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NO EMBEDDINGS OF SOLENOIDS INTO SURFACES

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Abstract. A quick proof of Bing's theorem indicated by the title is given. Indeed the inverse limit of a sequence of degree > 1 maps between closed oriented m-manifolds can never be embedded into any closed orientable $(m+1)$-manifold. The proof also concludes Gumerov's result on the covering of solenoids.

In this paper we always assume that a solenoid is not a manifold.

In his two important papers on solenoids, Bing proved first that no solenoid is planar [B1] and then if a solenoid can be embedded into surfaces, then it must be planar [B2], and therefore no solenoid can be embedded into surfaces. We will give a short proof of this result. Indeed we show that the inverse limit of a sequence of maps of degree > 1 between closed oriented m-manifolds can never be embedded into any closed orientable $(m+1)$-manifold. The proof also concludes a recent result of [Gu] on covering degrees of solenoids.

For the sequence $\{\phi_n : X_n \to X_{n-1}\}_{n\geq 1}$, its *inverse limit* $\varprojlim X_n$ is defined as the subspace

$$\{(x_0, x_1, ..., x_n, ...) \mid x_n \in X_n,\ x_{n-1} = \phi_n(x_n)\}$$

of the product space $\prod_{n=0}^{\infty} X_n$.

Identify S^1 with the abelian Lie group $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\phi_n : S^1 \to S^1$ be the homomorphism defined by $z \mapsto z^{w_n}$ where $w_n > 1$ is an integer. For the sequence $\{\phi_n : S^1 \to S^1\}_{n\geq 1}$, its inverse limit, denoted as Θ, is called the *solenoid* of type $\varpi = (w_1, w_2, ..., w_n, ...)$. By definition Θ is a connected, closed and hence compact subgroup of the abelian topological group $\prod_{n=0}^{\infty} S^1$ (cf. [Mc]).

Theorem 1. (1) *For a sequence $\{\phi_n : X_n \to X_{n-1}\}_{n\geq 1}$ of non-zero degree maps of closed oriented m-manifolds having infinitely many degree $\phi_n \neq \pm 1$, there is no embedding of $\varprojlim X_n$ into any closed oriented $(m+1)$-manifolds M.*

(2) *No solenoid Θ, can be embedded into a surface.*

Remark 2. In general, we call the inverse limit of maps $\{\phi_n : X_n \to X_{n-1}\}_{n\geq 1}$ an *m-dimensional solenoid* if all X_n are closed oriented m-manifolds and all ϕ_n are coverings of degree > 1. By Theorem 1 (1), there is no embedding of m-dimensional solenoids into $\mathbb{R}^{m+1}$. This fact has been proved for homogeneous solenoids by Prajs [Pr]. Note that solenoids of dimension > 1 need not be homogeneous unless the compositions of all coverings are regular.

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Theorem 3. *If X is a finite-fold covering of a solenoid Θ, then each component of X is homeomorphic to Θ.*

Remark 4. According to the description in [Gu, p. 2775], it seems Theorem 3 can be derived from several papers of Fox and Moore in the 1970's, and of Grigorian and Gumerov recently. We will give a direct proof of Theorem 3.

Proof of Theorem 1. (1) Suppose there is an embedding $\varprojlim X_n \subset M$ for some closed orientable $(m+1)$-manifold M. We have to seek a contradiction.

According to the continuity of Čech theory, for any commutative ring R, the Čech cohomology of the inverse limit is the direct limit of the singular cohomology, that is, $\check{H}^*(\varprojlim X_n; R) = \varinjlim H^*(X_n; R)$. See [ES, Chapter X, Theorems 2.2 and 3.1].

Clearly we have $H^m(X_n; R) = R$. Denote the degree of ϕ_n by w_n. Then the Čech cohomology group $\check{H}^m(\varprojlim X_n; R)$ is the direct limit of $R \xrightarrow{w_1} R \xrightarrow{w_2} R \xrightarrow{w_3} R \xrightarrow{w_4} \cdots$. Since all $w_n \neq 0$ and infinitely many $w_n \neq \pm 1$, it follows that $\check{H}^m(\varprojlim X_n; \mathbb{Z})$ is an infinitely generated $\mathbb{Z}$-module, but $\check{H}^m(\varprojlim X_n; \mathbb{Q}) \cong \mathbb{Q}$ is a finitely generated $\mathbb{Q}$-module.

