

## NEW CHARACTERIZATION OF $\Sigma$ -INJECTIVE MODULES

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ABSTRACT. We provide a new characterization for an injective module to be  $\Sigma$ -injective.

### 1. INTRODUCTION

In his paper [4], Carl Faith introduced the concept of  $\Sigma$ -injectivity and defined an injective module  $M$  to be  $\Sigma$ -injective if every direct sum of copies of  $M$  is injective. It turns out that such an  $R$ -module  $M$  provides a good deal of information about the structure of a ring  $R$ . For example,  $R$  is right noetherian if and only if every injective right  $R$ -module is  $\Sigma$ -injective [5]. If  $R$  is an integral domain, then the injective hull  $E(R_R)$  of  $R$  is  $\Sigma$ -injective if and only if  $R$  is a right Ore domain [4]. Goursaud-Valette showed that if a ring  $R$  admits a faithful  $\Sigma$ -injective module, then  $R$  is a right Goldie ring [6].

The following characterizations are well-known for an injective module to be  $\Sigma$ -injective.

**Theorem 1** (Cailleau [3], Faith [4]). *For an injective module  $M_R$ , the following are equivalent:*

- (1)  $M$  is  $\Sigma$ -injective.
- (2)  $M$  is countably  $\Sigma$ -injective.
- (3)  $R$  satisfies ACC on the set of right ideals  $I$  of  $R$  that are annihilators of subsets of  $M$ .
- (4)  $M$  is a direct sum of indecomposable  $\Sigma$ -injective modules.

The purpose of this paper is to provide the following new characterization for an injective module to be  $\Sigma$ -injective.

**Theorem 2.** *Let  $M_R$  be an injective module. Then the following statements are equivalent:*

- (a)  $M$  is  $\Sigma$ -injective.
- (b) Every essential extension of  $M^{(\aleph_0)}$  is a direct sum of injective modules.

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## 2. PRELIMINARIES

All rings considered in this paper have unity, and all modules are right unital. We denote by  $E(M)$  the injective hull of  $M$ . We shall write  $N \subseteq_e M$  whenever  $N$  is an essential submodule of  $M$ . A submodule  $L$  of  $M$  is called an essential closure of a submodule  $N$  of  $M$  if it is a maximal essential extension of  $N$  in  $M$ . A submodule  $K$  of  $M$  is called a complement if there exists a submodule  $U$  of  $M$  such that  $K$  is maximal with respect to the property that  $K \cap U = 0$ . Given a cardinal  $\alpha$  and a module  $N$ , we denote by  $N^{(\alpha)}$  the direct sum of  $\alpha$  copies of the module  $N$ . A module  $N$  is said to be  $\Sigma$ -injective provided that  $N^{(\alpha)}$  is injective for any cardinal  $\alpha$ . We say that the Goldie dimension  $G \dim_U(N)$  of  $N$  with respect to  $U$  is finite, written as  $G \dim_U(N) < \infty$ , if  $N$  does not contain an infinite independent family of nonzero submodules which are isomorphic to submodules of  $U$ . A module  $N$  is said to be *q.f.d.* relative to  $U$  if for any factor module  $\bar{N}$  of  $N$ ,  $G \dim_U(\bar{N}) < \infty$ . We say  $R$  is right *q.f.d.* relative to  $U$  if  $R_R$  is *q.f.d.* relative to  $U$ .

We first start with a key lemma.

**Lemma 3.** *Let  $M$  be an injective module and suppose that every essential extension of  $M^{(\aleph_0)}$  is a direct sum of injective modules. Then*

(a) *Given a direct sum  $G = \bigoplus_{i \in \mathbb{N}} M_i$ ,  $M_i \cong M$ , and nonzero injective submodules  $V_i$  of  $M_i$ , there exists an infinite subset  $\mathcal{J} \subseteq \mathbb{N}$  and nonzero injective submodules  $V'_j \subseteq V_j$ ,  $j \in \mathcal{J}$ , such that  $\bigoplus_{j \in \mathcal{J}} V'_j$  is injective.*

