every $x \in (0,1)$, $\lim_{n\to\infty} f^n(x) = 1$.*

Lemma 4.3. *Let $T : X \to X$, $\mu(X) = 1$, be a finite measure-preserving, weakly mixing and not lightly mixing transformation. Then there exist sequences of measurable sets $\{C_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ satisfying the following properties:*

(1) $\mu(C_i) > 0$ *and* $\mu(D_i) > 0$.
(2) $C_i \subset C_{i-1}$ *for* $i > 1$.
(3) $D_{i-1} \subset D_i$ *for* $i > 1$.
(4) $\lim_{i\to\infty} \mu(D_i) = 1$.
(5) $\lim_{i\to\infty} \mu(C_i) = 0$.
(6) *There is a sequence* $\{n_k\}_{k=1}^{\infty}$ *so that* $T^{n_k}(C_i) \cap D_i = \emptyset$.

Proof. We let C_1, D_1, and $\{n_k\}_{k=1}^{\infty}$ be as defined in Lemma 4.1. They clearly satisfy properties (1-3) and property (6). For the inductive step, assume that C_i and D_i have been chosen to satisfy these properties for all $i \le j$. As a consequence of weak mixing, there is a zero density subset $E_1 \subset \mathbb{N}$ such that

$$\lim_{n\to\infty, n\notin E_1} \mu(T^{-n}(C_j) \cap C_j) = \mu(C_j)^2.$$

As a result, the set of values n with $\mu(T^{-n}(C_j)\cap C_j) > 0$ has density 1. Similarly, there exists a zero density subset $E_2 \subset \mathbb{N}$ such that

$$\lim_{n\to\infty, n\notin E_2} \mu(T^{-n}(D_j) \cap D_j) = \mu(D_j)^2.$$

Consequently, there is a natural number m_j so that $\mu(T^{-m_j}(C_j) \cap C_j) > 0$ and $\mu(T^{-m_j}(D_j) \cap D_j) < \frac{1}{2}(\mu(D_j)^2 + \mu(D_j))$. Then

$$\mu(T^{-m_j}(D_j)\backslash D_j) > \frac{1}{2}(\mu(D_j) - \mu(D_j)^2).$$

For positive integers j, we let $C_{j+1} = C_j \cap T^{-m_j}(C_j)$ and let $D_{j+1} = D_j \cup T^{-m_j}(D_j)$. Properties (2) and (3) are clear from the definitions of C_{j+1} and D_{j+1}. Property (1) for D_{j+1} follows from the fact that D_0 has positive measure and $D_i \subset D_{i+1}$ for all $i \le j$. Property (1) for C_{j+1} follows from the fact that m_j is chosen to make C_{j+1} have positive measure. Property (4) follows from Lemma 4.2 applied to the function $\frac{3}{2}x - \frac{1}{2}x^2$ and the lower bound for the measure of D_{i+1} in terms of D_i. To show property (6), it suffices to see that

$$\begin{aligned} &T^{n_k}(T^{-m_j}(C_j) \cap C_j) \cap (T^{-m_j}(D_j) \cup D_j) \\ &\subset T^{-m_j}(T^{n_k}(C_j) \cap D_j) \cup (T^{n_k}(C_j) \cap D_j) = \emptyset. \end{aligned}$$

Property (5) follows from property (4) and the fact that $T^{-n_k}(D_i)$ and C_i must be disjoint. ∎

Proposition 4.4. *If $T : X \to X$, $\mu(X) = 1$, is weakly mixing, measure-preserving and not lightly mixing, then T is not strong measurably sensitive.*

