A COMPACTIFICATION OF THE MODULI SPACE OF POLYNOMIALS

MASAYO FUJIMURA AND MASAHIKO TANIGUCHI

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Abstract. In this paper, we introduce a compactification of the moduli space of polynomial maps with a fixed degree \( n \geq 2 \) such that the map from it to \( \mathbb{P}^{n-1}(\mathbb{C}) \) defined by using the elementary symmetric functions of multipliers at fixed points is a continuous surjection.

1. Introduction and main theorems

Let \( n \geq 2 \) and \( \text{Poly}_n \) be the set of all polynomial maps of \( \mathbb{C} \) to itself with degree \( n \). We say that two maps \( p_1 \) and \( p_2 \) in \( \text{Poly}_n \) are affine conjugate if there exists a biholomorphic automorphism \( g \) of \( \mathbb{C} \) such that \( g \circ p_1 \circ g^{-1} = p_2 \). The moduli space of polynomial maps with degree \( n \) is the set of all affine conjugacy classes of maps in \( \text{Poly}_n \), and is denoted by \( \text{MPoly}_n \). Here, recall that a natural complex orbifold structure can be introduced on \( \text{MPoly}_n \).

Remark 1.1. These spaces were investigated by Branner and Hubbard [2, 3], and Milnor [12] in the case of degree 3, and then by the first author in general cases (cf. [9]).

Several kinds of compactification of \( \text{MPoly}_n \) have been considered. One is given as the closure in the GIT compactification of the moduli space of rational maps, which is defined in [14]. (See also [4].) DeMarco and McMullen introduced one by using tree representations ([5]), and also DeMarco discussed another one in [4]. Here, we show that the extended moduli space introduced by the first author in [8] (also cf. [9], Introduction) gives a natural compactification of \( \text{MPoly}_n \) with a suitable topology.

Definition 1.2 (Extended moduli spaces). The extended moduli space \( \hat{\text{M}}_n \) of degree \( n \) is the set of all collections \( X = \{ \langle p_j \rangle \}_{j=1}^J \) of the affine conjugacy classes \( \langle p_j \rangle \) of polynomial maps \( p_j \) with degree \( n_j \geq 2 \) such that \( N_X = \sum_{j=1}^J n_j \leq n \). Here we include the empty collection in the extended moduli space.

By identifying the affine conjugacy class \( \langle p \rangle \) of \( p \in \text{Poly}_n \) with the collection \( \{ \langle p \rangle \} \) consisting of this class only, we consider \( \text{MPoly}_n \) as a subset of \( \hat{\text{M}}_n \).

Here, by definition, an element \( X \) of \( \hat{\text{M}}_n \) is an unordered set of affine conjugacy classes. But for the sake of convenience, we fix an order temporarily, and represent
we set $X$ as $\{ (p_j) \}_{j=1}^J$. Then, we can associate $X$ with a collection of sets of numbers as follows. For $X = \{ (p_j) \}_{j=1}^J$, assume that the fixed points of $p_j$, counted including multiplicities, are represented by numbers

$$n_j^X = n_{j-1}^X + 1, \cdots, n_1^X + \cdots + n_j^X$$

for every $j$. We call the partition of $\{1, \cdots, n\}$ into

$$E_0^X = \{ N_X + 1, \cdots, n \},$$
$$E_1^X = \{ 1, \cdots, n_1^X \},$$
$$\cdots,$$
$$E_J^X = \{ n_1^X + \cdots + n_J^X + 1, \cdots, N_X \}$$

the partition associated to $X$. For every $h$ with degree $n_h^Y$ such that $\mathcal{H} = \sum_{h=1}^H n_h^X \leq n$, which satisfy the following conditions:

1. (Marking) There is a family of mutually disjoint subsets $P_1, \cdots, P_H$ of $\{1, \cdots, J\}$ such that $\bigcup_{h=1}^H P_h = \{1, \cdots, J\}$ and

$$n_h^Y \geq n_h^X = \sum_{j \in P_h} n_j^X$$

for every $h = 1, \cdots, H$ (which implies that $N_Y \geq N_X$). Here, some of $P_h$ may be empty and then we set $N_h^X = 0$. (In particular, $H$ may be greater than $J$.)