*In particular, if  $\{V_i : i \in \mathbb{N}\}$  is an independent family of uniform injective submodules of  $M$ , then  $\bigoplus_{j \in \mathcal{J}} V_j$  is injective for some infinite subset  $\mathcal{J} \subseteq \mathbb{N}$ .*

(b)  *$R$  is right *q.f.d.* relative to  $M$ .*

*Proof.* (a) Set  $E = E(G)$ . Since  $V_i$  is an injective submodule of  $M_i$ ,  $M_i = V_i \oplus M'_i$  for some submodule  $M'_i \subseteq M_i$ . Therefore,  $G = (\bigoplus_{i \in \mathbb{N}} V_i) \oplus (\bigoplus_{i \in \mathbb{N}} M'_i)$ . Let  $H$  and  $H'$  be essential closures of  $\bigoplus_{i \in \mathbb{N}} V_i$  and  $\bigoplus_{i \in \mathbb{N}} M'_i$  in  $E$ , respectively. Clearly,  $E = H \oplus H'$ . If  $\bigoplus_{i \in \mathbb{N}} V_i = H$ , then there is nothing to prove.

Now consider the case when  $\bigoplus_{i \in \mathbb{N}} V_i \neq H$ . Pick  $x \in H \setminus \bigoplus_{i \in \mathbb{N}} V_i$ . Let  $Q$  be a submodule of  $H$  maximal with respect to the properties that  $\bigoplus_{i \in \mathbb{N}} V_i \subseteq Q$  and  $x \notin Q$ . Set  $P = Q \oplus H'$  and note that  $E/P = (H \oplus H')/(Q \oplus H') \cong H/Q$  is a subdirectly irreducible module.

Now, as  $G \subseteq_e E$  and  $G \subseteq P \subseteq E$ , we have  $G \subseteq_e P$ . Hence, by our assumption,  $P = \bigoplus_{k \in \mathcal{K}} W_k$ , where each  $W_k$  is a nonzero injective module. Since  $P \subseteq_e E$  and  $P \neq E$ ,  $P$  is not injective, and so  $|\mathcal{K}| = \infty$ .

We claim that for any finite subset  $\mathcal{L}$  of  $\mathcal{K}$  and for any positive integer  $n$  there exists  $i > n$  such that  $V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k)$  is not essential in  $V_i$ .

Suppose the above claim is not true. Then there exists a finite subset  $\mathcal{L} \subseteq \mathcal{K}$  and an integer  $n \geq 1$  such that  $V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k) \subseteq_e V_i$  for all  $i > n$ . Let  $A$  be an essential closure of  $\bigoplus_{i > n} (V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k))$  in  $\bigoplus_{k \in \mathcal{L}} W_k$  which is injective, and so  $A$  is also injective.

We have  $\bigoplus_{i > n} (V_i \cap \bigoplus_{k \in \mathcal{L}} W_k) \subseteq_e A \subseteq \bigoplus_{k \in \mathcal{L}} W_k$ . Setting  $B = V_1 \oplus V_2 \oplus \dots \oplus V_n \oplus A$ , we have  $V_1 \oplus V_2 \oplus \dots \oplus V_n \oplus \bigoplus_{i > n} (V_i \cap \bigoplus_{k \in \mathcal{L}} W_k) \subseteq_e B \subseteq E = H \oplus H'$ . Now,  $((\bigoplus_{i \leq n} V_i) \oplus \bigoplus_{i > n} (V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k))) \cap H \subseteq_e B \cap H \subseteq H$ , which gives  $(\bigoplus_{i \leq n} V_i) \oplus \bigoplus_{i > n} (V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k)) \subseteq_e B \cap H \subseteq H$ . Since  $V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k) \subseteq_e V_i$  for all  $i > n$ , we have  $(\bigoplus_{i \leq n} V_i) \oplus \bigoplus_{i > n} (V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k)) \subseteq_e \bigoplus_{i \in \mathbb{N}} V_i \subseteq_e H$ . Thus  $B \cap H$  is an essential

submodule of  $H$ . Furthermore, as  $(\bigoplus_{i \leq n} V_i) \oplus_{i > n} (V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k)) \subseteq_e B$ , we have  $B \cap H \subseteq_e B$ .