Proof. The space X can be decomposed as $X = \bigsqcup_{i,j=0}^{\infty}(C_i\backslash C_{i+1}) \cap (D_{j+1}\backslash D_j)$ (mod μ), where $C_0 = X$ and $D_0 = \emptyset$, and C_i and D_i are as in Lemma 4.3. Let $g_{i,j}$ be a nonsingular isomorphism from $(C_i\backslash C_{i+1}) \cap (D_{j+1}\backslash D_j)$ to $(2^{-j} + 2^{-i-1-j}, 2^{-j} + 2^{-i-j})$ with Lebesgue measure whenever $(C_i\backslash C_{i+1}) \cap (D_{j+1}\backslash D_j)$ has positive measure. Let N be the union of the backwards orbits of the points where no $g_{i,j}$ is defined. This set has measure zero, so the restriction of T to $X\backslash N$ is isomorphic to T. Let T' denote this restriction. Define function g on $X\backslash N$ by letting $g(x) = g_{i,j}(x)$. Then $d(x, y) = |g(x) - g(y)|$ is a metric on $X\backslash N$. A ball around a point $x \in X\backslash N$ must have positive measure since each of the maps $g_{i,j}$ is nonsingular, so the metric is μ-compatible. Note that $g(D_j) \subset (2^{-j}, 1)$ and $g(D_j^c) \subset (0, 2^{-j})$. Let x be a point in $(C_i\backslash N) \cap D_j$ for some j where such a point exists, and choose ε so that $B_\varepsilon^d(x) \subset C_i \cup D_j$. From property (6) of Lemma 4.3, there is a sequence $\{n_k\}_{k=1}^\infty$ so that $g((T')^{n_k}(B_\varepsilon(x))) \subset g^{-1}(0, 2^{-i})$. Since any strong sensitivity constant δ must be smaller than 2^{-i} for each positive integer i, there is no possible strong sensitivity constant. Hence, T' does not exhibit strong sensitive dependence for any μ'-compatible metric d, and so T is not strong measurably sensitive. $\square$

Lemma 4.5. *Suppose a strong measurably sensitive transformation $T : X \to X$, $\mu(X) = 1$, has finitely many invariant subsets A_1,A_2,..,A_n of positive measure. Then the restriction of T to each subset is strong measurably sensitive.*

Proof. Suppose the restriction $T|A_i = S$ is not strong measurably sensitive. Let d be a μ'-compatible metric on a set A_i' so that (A_i', μ', S') is measurably isomorphic to (A_i, μ, S) and assume that S' does not exhibit strong sensitivity. Let g be a measure-preserving isomorphism from $X\backslash A_i$ to (0,1) with Lebesgue measure. Let N be the set of points where g and the isomorphism from A_i' to A_i are not preserved as well as their backwards orbits. Then N must be a null set. For any two points $x, y \in X\backslash(A_i \cup N)$, extend d so that $d(x, y) = |g(x) - g(y)|$. Let $X' = X\backslash(A_i \cup N) \cup A_i'$ and let $T' : X' \to X'$ be equal to the restriction of T on $X\backslash(A_i \cup N)$ and equal to S' on A_i'. Then T' is measurably isomorphic to T. Now d is extended to a metric on X'. For any two points $x, y \in X\backslash(A_i \cup N)$, extend d so that $d(x, y) = |g(x) - g(y)|$. Choose a point $y_0 \in A_i'$. For points $y \in A_i'$ and for $x \in X\backslash(A_i \cup N)$, let $d(y, x) = d(x, y) = 1 + d(y, y_0)$. It is easy to verify that the extension of d is a μ-compatible metric on X'.

As S' does not exhibit strong sensitivity on A_i', for any $\varepsilon > 0$, there is a ball $B_r(x)$ with $x \in A_i'$ so that for some sequence $\{n_k\}$, almost every point $y \in B_r(x) \cap A_i'$ satisfies $d((S')^{n_k}(y), (S')^{n_k}(x)) < \varepsilon$. We may assume that $r < 1$. Then $B_r(x) = B_r(x) \cap A_i'$. As a consequence, almost every point $y \in B_r(x)$ satisfies $d((T')^{n_k}(y), (T')^{n_k}(x)) < \varepsilon$. Hence, T' is not strongly sensitive and so T is not strong measurably sensitive. $\square$

Theorem 4.6. *Let T be an ergodic transformation on a probability measure Lebesgue space X. If T is strong measurably sensitive, then there is some positive integer n so that T^n has n invariant sets of positive measure which cover almost all of X, and the restriction of T^n to each of the sets is lightly mixing.*

Proof. Suppose T is strong measurably sensitive. By Proposition 3.5, there are n invariant sets for T^n, each of positive measure. Every power of a strong measurably sensitive transformation is clearly strong measurably sensitive, so T^n must be strong measurably sensitive. By Lemma 4.5, the restriction of T^n to any of the sets must be strong measurably sensitive and, by Proposition 3.5, the restriction must be weakly mixing. Consequently, Proposition 4.4 indicates that the restriction must be lightly mixing. $\square$

5. Infinite measure spaces

While the existence of lightly mixing finite measure-preserving transformations implies the existence of finite measure-preserving, strong measurably sensitive transformations, there is no corresponding notion of light mixing for the infinite measure-preserving case. In fact, we show that there are no ergodic, infinite measure-preserving, strong measurably sensitive transformations.

