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THE FATOU SET FOR CRITICALLY FINITE MAPS

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Abstract. It is a classical result in complex dynamics of one variable that the Fatou set for a critically finite map on $\mathbf{P}^1$ consists of only basins of attraction for superattracting periodic points. In this paper, we deal with critically finite maps on $\mathbf{P}^k$. We show that the Fatou set for a critically finite map on $\mathbf{P}^2$ consists of only basins of attraction for superattracting periodic points. We also show that the Fatou set for a $k-$critically finite map on $\mathbf{P}^k$ is empty.

1. Introduction

A holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ is said to be critically finite if every component of the critical set for f is periodic or preperiodic. In [5], Thurston has given a topological classification of critically finite maps on $\mathbf{P}^1$. And it is well known that the Fatou set for a critically finite map on $\mathbf{P}^1$ consists of only basins of attraction for superattracting periodic points, i.e. points p with $f^n(p) = p$ and $(f^n)'(p) = 0$ for some $n \in \mathbf{N}$ (see [4]). In this paper, we show that the same is also true for critically finite maps on $\mathbf{P}^2$. More precisely, we have the following

Theorem 1.1. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is a critically finite holomorphic map, then the Fatou set for f consists of only basins of attraction for superattracting periodic points.*

With some extra assumptions, the above result has been obtained by Fornæss and Sibony ([2]).

We will also study critically finite maps on $\mathbf{P}^k$. In particular, we obtain the following theorem (see Section 2 for precise definitions).

Theorem 1.2. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map. If f is $k-$critically finite, then the Fatou set for f is empty.*

For $1-$critically finite maps on $\mathbf{P}^1$ and $2-$critically finite maps on $\mathbf{P}^2$, this was proved by Thurston ([5]) and Ueda ([6]), respectively.

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2. The Fatou set for critically finite maps

Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map of (algebraic) degree $d > 1$.

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Let C_1 be the critical set of f given by

$$C_1 = \{p \in \mathbf{P}^k | rank(df(p)) < k\},$$

where $df(p)$ denotes the differential of f at p.

We define the post-critical set D_1 of f by

$$D_1 = \bigcup_{j=1}^{\infty} f^j(C_1),$$

and the $\omega-$limit set E_1 of f by

$$E_1 = \bigcap_{j=1}^{\infty} f^j(\overline{D_1}).$$

By definition, a holomorphic map f on $\mathbf{P}^k$ is critically finite if the post-critical set D_1 is an analytic (hence algebraic) set in $\mathbf{P}^k$. This is equivalent to saying that there is an integer $l \geq 1$ such that $D_1 = \bigcup_{j=1}^{l} f^j(C_1)$. Hence, in the critically finite case, the set D_1 is an algebraic set of pure codimension 1.

Let us take a closer look at the structure of the post-critical set D_1 and the $\omega-$limit set E_1. If f is critically finite, then $f^{j-1}(D_1) = \bigcup_{l=j}^{\infty} f^l(C_1),\ j = 1, 2, \cdots,$ is a descending sequence of algebraic sets. Hence there is an integer $l_1 \geq 1$ such that $f^{l_1-1}(D_1) = f^{l_1}(D_1) = \cdots$. Consequently $E_1 = f^{l_1-1}(D_1)$ is an algebraic set of pure codimension 1. We can decompose E_1 into $E_1' \cup F_1$, where F_1 consists of those components in a critical cycle. (A periodic component L is said to be in a *critical cycle* if at least one of the forward images of L under f is contained in the critical set for f.)

Definition 2.1. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map. The map f is said to be critically finite of order 1 if D_1, hence E_1, is algebraic. And f is said to be 1-critically finite if C_1 and E_1 have no common irreducible component, i.e. $F_1 = \emptyset$.

We can now make the following inductive definition (cf. [3]).