Since  $B \cap H \subseteq_e B$ , we have  $B \cap H' = 0$ . As  $B \cap H \subseteq_e H$ , we have  $(B \cap H) \oplus H' \subseteq_e H \oplus H' = E$ . Therefore,  $B \oplus H' \subseteq_e E$ . But since both  $B$  and  $H'$  are injective,  $B \oplus H'$  is injective. Thus  $E = B \oplus H' = (V_1 \oplus V_2 \oplus \dots \oplus V_n \oplus A) \oplus H' \subseteq Q + P + H' = P$ , a contradiction because  $P \subset E$  and  $P \neq E$ .

This proves that for any finite subset  $\mathcal{L}$  of  $\mathcal{K}$  and for any positive integer  $n$  there exists  $i > n$  such that  $V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k)$  is not essential in  $V_i$ .

We now proceed by induction to construct a sequence of submodules  $\{W'_{k_j} : j = 1, 2, \dots, n, \dots\}$  such that each  $W'_{k_j}$  is a nonzero injective submodule of  $W_{k_j}$  isomorphic to a submodule  $V'_{i_j}$  of  $V_{i_j}$ , where  $k_1, k_2, \dots, k_n, \dots$  are distinct elements of  $\mathcal{K}$  and  $1 \leq i_1 < i_2 < \dots < i_n < \dots$ .

Let  $i_1 \geq 1$  be arbitrary. Now  $V_{i_1} \subset \bigoplus_{k \in \mathcal{K}} W_k$  implies that there exists a nonzero submodule  $V'_{i_1}$  of  $V_{i_1}$  such that  $V'_{i_1}$  is isomorphic to a submodule  $W'_{k_1}$  of  $W_{k_1}$  for some  $k_1 \in \mathcal{K}$ . Clearly, we may choose  $V'_{i_1}$  to be an injective submodule of  $V_{i_1}$ .

For  $n \geq 1$ , assume that we have a sequence  $\{W'_{k_j} : j = 1, 2, \dots, n\}$  with the above stated property. By the fact proved above, there exists  $i_{n+1} > i_n$  such that  $X = V_{i_{n+1}} \cap (\bigoplus_{k \in \mathcal{K}_1} W_k)$  is not essential in  $V_{i_{n+1}}$ , where  $\mathcal{K}_1 = \{k_1, k_2, \dots, k_n\}$ . Let  $X'$  be a complement of  $X$  in  $V_{i_{n+1}}$ . Then  $X' \neq 0$  and  $X' \cap (\bigoplus_{k \in \mathcal{K}_1} W_k) = X' \cap X = 0$ . We have  $X' \subset V_{i_{n+1}} \subset (\bigoplus_{k \in \mathcal{K}_1} W_k) \oplus (\bigoplus_{k \in \mathcal{K}_2} W_k)$ , where  $\mathcal{K}_2 = \mathcal{K} \setminus \mathcal{K}_1$ . Let  $\pi : (\bigoplus_{k \in \mathcal{K}_1} W_k) \oplus (\bigoplus_{k \in \mathcal{K}_2} W_k) \rightarrow \bigoplus_{k \in \mathcal{K}_2} W_k$  be the projection. Then  $\ker(\pi|_{X'}) = X' \cap (\bigoplus_{k \in \mathcal{K}_1} W_k) = 0$ . Therefore,  $X'$  is isomorphic to some submodule of  $\bigoplus_{k \in \mathcal{K}_2} W_k$ . So,  $X'$  contains a nonzero submodule which is isomorphic to a submodule  $F$  of  $W_{k_{n+1}}$  for some  $k_{n+1} \in \mathcal{K}_2$ . Denote by  $W'_{k_{n+1}}$  an essential closure of  $F$  in  $W_{k_{n+1}}$ . Since  $F$  is isomorphic to a submodule of the injective module  $V_{i_{n+1}}$ , we conclude that  $W'_{k_{n+1}}$  is isomorphic to a submodule of  $V_{i_{n+1}}$  as well. Obviously the family  $\{W'_{k_j} : j = 1, 2, \dots, n + 1\}$  satisfies the required property. This completes the induction argument.