By the Alexander duality $\check{H}^m(\varprojlim X_n; R) \cong H_1(M, M \setminus \varprojlim X_n; R)$ (cf. [GH, p. 233]), we have the exact sequence

$$\cdots \to H_1(M; R) \to \check{H}^m(\varprojlim X_n; R) \to H_0(M \setminus \varprojlim X_n; R) \to H_0(M; R) \to \cdots .$$

Since $H_*(M; R)$ is a finitely generated R-module for any commutative ring R, the above exact sequence implies that $H_0(M \setminus \varprojlim X_n; \mathbb{Z})$ is an infinitely generated $\mathbb{Z}$-module and $H_0(M \setminus \varprojlim X_n; \mathbb{Q})$ is a finitely generated $\mathbb{Q}$-module. But on the other hand $H_0(M \setminus \varprojlim X_n; \mathbb{Z})$ is a free abelian group and $H_0(M \setminus \varprojlim X_n; \mathbb{Q}) = H_0(M \setminus \varprojlim X_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, so $H_0(M \setminus \varprojlim X_n; \mathbb{Z})$ and $H_0(M \setminus \varprojlim X_n; \mathbb{Q})$ should have the same basis, a contradiction.

(2) Suppose there is an embedding $\Theta \subset F$ of a solenoid Θ into a surface F. First, we may assume F is a closed surface. If not, since Θ is compact, Θ is contained in the interior of some connected, compact subsurface $F' \subset F$. Capping a disc on each component of $\partial F'$, we get an embedding of Θ into a closed surface.

According to Theorem 1 (1), F cannot be orientable. If F is not orientable, we consider the orientable double covering $\pi : \tilde{F} \to F$. Since $\pi^{-1}(\Theta)$ is a double covering of Θ, by Theorem 3 each component of $\pi^{-1}(\Theta)$ is homeomorphic to Θ; hence we get an embedding $\Theta \subset \tilde{F}$ into a closed, orientable surface, which still contradicts Theorem 1 (1). □

Proof of Theorem 3. Suppose Θ is a solenoid of type $\varpi = (w_1, w_2, ..., w_n, ...)$ and $\pi : X \to \Theta$ is an r-fold covering.

Let Γ_n be the kernel of the projection $p_n : \Theta \to S^1$ onto the nth coordinate, i.e.

$$\Gamma_n = \{(x_0, x_1, \ldots, x_n, \ldots) \in \Theta \mid x_0 = x_1 = \cdots = x_n = 1\}.$$

We have the infinite sequence of closed, and hence compact, subgroups (indeed each Γ_n is homeomorphic to the Cantor set; cf. [Mc])

$$\Theta > \Gamma_0 > \Gamma_1 > \Gamma_2 > \cdots > \Gamma_n > \cdots .$$

The right multiplication of Γ_n on Θ makes the projection $p_n : \Theta \to S^1$ a principal fiber bundle (cf. [Mc]). It follows that Θ is the mapping torus of a left transformation $\psi_n : \Gamma_n \to \Gamma_n$. We choose the left transformation ψ_n as follows. Fix a closed path $\alpha(t) = e^{2\pi it}$, $t \in [0,1]$, in S^1. Then $\alpha_n(t) = \gamma(w_1 \cdots w_n t)$ is a lift of α via the projection p_n, where

$$\gamma(t) = (e^{2\pi it}, e^{2\pi it/w_1}, e^{2\pi it/w_1 w_2}, \dots, e^{2\pi it/w_1\cdots w_n}, \dots)$$

is the one-parameter subgroup of Θ. Let ψ_n be given by $x \mapsto \alpha_n(1)x$.

Moreover, as the covering space of Θ, X is the mapping torus of a lift $\tilde{\psi}_n$ of ψ_n which is uniquely determined as follows. For each $\tilde{x} \in \pi^{-1}(\Gamma_n)$, let $\tilde{\alpha}$ be the unique lift starting from $\tilde{x}$ of the path $\alpha_n(t)\pi(\tilde{x})$ in Θ via the covering map π; then $\tilde{\psi}_n(\tilde{x}) = \tilde{\alpha}(1)$. Note that $\psi_n = \psi_0^{w_1\cdots w_n}|_{\Gamma_n}$ and hence $\tilde{\psi}_n = \tilde{\psi}_0^{w_1\cdots w_n}|_{\pi^{-1}(\Gamma_n)}$.

Claim 1. Let $\mathcal{U}$ be an open covering of Γ_0. Then for sufficiently large n each coset of Γ_n in Γ_0 is contained in some element of $\mathcal{U}$.