Furthermore, if $N_X > 0$, then there is an injection

$$\iota : \{1, \cdots, N_X\} \to \{1, \cdots, N_Y\}$$

such that the image of $\bigcup_{j \in P_h} E_j^X$ by $\iota$ is contained in $E_h^Y$ for every $h$, where $\{E_0^Y, \cdots, E_H^Y\}$ is the partition associated to $Y = \{ (q_h) \}_{h=1}^H$.

2. (Convergence) For every $k$ with $1 \leq k \leq N_X$, denote by $p_{j(k)}$ and $q_{h(k)}$ the polynomial maps in $\{ p_j \}_{j=1}^J$ and $\{ q_h \}_{h=1}^H$ having fixed points represented by $k$ and $\iota(k)$, respectively. We normalize $p_{j(k)}$ and $q_{h(k)}$ so that these fixed points are 0 and all coefficients of $z^r$ with $r \neq 1$ have the absolute values not greater than 1 and at least one of them is 1. Then, among polynomial maps obtained from $p_{j(k)}$ and $q_{h(k)}$ by this normalization, we can find polynomial maps $\tilde{p}_k$ and $\tilde{q}_k$, respectively, such that

$$|\tilde{p}_k - \tilde{q}_k| < \epsilon$$

on $\{ z \in \mathbb{C} \mid |z| < 1/\epsilon \}$.

3. (Degeneration) For every $l$ with $1 \leq l \leq N_Y$ not in the image of $\iota$, let $q_{h(l)}$ be the polynomial map in $\{ q_h \}_{h=1}^H$ having a fixed point represented by $l$, and $m_l(Y)$ be the multiplier of $q_{h(l)}$ at the fixed point. Then we have

$$|m_l(Y)| > \frac{1}{\epsilon}.$$
Finally, take \( \{U_\epsilon(X) \mid \epsilon > 0, X \in \hat{M}_n\} \) as basic open sets of \( \hat{M}_n \), and we have a topology on \( \hat{M}_n \).

**Remark 1.4.** If \( J = 0 \), i.e. \( X \) is the empty collection, then we assume that \( N_X = 0 \) and \( P_h = \emptyset \) for every \( h \), and hence conditions (1) and (2) hold trivially.

**Proposition 1.5.** The set \( \text{MPoly}_n \) is open in \( \hat{M}_n \).

**Proof.** Fix \( X \in \text{MPoly}_n \). Then \( X \) consists of a single affine conjugacy class of a polynomial map \( p \) of degree \( n \). Fix \( \epsilon > 0 \) arbitrarily. Then the condition (1) implies that every point of \( U_\epsilon(X) \) again consists of a single affine conjugacy class of a polynomial \( p \) of degree \( n \). Thus \( U_\epsilon(X) \subset \text{MPoly}_n \). \( \square \)

**Proposition 1.6.** Points \( Y_m \in \text{MPoly}_n \) tend to the empty collection if and only if the multipliers \( m_h(Y_m) \) tend to \( \infty \) as \( m \) tend to \( \infty \) for every \( h \) (= 1, \ldots, n).

**Proof.** By definition, \( Y_m \in \text{MPoly}_n \) tend to the empty collection if and only if, for every \( \epsilon > 0 \), there is an \( m_0 \) such that

\[
|m_h(Y_m)| > \frac{1}{\epsilon}
\]

for every \( h \) and every \( m \geq m_0 \), which is clearly equivalent to the fact that the multipliers \( m_h(Y_m) \) tend to \( \infty \) as \( m \) tend to \( \infty \) for every \( h \). \( \square \)

**Corollary 1.7.** The points \( Y_m = \{ \langle q_m \rangle \} \in \hat{M}_n \) with \( q_m(z) = z^n + z - m^n \) tend to the empty collection as \( m \) tends to \( \infty \).

Now the main theorem of this paper is the following one.

**Theorem 1.8.** The extended moduli space \( \hat{M}_n \) of degree \( n \) equipped with the topology defined above is compact, and \( \text{MPoly}_n \) is dense in it.

The proof will be given in \( \S 2 \).