Now set  $\mathcal{K}' = \{k_1, k_2, \dots, k_n, \dots\}$ . Choose disjoint subsets  $\mathcal{K}'_1$  and  $\mathcal{K}'_2$  of  $\mathcal{K}$  such that  $\mathcal{K} = \mathcal{K}'_1 \cup \mathcal{K}'_2$  and  $\mathcal{K}' \cap \mathcal{K}'_1 = \{k_1, k_3, \dots, k_{2n+1}, \dots\}$ . Clearly,  $\mathcal{K}' \cap \mathcal{K}'_2 = \{k_2, k_4, \dots, k_{2n}, \dots\}$ .

Now we claim that either  $\bigoplus_{k \in \mathcal{K}'_1} W_k$  is injective or  $\bigoplus_{k \in \mathcal{K}'_2} W_k$  is injective.

Set  $V = \bigoplus_{k \in \mathcal{K}'_1} W_k$  and  $W = \bigoplus_{k \in \mathcal{K}'_2} W_k$ . We have  $P = V \oplus W$ . Let  $\widehat{V}$  and  $\widehat{W}$  be essential closures of  $V$  and  $W$ , respectively, in  $E$ . Clearly,  $E = \widehat{V} \oplus \widehat{W}$ . Therefore,  $E/P = (\widehat{V} \oplus \widehat{W}) / (V \oplus W) \cong (\widehat{V}/V) \times (\widehat{W}/W)$ . Since  $E/P$  is shown to be subdirectly irreducible in the beginning of the proof, we have either  $V = \widehat{V}$  or  $W = \widehat{W}$ . This proves our claim.

Thus, we may assume, without loss of generality, that the module  $\bigoplus_{k \in \mathcal{K}'_1} W_k$  is injective. Since  $\bigoplus_{n=0}^\infty W'_{k_{2n+1}}$  is a direct summand of  $\bigoplus_{k \in \mathcal{K}'_1} W_k$ , we get that  $\bigoplus_{n=0}^\infty W'_{k_{2n+1}}$  is injective. Recalling that  $\bigoplus_{n=0}^\infty V'_{i_{2n+1}} \cong \bigoplus_{n=0}^\infty W'_{k_{2n+1}}$ , we conclude that  $\bigoplus_{n=0}^\infty V'_{i_{2n+1}}$  is an injective module. This completes the proof.

(b) Assume to the contrary that  $R$  is not right *q.f.d.* relative to  $M$ . Then there exists a cyclic right  $R$ -module  $C$  with an infinite independent family  $\{C_i : i \in \mathbb{N}\}$  of

nonzero submodules of  $C$  such that each  $C_i$  is isomorphic to a submodule  $B_i$  of  $M$ . Set  $D_i$  equal to an essential closure of  $B_i$  in  $M$ . Then  $\{D_i : i \in \mathbb{N}\}$  is a family of injective submodules of  $M$ . Therefore by (a), there exists an infinite subset  $\mathcal{J} \subseteq \mathbb{N}$  and nonzero injective submodules  $D'_j \subseteq D_j$ ,  $j \in \mathcal{J}$ , such that  $\bigoplus_{j \in \mathcal{J}} D'_j$  is injective. Set  $B'_j = B_j \cap D'_j$ ,  $j \in \mathcal{J}$  and note that  $B'_j \neq 0$ . Let  $C'_j$  be the inverse image of  $B'_j$  under the isomorphism  $C_j \rightarrow B_j$  stated above. This induces a canonical isomorphism between  $\bigoplus_{j \in \mathcal{J}} C'_j$  and  $\bigoplus_{j \in \mathcal{J}} B'_j$ , say  $\theta$ . Let  $\sigma$  be the inclusion map  $\bigoplus_{j \in \mathcal{J}} B'_j \rightarrow \bigoplus_{j \in \mathcal{J}} D'_j$ . Then, since  $\bigoplus_{j \in \mathcal{J}} D'_j$  is injective, the map  $f = \sigma\theta : \bigoplus_{j \in \mathcal{J}} C'_j \rightarrow \bigoplus_{j \in \mathcal{J}} D'_j$  can be extended to a homomorphism  $\widehat{f} : C \rightarrow \bigoplus_{j \in \mathcal{J}} D'_j$ . Because  $C$  is cyclic, there exists a finite subset  $\mathcal{K} \subseteq \mathcal{J}$  such that  $\widehat{f}(C) \subseteq \bigoplus_{k \in \mathcal{K}} D'_k$ . Now,  $\widehat{f}(C'_j) = f(C'_j) = \sigma\theta(C'_j) = \sigma(B'_j) = B'_j$ . But  $\widehat{f}(C'_j) \subseteq \widehat{f}(C) \cap D'_j = 0$  for all  $j \notin \mathcal{K}$ , a contradiction.