Definition 2.2. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map. Suppose f is critically finite of order $n-1$, $1 < n \leq k$. Denote $C_n = C_1 \cap E_{n-1}$, $D_n = \bigcup_{j=1}^{\infty} f^j(C_n)$, and $E_n = \bigcap_{j=1}^{\infty} f^j(\overline{D_n})$. We say that f is critically finite of order n if D_n, hence E_n, is algebraic. Let l_n be the least integer such that $E_n = f^{l_n-1}(D_n) = \bigcup_{j=l_n}^{\infty} f^j(C_n)$. We can decompose E_n into $E_n' \cup F_n$, where F_n consists of those components, of codimension less or equal to n, in a critical cycle. If in addition f is $(n-1)-$critically finite, then we say that f is $n-$critically finite if E_n has no irreducible component contained in C_1, i.e. $F_n = \emptyset$.

Remark 2.3. A critically finite map is by definition critically finite of order 1. Jonsson ([3, Remark 2.10]) noted that a critically finite map on $\mathbf{P}^2$ is always strictly critically finite in the sense of Fornæss and Sibony ([2]). Recall that a critically finite map f is said to be strictly critically finite if f, when restricted to each irreducible periodic component of E_1, is also critically finite. We will use this remark implicitly in the proof of Theorem 1.1.

Before we go further, let us recall some definitions and results from [6].

Definition 2.4. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map and let U be a Fatou component, i.e. a connected component of the Fatou set for f. A holomorphic map $\varphi : U \to \mathbf{P}^k$ is called a limit map on U if there is a sequence $\{f^{n_j}|U\}$ which converges to φ uniformly on compact sets in U. A point $q \in \mathbf{P}^k$ is called a *Fatou limit point* if there is a limit map φ on a Fatou component U such that $q \in \varphi(U)$. The set of all Fatou limit points is called the *Fatou limit set*.

Definition 2.5. A Fatou component U is called a *rotation domain* if the identity map $id_U : U \to U$ is a limit map on U.

Definition 2.6. A point $q \in \mathbf{P}^k$ is said to be a *point of bounded ramification* with respect to f if the following conditions are satisfied:

(i) There is a neighborhood W of q such that $D_1 \cap W$ is an analytic subset of W.

(ii) There exists an integer l such that, for every integer $j > 0$ and every $p \in f^{-j}(q)$, the cardinality $\sharp(I)$ of the set

$$I = \{i | 0 \le i \le j-1, f^i(p) \in C_1\}$$

is not greater than l.

The following two theorems by Ueda are crucial.

Theorem 2.7 ([6, Theorem 4.8])**.** *Suppose that $q \in \mathbf{P}^k$ is a point of bounded ramification and also a Fatou limit point. Then q is contained in a rotation domain.*

Theorem 2.8 ([6, Proposition 5.1, (1)])**.** *If $f : \mathbf{P}^k \to \mathbf{P}^k$ is critically finite, then there is no rotation domain.*

Remark 2.9. Since the set D_1 is an analytic set in the critically finite case, condition (i) in Definition 2.6 is automatically true. So we only need to check condition (ii) in Definition 2.6 to see if a point $p \in \mathbf{P}^k$ is of bounded ramification.

We need the following lemma, whose proof is an elaboration of the proof of [6, Lemma 5.7].

Lemma 2.10. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map. If f is critically finite of order n, $1 \le n < k$, then every point in $\mathbf{P}^k \backslash E_n$ is a point of bounded ramification. If f is critically finite of order k, then every point in $\mathbf{P}^k \backslash F_k$ is a point of bounded ramification.*

Proof. First assume that f is critically finite of order n, $1 \le n < k$. Let $q \in \mathbf{P}^k \backslash E_n$ and let $p \in f^{-j}(q)$ for some integer $j > 0$. By Remark 2.9, we only need to show that the cardinality $\sharp(I)$ of the set

$$I = \{i | 0 \le i \le j-1, f^i(p) \in C_1\}$$

is not greater than some integer $l > 0$. Let

$$I_m = \{i | 0 \le i \le j-1, f^i(p) \in C_m \backslash C_{m+1}\}, \quad m = 1, \cdots, n-1,$$
$$I_n = \{i | 0 \le i \le j-1, f^i(p) \in C_n\}.$$

We claim that $\sharp(I_m) \le l_m$, $m = 1, \cdots, n$.

For each $1 \le m < n$, suppose that I_m is non-empty and let i_m be the least index in I_m. Then $f^{i_m}(p) \in C_m$. For $i \ge i_m + l_m$, we have $f^i(p) \in E_m$ and hence $f^i(p) \notin C_m \backslash C_{m+1}$. Thus I_m is a subset of $\{i_m, \cdots, i_m + l_m - 1\}$, and $\sharp(I_m) \le l_m$.