Next, we define a map

\[
\hat{\Psi}_n : \hat{M}_n \rightarrow \mathbb{P}^{n-1}(\mathbb{C}).
\]

**Definition 1.9** (Extended multiplier representation). Let \( X = \{ \langle p_j \rangle \}_{j=1}^r \) be an element of \( \hat{M}_n \), and let \( n_j \) and \( N_X \) be as before. Let \( \sigma_k \) (\( k = 1, \ldots, N_X \)) be the elementary symmetric functions of multipliers \( \{ m_k \}_{k=1}^{N_X} \) at finite fixed points of all \( p_j \) counted including multiplicities:

\[
\begin{align*}
\sigma_1 &= m_1 + m_2 + \cdots + m_{N_X}, \\
\sigma_r &= \sum_{j_1 < j_2 < \cdots < j_r} m_{j_1} m_{j_2} \cdots m_{j_r}, \\
&\quad \cdots, \\
\sigma_{N_X} &= m_1 m_2 \cdots m_{N_X}.
\end{align*}
\]

We define the map \( \hat{\Psi}_n \) by sending \( X \) to the point of \( \mathbb{P}^{n-1}(\mathbb{C}) \) represented by

\[
(0 : \cdots : 0 : 1 : \sigma_1 : \cdots : \sigma_{N_X-2} : \sigma_{N_X}).
\]

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Finally, we map the empty collection to the point represented by
\[
(0 : \cdots : 0 : 1).
\]

Here it is well-known that for every polynomial with degree \(n\) the unique linear relation between the elementary symmetric functions \(\{\sigma_k\}_{k=1}^n\) of multipliers at finite fixed points is
\[
n + \sum_{k=1}^n (-1)^k (n - k)\sigma_k = 0,
\]
which is called Fatou's index theorem. This relation still holds in the circumstance as in the above definition when we replace \(n\) by \(N_X\).

Now, we conclude the following theorem.

**Theorem 1.10** (Cf. [8], Theorem 3.2). The map
\[
\hat{\Psi}_n : \hat{M}_n \to \mathbb{P}^{n-1}(\mathbb{C})
\]
defined as above is a continuous surjection.

Here recall that \(\hat{\Psi}_n\) restricted to \(\text{MPoly}_n\) can be considered as a map into \(\mathbb{C}^{n-1}\), which is never surjective if \(n \geq 4\) ([9], Theorem 2). Also we have the following corollary of [9], Theorem 6.

**Theorem 1.11.** For a generic point \(x\) of \(\mathbb{P}^{n-1}(\mathbb{C})\), the preimage \(\hat{\Psi}_n^{-1}(x)\) consists of exactly \((n-2)!\) points of \(\hat{M}_n\).

The proofs of Theorems 1.10 and 1.11 will be given in §3.

**Remark 1.12.** 1) The construction of our compactification resembles that of the Deligne-Mumford one due to Bers for the moduli space of compact Riemann surfaces ([1] and cf. [11]). Also see [14], the Remark after Corollary 1.6.

2) There is another deformation space concerning \(\text{Poly}_n\). We say that two maps \(p_1\) and \(p_2\) in \(\text{Poly}_n\) determine the same isomorphism class if there is a biholomorphic automorphism \(g\) of \(\mathbb{C}\) such that \(p_2 = p_1 \circ g\). The (classical) Hurwitz space \(H_{0,n}[n]\) of genus 0 and degree \(n\) with type \((n)\) is the set consisting of all isomorphism classes of polynomial maps of degree \(n\) in general position. Recall that a natural compactification of the Hurwitz space was introduced in [7] and [13]. (Also cf. [6].)

2. Proof of Theorem 1.8

First, we show that \(\text{MPoly}_n\) is dense in \(\hat{M}_n\). By Corollary 1.7 we start with a point \(X = \{(p_j)\}_{j=1}^J\) in \(\hat{M}_n\) which is not the empty collection, i.e. with \(J \geq 1\).

First, assume that \(J = 1\) but \((2 \leq)\ n_1 = n_1^X < n\). Here, we may write \(p_1\) as
\[
p_1(z) = z + \prod_{\mu=1}^{n_1} (z - a_\mu).
\]
Set
\[
p_1^r(z) = z + \frac{r - z}{r} \prod_{\mu=1}^{n_1} (z - a_\mu).
\]
Then $p_1^r$ are polynomials with degree $n_1 + 1$ converging to $p_1$ uniformly on compact sets in $\mathbb{C}$ as $r$ tend to $\infty$. Also, since $n_1 \geq 2$, the multipliers at $r$ tend to $\infty$ as do $r$. Thus the following lemma is clear.