Therefore,  $R$  is right *q.f.d.* relative to  $M$ .  $\square$

### 3. PROOF OF THEOREM 2

*Proof.* (b)  $\implies$  (a). Suppose that  $M^{(\lambda)}$  is not injective for some infinite cardinal  $\lambda$ . Set  $E = E(M^{(\lambda)})$ , pick  $x \in E \setminus M^{(\lambda)}$  and let  $L = xR$ . By Lemma 3 (b),  $R$  is right *q.f.d.* relative to  $M$ . From this it follows that every nonzero cyclic and hence every nonzero submodule of  $M$  contains a uniform submodule. Now, consider the set  $\mathcal{S}$  of independent families  $(M_k)_{k \in \mathcal{K}}$  of uniform injective modules  $0 \neq M_k \subseteq M$ . Suppose  $\mathcal{S}$  is partially ordered by  $(M_k)_{k \in \mathcal{K}} \leq (N_l)_{l \in \mathcal{L}}$  if and only if  $\mathcal{K} \subseteq \mathcal{L}$  and  $M_k = N_k$  for  $k \in \mathcal{K}$ . By Zorn's lemma we get a maximal independent family  $(M_i)_{i \in \mathcal{I}}$  of uniform injective submodules. Clearly  $\bigoplus_{i \in \mathcal{I}} M_i \subseteq_e M$ , because otherwise we will get a contradiction to the maximality of this independent family of submodules. This yields that we have an independent family  $\{W_i : i \in \mathcal{I}\}$  of uniform injective submodules of  $M^{(\lambda)}$  such that each  $W_i$  is isomorphic to a submodule of  $M$  and  $\bigoplus_{i \in \mathcal{I}} W_i \subseteq_e M^{(\lambda)}$ .

Now we proceed to show that there is a sequence of pairwise distinct elements  $i_1, i_2, \dots$  in  $\mathcal{I}$  and an independent family of direct summands  $V_1, V_2, \dots$  of  $E$  such that  $V_j \cong W_{i_j}$  with  $V_j \oplus (\bigoplus_{i \in \mathcal{I}_j} W_i) = \bigoplus_{i \in \mathcal{I}_{j-1}} W_i$ ,  $E = E_j \oplus (\bigoplus_{k=1}^j V_k)$  and  $\pi_{j-1}(L) \cap V_j \neq 0$  for all  $j \in \mathbb{N}$ , where  $\mathcal{I}_0 = \mathcal{I}$ ,  $\mathcal{I}_j = \mathcal{I}_{j-1} \setminus \{i_j\}$  for  $i_j \in \mathcal{I}$ ,  $E_0 = E$ ,  $E_j$  is an essential closure of  $\bigoplus_{i \in \mathcal{I}_j} W_i$  in  $E_{j-1}$ ,  $\pi_0 = id_E$ , and  $\pi_j$  is the projection of  $E$  onto  $E_j$  along  $V_1 \oplus \dots \oplus V_j$ .