There exist nonconservative ergodic nonsingular transformations that are strong measurably sensitive; for example let $T : X \to X$ be a finite measure-preserving mixing transformation and define $S : X \times \mathbb{N} \to X \times \mathbb{N}$ by $S(x, n) = (Tx, n-1)$ if $n > 1$ and $S(x, 1) = (Tx, 2)$.

Proposition 5.1. *There are no ergodic, infinite measure-preserving, strong measurably sensitive transformations.*

Proof. Let T be an ergodic, measure-preserving transformation on a σ-finite measure space X with infinite measure. Then $X = \bigsqcup_{i=1}^{\infty} A_i$, where each set A_i has positive finite measure. Let $D_i = \bigcup_{j=1}^{i} A_j$. Then, as both A_i and D_i have finite measure, $\liminf_{n\to\infty} \mu(T^{-n}(D_i) \cap A_i) = 0$. Choose an increasing sequence $\{n_{i,k}\}_{k=1}^{\infty}$ so that $\mu(T^{-n_{i,k}}(D_i) \cap A_i) < 2^{-k-1}\mu(A_i)$. Then let $C_i = A_i \setminus \bigcup_{k=1}^{\infty} T^{-n_{i,k}}(D_i)$. The set C_i has positive measure and $T^{n_{i,k}}(C_i) \cap D_i = \emptyset$ for every natural number k.

Let g_i be a nonsingular isomorphism from $A_i \setminus C_i$ to $(2^{-2i+1}, 2^{-2i+2})$ with Lebesgue measure and let h_i be a nonsingular isomorphism from C_i to $(2^{-2i}, 2^{-2i+1})$ with Lebesgue measure whenever $A_i \setminus C_i$ has positive measure. Let N denote the set of points in X where neither g_i nor h_i is defined for all i together with the backwards orbits of these points. N must have measure zero due to the nonsingularity of T and so T', the restriction of T to $X \setminus N$, is measurably isomorphic to T. Define $g : X \setminus N \to (0, 1)$ piecewise to be whichever of h_i, g_i is defined. Note that $g(D_i) \subset (2^{-2i}, 1)$. Define a metric d on $X \setminus N$ by $d(x, y) = |g(x) - g(y)|$. A ball in metric d around any point in $X \setminus N$ must have positive measure since each isomorphism is nonsingular, and so the metric is μ-compatible.

Let $x \in C_i \setminus N$. For sufficiently small $\varepsilon > 0$, $B_\varepsilon(x) \subset C_i$. Then $(T')^{n_{i,k}}(B_\varepsilon(x)) \subset D^c \setminus N$ and, as any two points in $D_i^c \setminus N$ have a maximum distance of 2^{-2i} between them, any strong sensitivity constant must be at most 2^{-2i}. Consequently, T' does not exhibit strong sensitive dependence under the metric d. As the metric is μ-compatible, T is not strong measurably sensitive. □

While an infinite, ergodic, measure-preserving dynamical system $(X, \mathcal{S}(X), \mu, T)$ cannot be strong measurably sensitive, it can, however, exhibit the desired property with respect to a μ-compatible metric.

Proposition 5.2. *Let $(X, \mathcal{S}(X), \mu, T)$ be the Hajian-Kakutani Skyscraper and d the standard Euclidean metric X. Then there exists $\delta > 0$ such that for all $x \in X$ and all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu\{y \in B_\varepsilon(x) : d(T^n(x), T^n(y)) > \delta\} > 0$ for all integers $n \geq N$.*

Proof. The Hajian-Kakutani Skyscraper is an infinite measure-preserving, invertible, ergodic, rank-one transformation constructed from a recursively defined sequence of columns consisting of left-open, right-closed intervals (see [9], [11]). C_0 consists of $(0, 1]$, and C_{n+1} is formed from C_n by cutting C_n into two equal pieces, placing $2h_n$ spacers, which we denote S_{n+1}, above the right-hand half of C_n, and stacking right-over-left. The number of levels in column C_n is denoted h_n.