**Lemma 2.1.** $Y_r = \{\langle p_1^r \rangle \}$ converge to $X$ as $r$ tend to $\infty$.

If $J > 1$, then we also consider $p_2$ and set $n_2 = n_2^X (\geq 2)$. We may write $p_2$ as

$$p_2(z) = z + \prod_{\nu = 1}^{n_2} (-z + b_{\nu}).$$

We set

$$p_1^r p_2(z) = z + \frac{1}{r(n_2 - 1)n_2} \prod_{\mu = 1}^{n_1} (z - a_{\mu}) \prod_{\nu = 1}^{n_2} (-z + r^{n_2 - 1} + r^{n_2 - n_1} b_{\nu}).$$

Then $p_1^r p_2$ are polynomials with degree $n_1 + n_2$ converging to $p_1$ uniformly on compact sets in $\mathbb{C}$ as $r$ tend to $\infty$. Also, we have

**Lemma 2.2.** $Y_r = \{(p_1^r p_2), (p_3), \ldots, (p_J)\}$ converge to $X$ as $r$ tend to $\pm \infty$.

*Proof.* The polynomials $p_1^r p_2(z)$ are conjugate to

$$z + \frac{1}{r(n_1 - 1)n_1} \prod_{\mu = 1}^{n_1} (z + r^{n_1 - 1} - r^{n_1 - n_2} a_{\mu}) \prod_{\nu = 1}^{n_2} (-z + b_{\nu})$$

by the affine automorphisms

$$V_r(z) = \frac{z - r^{n_2 - 1}}{r^{n_2 - n_1}},$$

which converge to $p_2$ uniformly on compact sets in $\mathbb{C}$ as $r$ tend to $\infty$. Thus the assertion follows. □

*Proof of denseness.* Take a sufficiently large $r$, and we have an element $\{\langle p_1^r \rangle \}$ or $\{(p_1^r p_2), (p_3), \ldots, (p_J)\}$ of $\hat{M}_n$ which is arbitrarily near to $X$ by Lemmas 2.1 and 2.2 and which satisfies that the degree of $p_1^r$ is $n_1 + 1$ or (the degree of $p_1^r p_2$ is $n_1 + n_2$ but) the number of the conjugacy classes decreases from $J$ to $J - 1$, respectively.

Repeating such an approximation by a finite number of times, we have an element of $\text{MPoly}_n$ arbitrarily near to $X$. Thus we conclude that $\text{MPoly}_n$ is dense in $\hat{M}_n$. □

**Example 2.3.** The points $Y_m = \{\langle q_m \rangle \} \in \hat{M}_n$ $(n \geq 3)$ with $q_m(z) = z^3 - mz^2 + z$ converge to $X = \{(q)\}$ with $q(z) = z^2 + z$ as $m$ tend to $\infty$.

Indeed, the fixed points of $q_m$ are $0$ (double) and $m$, with multipliers $1$ and $m^2 + 1$, respectively. Associate the numbers $1, 2$ to $0$ and normalize $q_m$ as before with respect to $0$, and we have $q_1[m](z) = m^{-2} z^3 + z^2 + z$, which converge to $q(z)$ uniformly on compact sets in $\mathbb{C}$.

**Example 2.4.** The points $Y_m = \{\langle q_m \rangle \} \in \hat{M}_n$ $(n \geq 4)$ with $q_m(z) = z^4 - 2m z^3 + m^2 z^2 + z$ converge to $\{\langle q \rangle, \langle q \rangle\}$ with $q(z) = z^2 + z$ as $m$ tend to $\infty$.