Since  $\bigoplus_{i \in \mathcal{I}} W_i \subseteq_e M^{(\lambda)} \subseteq_e E$  and  $L$  is a nonzero submodule of  $E$ , we have  $L \cap (\bigoplus_{i \in \mathcal{I}} W_i) \neq 0$ . So  $L \cap (\bigoplus_{i \in \mathcal{I}} W_i)$  contains a nonzero cyclic uniform submodule, say,  $C_1$ . This implies that there exists a finite subset  $\mathcal{K}_1 \subset \mathcal{I}$  such that  $C_1 \subseteq \bigoplus_{i \in \mathcal{K}_1} W_i$ . Let  $V_1$  be an essential closure of  $C_1$  in  $\bigoplus_{i \in \mathcal{K}_1} W_i$ . Since  $\bigoplus_{i \in \mathcal{K}_1} W_i$  is injective,  $V_1$  is injective. So,  $\bigoplus_{i \in \mathcal{K}_1} W_i = V_1 \oplus D_1$  for some submodule  $D_1$  of  $\bigoplus_{i \in \mathcal{K}_1} W_i$ . Since  $V_1$  is injective, it has the exchange property. Therefore,  $\bigoplus_{i \in \mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i \in \mathcal{K}_1} W'_i)$  for some submodules  $W'_i$  of  $W_i$ . Since  $W'_i$  are injective and each  $W_i$  is indecomposable, either  $W'_i = 0$  or  $W'_i = W_i$ . We recall that  $V_1$  is uniform because it is the closure of the uniform module  $C_1$ . Comparing the Goldie dimension on each side of  $\bigoplus_{i \in \mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i \in \mathcal{K}_1} W'_i)$ , we get that there exists exactly one index, say  $i_1 \in \mathcal{K}_1$ , such that  $W'_{i_1} = 0$ , and for all  $i (\neq i_1) \in \mathcal{K}_1$ ,  $W'_i = W_i$ . So,  $\bigoplus_{i \in \mathcal{K}_1} W_i = V_1 \oplus (\bigoplus_{i \in \mathcal{K}_1 \setminus \{i_1\}} W_i)$ . This yields

$V_1 \cong (\bigoplus_{i \in \mathcal{K}_1} W_i) / (\bigoplus_{i \in \mathcal{K}_1 \setminus \{i_1\}} W_i) \cong W_{i_1}$ . Also, we have  $V_1 \oplus (\bigoplus_{i \in \mathcal{K}_1 \setminus \{i_1\}} W_i) \oplus (\bigoplus_{i \in \mathcal{I} \setminus \mathcal{K}_1} W_i) = (\bigoplus_{i \in \mathcal{K}_1} W_i) \oplus (\bigoplus_{i \in \mathcal{I} \setminus \mathcal{K}_1} W_i)$ . This yields  $V_1 \oplus (\bigoplus_{i \in \mathcal{I}_1} W_i) = \bigoplus_{i \in \mathcal{I}} W_i$ . Taking injective hulls of both sides, we get  $E_1 \oplus V_1 = E$ . Clearly,  $L \cap V_1 \neq 0$  as it contains  $C_1$ .