Now suppose that I_n is non-empty and let i_n be the least index in I_n. Then $f^{i_n}(p) \in C_n$. For $i \ge i_n + l_n$, we have $f^i(p) \in E_n$. Since $f^j(p) = q \notin E_n$, we have $i_n + l_n > j$. Thus I_n is a subset of $\{i_n, \cdots, i_n + l_n - 1\}$ and $\sharp(I_n) \le l_n$.

Next assume that f is critically finite of order k. Let $q \in \mathbf{P}^k \backslash F_k$ and let $p \in f^{-j}(q)$ for some integer $j > 0$. Let

$$I_m = \{i | 0 \leq i \leq j-1, f^i(p) \in C_m \backslash C_{m+1}\}, \quad m = 1, \cdots, k-1,$$
$$I_k = \{i | 0 \leq i \leq j-1, f^i(p) \in C_k\}.$$

For the same reason as above we have that $\sharp(I_m) \leq l_m$ for $1 \leq m < k$. Now suppose that I_k is non-empty and let i_k be the least index in I_k. Then $f^{i_k}(p) \in C_k$. For $i \geq i_k + l_k$, we have $f^i(p) \in E_k$. Note that $f(F_k) = F_k$ and $(E_k \backslash F_k) \cap C_k = \emptyset$. Since $f^j(p) = q \notin F_k$, we have $i_k + l_k > j$. Thus I_k is a subset of $\{i_k, \cdots, i_k + l_k - 1\}$ and $\sharp(I_k) \leq l_k$. □

Combining this lemma with Theorems 2.7 and 2.8, we obtain the following result, which generalizes [6, Theorem 5.8].

Theorem 2.11. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map. If f is critically finite of order n, $1 \leq n < k$, then the Fatou limit set is contained in E_n. If f is critically finite of order k, then the Fatou limit set is contained in F_k.*

By definition, a $k-$critically finite map on $\mathbf{P}^k$ has $F_k = \emptyset$. Therefore we obtain Theorem 1.2 as a corollary to the above theorem.

Now we turn to critically finite maps on $\mathbf{P}^2$. We say that a point p is a super-attracting periodic point for a holomorphic map f on $\mathbf{P}^2$ if there exists an $n \in \mathbf{N}$ such that $f^n(p) = p$ and both of the eigenvalues of the differential $df^n(p)$ are equal to zero; i.e. $df^n(p)$ is nilpotent.

We now prove Theorem 1.1.

Proof of Theorem 1.1. If f is not $1-$critically finite, we are done by [2, Theorem 7.8]. Therefore, we can assume that f is $1-$critically finite. Then by [6, Theorem 5.8], the Fatou limit set for f consists of finitely many periodic critical points in F_2. It is obvious from the definition of F_2 that these periodic points belong to the singular set of $V = \bigcup_{j=0}^{\infty} f^j(C_1)$. Hence, arguing as in the first part of the proof of [2, Theorem 7.7], we are done. Note that the assumption of $\mathbf{P}^2 \backslash V$ being hyperbolic in [2, Theorem 7.7] is only needed in the second part of its proof. □

Remark 2.12. Bonifant and Dabija ([1, Theorem 4.1]) showed that an invariant critical component for a holomorphic map on $\mathbf{P}^2$ must be a rational curve. While most known examples of critically finite maps on $\mathbf{P}^2$ have only smooth rational curves as invariant critical components, here we give a family of critically finite maps on $\mathbf{P}^2$ with singular rational curves as invariant critical components:

$$g_d : [z : w : t] \mapsto [z^d - w^{d-1}t : -w^d : -t^d], \quad d > 2.$$

Note that g_d maps the critical component $\{z = 0\}$ to the singular rational curve $\{z^d = w^{d-1}t\}$ and maps $\{z^d = w^{d-1}t\}$ back to $\{z = 0\}$. So g_d^2 will have $\{z^d = w^{d-1}t\}$ as a fixed critical component and obviously g_d^2 is a critically finite map.

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