Indeed, the fixed points of $q_m$ are $0$ (double) and $m$ (double) with multiplier $1$. We associate the number $1, 2$ to $0$ and $3, 4$ to $m$. Normalize $q_m$ as before with respect to $0$ and $m$, and get

$q_1[m](z) = m^{-6} z^4 - 2m^{-3} z^3 + z^2 + z$ and $q_3[m](z) = m^{-6} z^4 + 2m^{-3} z^3 + z^2 + z$, respectively, both of which converge to $q(z)$ uniformly on compact sets in $\mathbb{C}$.
Next, the space $\hat{M}_n$ equipped with the topology defined above satisfies the second countability axiom. (Actually, the topology has a countable base
\[
\{ U_{\epsilon_u}(X_v) \mid u, v \in \mathbb{Z} \},
\]
where $\epsilon_u$ moves over all positive rational numbers and $X_v$ over all collections of the affine conjugacy classes of polynomial maps with rational coefficients.) Hence to finish the proof of Theorem 1.8 it is enough to show sequential compactness of $\hat{M}_n$.

Let $\{ Y_l \}_{l=1}^{\infty}$ be an arbitrary sequence of distinct points in $\hat{M}_n$. Here taking a subsequence if necessary, we may assume that every $Y_l$ consists of the same number of affine conjugacy classes, say $\{ \langle q_h[l] \rangle \}_{h=1}^H$, and that the degree of $q_h[l]$ is a constant $n_h$ for every $l$. Set $N = \sum_{h=1}^H n_h$. We may also assume that all fixed points of all $q_h[l]$ are numbered so that the fixed point $z_k[l]$ represented by $k$ is a fixed point of $q_{h(k)}[l]$ with $h(k)$ independent of $l$ for every $k = 1, \cdots, N$.

Let $m_k[l]$ be the multipliers of $q_{h(k)}[l]$ at the fixed point $z_k[l]$. Here, again taking a subsequence and renumbering the fixed points if necessary, we may assume that $m_k[l]$ converge to some value $m_k$ in $\mathbb{P}^1(\mathbb{C})$ as $l \to \infty$ for every $k$, and that $m_k = \infty$ if and only if $k$ satisfies $N' < k \leq N$ with some $N' \leq N$. Here, if $N' = 0$, then as in the proof of Proposition 1.6 $Y_l$ converge to the empty collection as $l \to \infty$.

If $N' > 0$, then $m_1[l]$ converge to a finite $m_1$ as $l \to \infty$. Let $\tilde{q}_1[l]$ be a polynomial map obtained from $q_{h(1)}[l]$ by the normalization as before with respect to $z_1[l]$ for every $l$. Then taking a subsequence if necessary, we may assume that $\tilde{q}_1[l]$ converge to a polynomial $\tilde{p}_1$ uniformly on compact sets in $\mathbb{C}$ as $l \to \infty$. Here from the definition, the degree of $\tilde{p}_1$ is not less than 2, and the multiplier at the fixed point 0 is $m_1$.

Let $k$ satisfy that $1 < k \leq N'$, $q_{h(k)}[l] = q_{h(1)}[l]$ and the fixed points $\tilde{z}_k[l]$ of $\tilde{q}_1[l]$ corresponding to $z_k[l]$ are bounded. Then, we may further assume that $\tilde{z}_k[l]$ converge to a fixed point $\tilde{z}_k$ of $\tilde{p}_1$, which is represented also by $k$, as $l \to \infty$. Let Fix$_1$ be the set of all numbers representing fixed points of $\tilde{p}_1$ including multiplicities, and we have the following lemma.

**Lemma 2.5.** For every $k \in$ Fix$_1$, we normalize $q_{h(1)}[l]$ and $\tilde{p}_1$ as before with respect to $z_k[l]$ and $\tilde{z}_k$, respectively. Then among polynomial maps obtained by the normalization, we can find polynomial maps $\tilde{q}_k[l]$ and $\tilde{p}_k$, respectively, such that $\tilde{q}_k[l]$ converge to $\tilde{p}_k$ locally uniformly on $\mathbb{C}$ as $l \to \infty$.

**Proof.** If $k > 1$ and $k \in$ Fix$_1$, take the conjugates of $\tilde{q}_1[l]$ by the affine automorphisms $U_k(z) = z - \tilde{z}_k[l]$. Then we have polynomials with bounded coefficients which converge to the conjugate of $\tilde{p}_1$ by $U(z) = z - \tilde{z}_k$, which implies the assertion. \(\square\)

Now let $k_2 > 1$ be the least number not contained in Fix$_1$. Then starting from $q_{h(k_2)}[l]$, we also have a polynomial map $\tilde{p}_{k_2}$ satisfying the same assertion as in Lemma 2.5 with $k_2$ instead of 1.