For  $n \geq 1$ , assume that we have a sequence  $\{V_j\}$ ,  $1 \leq j \leq n$ , of submodules of  $E$  with the above stated properties. Since  $x \notin M^{(\lambda)}$ ,  $L = xR \not\subseteq \bigoplus_{i=1}^n V_i = \ker(\pi_n)$ , if  $x \in \bigoplus_{i=1}^n V_i$ , then  $V_1 \oplus \dots \oplus V_n \oplus (\bigoplus_{i \in \mathcal{I}_n} W_i) = \bigoplus_{i \in \mathcal{I}_0} W_i$  implies that  $x$  belongs to  $\bigoplus_{i \in \mathcal{I}_0} W_i$  and hence to  $M^{(\lambda)}$ , a contradiction. So  $\pi_n(L) \neq 0$ . Now  $\bigoplus_{i \in \mathcal{I}_n} W_i \subseteq_e E_n$  and because  $\pi_n : E \rightarrow E_n$ , we have  $\pi_n(L) \cap (\bigoplus_{i \in \mathcal{I}_n} W_i) \neq 0$ . So  $\pi_n(L) \cap (\bigoplus_{i \in \mathcal{I}_n} W_i)$  contains a nonzero cyclic uniform submodule, say,  $C_{n+1}$ . This implies, there exists a finite subset  $\mathcal{K}_{n+1} \subseteq \mathcal{I}_n$  such that  $C_{n+1} \subseteq \bigoplus_{i \in \mathcal{K}_{n+1}} W_i$ . Let  $V_{n+1}$  be an essential closure of  $C_{n+1}$  in  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i$ . Since  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i$  is injective,  $V_{n+1}$  is injective. So,  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i = V_{n+1} \oplus D_{n+1}$  for some submodule  $D_{n+1}$  of  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i$ . Since  $V_{n+1}$  is injective, it has the exchange property. Therefore,  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i \in \mathcal{K}_{n+1}} W'_i)$  for some submodules  $W'_i$  of  $W_i$ . Since  $W'_i$  are injective and each  $W_i$  is indecomposable, either  $W'_i = 0$  or  $W'_i = W_i$ . Again note that  $V_{n+1}$  is uniform because it is the closure of the uniform module  $C_{n+1}$ . Comparing the Goldie dimension on each side of  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i \in \mathcal{K}_{n+1}} W'_i)$ , we get that there exists exactly one index, say  $i_{n+1} \in \mathcal{K}_{n+1}$ , such that  $W'_{i_{n+1}} = 0$ , and for all  $i (\neq i_{n+1}) \in \mathcal{K}_{n+1}$ ,  $W'_i = W_i$ . So,  $\bigoplus_{i \in \mathcal{K}_{n+1}} W_i = V_{n+1} \oplus (\bigoplus_{i \in \mathcal{K}_{n+1} \setminus \{i_{n+1}\}} W_i)$ . This yields  $V_{n+1} \cong (\bigoplus_{i \in \mathcal{K}_{n+1}} W_i) / (\bigoplus_{i \in \mathcal{K}_{n+1} \setminus \{i_{n+1}\}} W_i) \cong W_{i_{n+1}}$ . Also, we get  $V_{n+1} \oplus (\bigoplus_{i \in \mathcal{K}_{n+1} \setminus \{i_{n+1}\}} W_i) \oplus (\bigoplus_{i \in \mathcal{I}_n \setminus \mathcal{K}_{n+1}} W_i) = (\bigoplus_{i \in \mathcal{K}_{n+1}} W_i) \oplus (\bigoplus_{i \in \mathcal{I}_n \setminus \mathcal{K}_{n+1}} W_i)$ . This yields  $V_{n+1} \oplus (\bigoplus_{i \in \mathcal{I}_{n+1}} W_i) = \bigoplus_{i \in \mathcal{I}_n} W_i$ . Taking injective hulls of both sides, we get  $E_{n+1} \oplus V_{n+1} = E_n$ . Thus, we have  $E = E_{n+1} \oplus (\bigoplus_{k=1}^{n+1} V_k)$ . Note that  $\pi_n(L) \cap V_{n+1} \neq 0$  as it contains  $C_{n+1}$ . Thus, we have obtained a sequence of submodules  $\{V_j\}$ ,  $j = 1, 2, \dots$ , with the required properties. This completes the induction argument.

Now we claim that there exists a properly ascending chain  $N_0 \subset N_1 \subset \dots \subset N_j \subset \dots$  of submodules of  $L$  such that  $N_0 = 0$  and  $E(N_j/N_{j-1}) \cong V_j$  for all  $j \geq 1$ .