Proof. Note that $\mathcal{B} = \{x\Gamma_n \mid n \geq 0,\ x \in \Gamma\}$ forms a basis for the open sets of Γ_0. So the covering $\mathcal{U}$ has a refinement $\mathcal{U}'$ which consists of elements of $\mathcal{B}$. Since Γ_0 is compact we may assume $\mathcal{U}'$ is finite. Therefore, for sufficiently large n each coset of Γ_n in Γ_0 is contained in some element of $\mathcal{U}'$, hence in some element of $\mathcal{U}$. □

Claim 2. We have $\pi^{-1}(\Gamma_0) \cong \Gamma_0 \times \{1, 2, \dots, r\}$ as covering spaces of Γ_0.

Proof. Let $\mathcal{U}$ be the open covering of Γ_0 which consists of the open fundamental regions of the covering map $\pi|_{\pi^{-1}(\Gamma_0)}$ and let n be sufficiently large as in Claim 1. Then each coset $x\Gamma_n$ in Γ_0 is contained in some element $\mathcal{U}$; hence $\pi^{-1}(x\Gamma_n) \cong x\Gamma_n \times \{1, 2, \dots, r\}$ as covering spaces of $x\Gamma_n$. Since the cosets of Γ_n are disjoint and open in Γ_0, the claim follows. □

In what follows we fix a homeomorphism from Claim 2 and identify both sets.

Claim 3. For each sufficiently large n there exists a permutation σ_n of $\{1, 2, \dots, r\}$ such that $\tilde{\psi}_n(x, j) = (\psi_n(x), \sigma_n(j))$.

Proof. Applying Claim 1 on the the open covering of Γ_0 which consists of the open (and closed) sets $U_\sigma = \{x \in \Gamma_0 \mid \tilde{\psi}_0(x, j) = (\psi_0(x), \sigma(j))\}$ with σ running over all permutations of $\{1, 2, \dots, r\}$, one notices that for sufficiently large n each coset of Γ_n in Γ_0 is contained in some U_σ. Since $\tilde{\psi}_n = \tilde{\psi}_0^{w_1\cdots w_n}|_{\pi^{-1}(\Gamma_n)}$, the claim follows. □

By Claim 3 X is the disjoint union of the mapping tori of $\tilde{\psi}_n|_{\Gamma_n \times J}$ where J runs over all σ_n-orbits. Since Θ is connected, the following two claims eventually establish the theorem. Below we denote by T_f the mapping torus of a self homeomorphism f.

Claim 4. For sufficiently large n the length of each σ_n-orbit is relatively prime to $w_{n'}$ for all $n' > n$.

Proof. Note that $\sigma_{n+1} = \sigma_n^{w_{n+1}}$. Therefore, each $\sigma_{n'}$-orbit is contained in some σ_n-orbit for $n' > n$ and if the length of a σ_n-orbit J is not relatively prime to $w_{n'}$ for some $n' > n$, then J splits into several $\sigma_{n'}$-orbits. Since a permutation of $\{1, 2, \dots, r\}$ has at most r orbits, the claim follows. □

Claim 5. Let n be sufficiently large as in Claim 3. If the length of a σ_n-orbit J is relatively prime to $w_{n'}$ for all $n' > n$, then $T_{\tilde{\psi}_n|_{\Gamma_n \times J}} \cong \Theta$.

Proof. Let $l = |J|$ be the length of J. It is clear that $T_{\tilde{\psi}_n|_{\Gamma_n \times J}} \cong T_{\psi_n^l}$. Note that Γ_n is the inverse limit of the sequence

$$\{\phi_{n+k} : \mathrm{Ker}(\phi_{n+1} \cdots \phi_{n+k}) \to \mathrm{Ker}(\phi_{n+1} \cdots \phi_{n+k-1})\}_{k \geq 1}.$$

Since l is relatively prime to $w_{n'}$ for all $n' > n$, the homomorphisms

$$\mathrm{Ker}(\phi_{n+1} \cdots \phi_{n+k}) \to \mathrm{Ker}(\phi_{n+1} \cdots \phi_{n+k})$$

defined by $x \mapsto x^l$ are isomorphic. It follows that the homomorphism $\Gamma_n \to \Gamma_n$ defined by $x \to x^l$ is isomorphic, via which one notices that ψ_n is topologically conjugate to ψ_n^l. So we have $T_{\psi_n} \cong T_{\psi_n^l}$ and therefore, $T_{\tilde{\psi}_n|_{\Gamma_n \times J}} \cong T_{\psi_n^l} \cong T_{\psi_n} \cong \Theta$. □

From Claim 4 and Claim 5 we also have

Corollary 5 ([Gu]). *A solenoid of type* $\varpi = (w_1, w_2, ..., w_n, ...)$ *has a connected* r*-fold covering if and only if* r *is relatively prime to all but finitely many* w_n.

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