**Proof of compactness.** Repeating the above arguments a finite number of times, we finally get a set of polynomial maps $\{ \tilde{p}_k \}_{k=1}^{N'}$. Let
\[
\{ \text{Fix}_1, \text{Fix}_{k_2}, \cdots, \text{Fix}_{k_K} \}
\]
be the corresponding partition of $\{ 1, \cdots, N' \}$, and set
\[
X = \{ \langle \tilde{p}_1 \rangle, \langle \tilde{p}_{k_2} \rangle, \cdots, \langle \tilde{p}_{k_K} \rangle \}.
\]
Then from the construction, \( Y_l \) converge to \( X \) in \( \hat{M}_n \) as \( l \) tend to \( \infty \).

\[ \square \]

### 3. Proofs of Theorems \[1.10\] and \[1.11\]

First we show the following lemma.

**Lemma 3.1.** \( \hat{\Psi}_n \) is continuous.

**Proof.** If the multipliers \( m_k(X) \) at the fixed point represented by \( k \) are bounded for all \( k \), then by condition (2), \( m_k(X) \) are continuous. Since we define \( \hat{\Psi}_n \) by using the elementary symmetric functions of these multipliers, the assertion follows.

The case that some \( m_k(X) \) are \( \infty \) or tend to \( \infty \) is similar. For the sake of simplicity, we give a proof only for the case that \( m_1(X) \) are finite but tend to \( \infty \), while the others are bounded, as \( X \) tend to \( X_\infty \). In this case, \( \hat{\Psi}_n(X) \) can also be represented as

\[
\left( \frac{1}{m_1(X)} : 1 + \frac{\sigma_1'(X)}{m_1(X)} : \cdots : \sigma_n'(X) \right)
\]

with \( \sigma_1'(X) = \sum_{k=2}^n m_k(X), \cdots, \sigma_n'(X) = \prod_{k=2}^n m_k(X) \), which converge to \( \hat{\Psi}_n(X_\infty) \) from the definition.

**Proof of Theorem \[1.10\]** Since \( \hat{\Psi}_n \) is continuous by Lemma 3.1 and \( \hat{M}_n \) is compact by Theorem \[1.8\] the image \( \hat{\Psi}_n(M_n) \) is also compact in \( \mathbb{P}^{n-1}(\mathbb{C}) \). Hence to prove Theorem \[1.10\] it remains only to show that the image \( \hat{\Psi}_n(M_n) \) is dense. But by Lemma 3.2 below, \( \hat{\Psi}_n(M_{\text{Poly}_n}) \) is already dense in \( \mathbb{P}^{n-1}(\mathbb{C}) \).

**Lemma 3.2.** \( \hat{\Psi}_n(M_{\text{Poly}_n}) \) is dense in \( \mathbb{P}^{n-1}(\mathbb{C}) \).

**Proof.** Let \( \Omega \) be the set of all points \( (s_1, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1} \) such that there is a solution \( \{m_1, \cdots, m_n\} \) of the equation system

\[
\begin{align*}
s_1 &= m_1 + m_2 + \cdots + m_n, \\
\cdots \\
s_r &= \sum_{j_1 < j_2 < \cdots < j_r} m_{j_1} m_{j_2} \cdots m_{j_r}, \\
\cdots \\
s_n &= m_1 m_2 \cdots m_n,
\end{align*}
\]

with \( m_j \neq 1 \) for every \( j \) and \( \sum_{j \in S} \frac{1}{1-m_j} \neq 0 \) for every proper subset \( S \) of \( \{1, \cdots, n\} \).

Here, the unique linear relation between the elementary symmetric functions corresponds to \( \sum_{j=1}^n \frac{1}{1-m_j} = 0 \). This equation between \( \{m_j\}_{j=1}^n \) is independent of the equations \( m_j = 1 \) and \( \sum_{j \in S} \frac{1}{1-m_j} = 0 \) for every proper subset \( S \) of \( \{1, \cdots, n\} \).