Set  $N_j = L \cap (V_1 \oplus \dots \oplus V_j)$ . Clearly,  $N_0 \subseteq N_1 \subseteq \dots \subseteq N_j \subseteq \dots$ . Since  $N_j \cap \ker(\pi_{j-1}) = N_{j-1}$ , we have  $N_j/N_{j-1} \cong \pi_{j-1}(N_j)$ . If  $l \in N_j$ , then  $l = v_1 + \dots + v_j$  with  $v_i \in V_i$ , so  $\pi_{j-1}(l) = v_j$  and  $v_j \in \pi_{j-1}(L) \cap V_j$ . This shows that  $\pi_{j-1}(N_j) \subseteq \pi_{j-1}(L) \cap V_j$ . Conversely, if  $v_j \in \pi_{j-1}(L) \cap V_j$ , then  $v_j = \pi_{j-1}(l)$  with  $l \in L \cap (V_1 \oplus \dots \oplus V_j) = N_j$ , so  $v_j \in \pi_{j-1}(N_j)$ . Therefore  $\pi_{j-1}(N_j) = \pi_{j-1}(L) \cap V_j \neq 0$ . Because  $\pi_{j-1}(N_{j-1}) = 0$  and  $\pi_{j-1}(N_j) \neq 0$ , it follows that  $N_{j-1} \subsetneq N_j$ . Since  $N_j/N_{j-1} \cong \pi_{j-1}(N_j) = \pi_{j-1}(L) \cap V_j$ , we have  $E(N_j/N_{j-1}) \cong V_j$ .

Since  $\{V_j : j \in \mathbb{N}\}$  is an independent family of uniform injective modules isomorphic to submodules of  $M$ , by the above lemma there exists an infinite subset  $\mathcal{J} \subseteq \mathbb{N}$  such that  $\bigoplus_{j \in \mathcal{J}} V_j$ , and hence  $\bigoplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$  is injective. Set  $N = \bigcup_{j \in \mathcal{J}} N_j$ . Given  $j \in \mathcal{J}$ , the canonical map  $N_j \rightarrow N_j/N_{j-1} \subset E(N_j/N_{j-1})$  induces a map  $\alpha_j : N \rightarrow E(N_j/N_{j-1})$ . Let  $\alpha : N \rightarrow \bigoplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$  be defined by  $\alpha(x) = \{\alpha_j(x)\}_{j \in \mathcal{J}}$  for all  $x \in N$ . Since  $\bigoplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$  is injective, we may extend  $\alpha$  to  $\alpha^* : L \rightarrow \bigoplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$ . As  $L$  is finitely generated, there exists a finite subset  $\mathcal{K} \subseteq \mathcal{J}$  such that  $\alpha^*(L) \subseteq \bigoplus_{k \in \mathcal{K}} E(N_k/N_{k-1})$ . For  $j \in \mathcal{J} \setminus \mathcal{K}$  and  $x \in N_j$  we have  $0 = \alpha_j(x) = x + N_{j-1}$ , showing that  $N_{j-1} = N_j$ , a contradiction.

Therefore,  $M^{(\lambda)}$  is injective for any cardinal  $\lambda$ , and hence  $M$  is  $\Sigma$ -injective.

(a)  $\implies$  (b) is obvious.

This completes the proof of Theorem 2.  $\square$

As a consequence of Theorem 2, we have the following characterization for a right noetherian ring.

**Theorem 4.** *Let  $R$  be a ring. Then the following are equivalent:*

(i)  $R$  is right noetherian.

(ii) For each injective module  $M_R$ , every essential extension of  $M^{(\aleph_0)}$  is a direct sum of injective modules.

*Proof.* (i)  $\implies$  (ii) is obvious. (ii)  $\implies$  (i) follows from Theorem 2 and by Faith-Walker [5] that a ring  $R$  is right noetherian if and only if every injective right  $R$ -module is  $\Sigma$ -injective.  $\square$

*Remark 5.* The above result generalizes a result of Beidar-Ke [2] which states that a ring  $R$  is right noetherian if and only if every essential extension of a direct sum of injective right  $R$ -modules is again a direct sum of injective right  $R$ -modules. Note that [2] indeed generalizes a result of Bass [1] that a ring is right noetherian if and only if every direct sum of injective modules is injective.

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