We begin with some remarks about the structure of column C_k, for $k \geq 1$. C_k consists of 4 subcolumns of height h_{k-1}; the bottom two subcolumns are points in C_{k-1}, and the top two are points in S_k. S_{k+1} may be thought of as consisting of 8 subcolumns of height h_{k-1} and width half that of those in C_k positioned above the right half of C_k. For convenience, we refer to these subcolumns in order from bottom to top by K_1 to K_8.

Let $\delta < \frac{3}{8}$, $x \in X$ and $\varepsilon > 0$. There exists an integer $\alpha > 1$ such that $B_\varepsilon(x)$ contains a level, which we denote I, of C_α. Define a sequence $\{l_i\}_{i=\alpha}^{\infty}$, where l_i is defined to be the first time for which the top level of column C_i is contained in $T^{l_i}(I)$. Set $N = l_\alpha$. Then for an integer $n > N$, there exists a unique integer $k \geq 1$ such that $l_k < n \leq l_{k+1}$.

Denote the left half of I by L and the right half by R. Since $T^{l_k+8h_{k-1}}(I)$ contains the top level of column C_{k+1}, there exists a unique $j \in \{1, ..., 8\}$ such that $(j-1)h_{k-1} < n - l_k \leq jh_{k-1}$. Then $T^n(L) \subset K_j$ and $T^n(R) \subset K_{j+4}$. When $j = 1$, $T^n(L) \subset K_1 \subset C_{k-1}$ and $T^n(R) \subset K_{j+4} \subset S_{k+1}$, so S_k lies between $T^n(R)$ and $T^n(L)$ on the real line. As $\mu(S_k) > 1$, $d(T^n(L), T^n(R)) \geq 1$. For $2 \leq j \leq 8$, K_{j+3} lies between $T^n(R)$ and $T^n(L)$ on the real line, and as $\mu(K_{j+3}) \geq \frac{1}{4}$, $d(T^n(L), T^n(R)) \geq \frac{1}{4}$. As $\delta < \frac{1}{8}$, $T^n(x)$ cannot be within δ of both $T^n(L)$ and $T^n(R)$. Hence $\mu\{y \in B_\epsilon(x) : d(T^n(x), T^n(y)) > \delta\} > 0$. $\square$

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References

[1] Christophe Abraham, Gérard Biau, and Benoît Cadre. Chaotic properties of mappings on a probability space. *J. Math. Anal. Appl.*, 266(2):420–431, 2002. MR1880515 (2002j:37009)

[2] Ethan Akin, Joseph Auslander, and Kenneth Berg. When is a transitive map chaotic? In *Convergence in Ergodic Theory and Probability (Columbus, OH, 1993)*, volume 5 of Ohio State Univ. Math. Res. Inst. Publ., pages 25–40, de Gruyter, Berlin, 1996. MR1412595 (97i:58106)

[3] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992. MR1157223 (93d:54059)

[4] François Blanchard, Eli Glasner, Sergiĭ Kolyada, and Alejandro Maass. On Li-Yorke pairs. *J. Reine Angew. Math.*, 547:51–68, 2002. MR1900136 (2003g:37014)

[5] Amie Bowles, Lukasz Fidkowski, Amy E. Marinello, and Cesar E. Silva. Double ergodicity of nonsingular transformations and infinite measure-preserving staircase transformations. *Illinois J. Math.*, 45(3):999–1019, 2001. MR1879249 (2002j:37014)

[6] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton Univ. Press, Princeton, NJ, 1981. MR603625 (82j:28010)

[7] Eli Glasner and Benjamin Weiss. Sensitive dependence on initial conditions. *Nonlinearity*, 6(6):1067–1075, 1993. MR1251259 (94j:58109)

[8] John Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, 70(2):133–160, 1979. MR553966 (82c:58037)

[9] A. Hajian and S. Kakutani. Example of an ergodic measure preserving transformation on an infinite measure space. Lecture Notes in Math., 160:45–52, Springer, Berlin, 1970. MR0269807 (42:4702)

[10] Lianfa He, Xinhua Yan, and Lingshu Wang. Weak-mixing implies sensitive dependence. *J. Math. Anal. Appl.*, 299(1):300–304, 2004. MR2091290 (2005i:37005)

[11] C.E. Silva. *Invitation to Ergodic Theory.* Student Math. Library, vol. 42, Amer. Math. Soc., Providence, RI, 2008. MR2371216

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