Since the map sending \( (m_1, \cdots, m_n) \) to \( (s_1, \cdots, s_n) \) is a finite-sheeted holomorphic branched covering projection of \( \mathbb{C}^n \) onto itself, we see that \( \Omega \) is a generic set in \( \mathbb{C}^{n-1} \).

On the other hand, [9], Theorem 4, implies that \( \hat{\Psi}_n(M_{\text{Poly}_n}) \) contains \( \Omega \), which shows the assertion.

**Proof of Theorem \[1.11\]** Every element of \( \hat{M}_n - M_{\text{Poly}_n} \) consists of either a single affine conjugacy class of a polynomial map with degree less than \( n \) or of more than one class. Hence the set \( M_n - M_{\text{Poly}_n} \) is, at most, complex \( n-2 \) dimensional, which implies that \( \mathbb{C}^{n-1} - \hat{\Psi}_n(M_n - M_{\text{Poly}_n}) \) is a generic set in \( \mathbb{P}^{n-1}(\mathbb{C}) \). Thus it is enough to show that for a generic point in \( \mathbb{C}^{n-1} \) the preimage of the point by
\[ \hat{\Psi}_n \text{ contains exactly } (n-2)! \text{ points counted including multiplicities. But this is } [9], \text{ Theorem 6.} \]

As closing remarks, we discuss more details in the case that \( n = 2, 3, \) and 4.

First, \( \hat{M}_2 - \text{MPoly}_2 \) consists of the empty collection only. Thus \( \hat{M}_2 \) is the one-point \( \hat{\mathbb{C}} \)-compactification of \( \text{MPoly}_2 \), and \( \hat{M}_2 \) is homeomorphic to \( \mathbb{P}^1(\mathbb{C}) \).

Next, \( \hat{M}_3 - \text{MPoly}_3 \) consists of the empty collection and a set which can be identified with \( \text{MPoly}_2 \). Hence \( \hat{M}_3 \) is homeomorphic to \( \mathbb{P}^2(\mathbb{C}) \) and \( \hat{\Psi}_3 \) gives a homeomorphism. Compare with [5], Theorem 11.1.

Now, \( \hat{M}_4 - \text{MPoly}_4 \) consists of the empty collection and sets

\[ E_2, E_3, E_{2,2} \]

which can be identified with \( \text{MPoly}_2, \text{MPoly}_3, \) and the symmetric product of \( \text{MPoly}_2 \) with itself, respectively. Also we can describe the map \( \hat{\Psi}_4 \) completely as follows (cf. [10], Theorem 3):

1. If \( x \in \mathbb{P}^3(\mathbb{C}) \) is represented by \((1 : 4 : 6 : 1)\), then \( \hat{\Psi}_4^{-1}(x) \) consists of infinite number of points.
2. If \( x \) is represented by either
   (a) \((0 : s_1 : s_2 : s_4)\)
   (b) \((1 : s_1 : s_2 : s_4)\) \((\neq (1 : 4 : 6 : 1))\) satisfying the equation
   \[
   54s_1^3 + (-81s_2 - 27s_4 - 135)s_1^2 + (36s_2^2 - 144s_2 - 1008)s_1^3
   + (-4s_2^3 + 360s_2^2 + (144s_4 + 2976)s_2 + 576s_4 + 4192)s_1^2
   + (-160s_3^2 + 2176s_2^2 + (-384s_4 - 6400)s_2 - 1280s_4 - 5376)s_1
   + 16s_2^4 + 448s_3^2 + (-128s_4 + 2176)s_2^2 + (256s_4 + 3840)s_2
   + 256s_4^2 + 768s_4 + 2304 = 0,
   \]

   then \( \hat{\Psi}_4^{-1}(x) \) consists of a single point.
3. Otherwise \( \hat{\Psi}_4^{-1}(x) \) consists of two points.

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DEPARTMENT OF MATHEMATICS, NATIONAL DEFENSE ACADEMY, YOKOSUKA 239-8686, JAPAN
E-mail address: masayo@nda.ac.jp

DEPARTMENT OF MATHEMATICS, NARA WOMEN’S UNIVERSITY, NARA 630-8506, JAPAN
E-mail address: tanig@cc.nara-wu.ac.jp