

ON THE UPPER BOUND OF THE MULTIPLICITY CONJECTURE

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Dedicated to Juergen Herzog on the occasion of his 65th birthday

ABSTRACT. Let $A = K[X_1, \dots, X_n]$ and let I be a graded ideal in A . We show that the upper bound of the multiplicity conjecture of Herzog, Huneke and Srinivasan holds asymptotically (i.e., for I^k and all $k \gg 0$) if I belongs to any of the following large classes of ideals:

- (1) radical ideals,
- (2) monomial ideals with generators in different degrees,
- (3) zero-dimensional ideals with generators in different degrees.

Surprisingly, our proof uses local techniques like analyticity, reductions, equimultiplicity and local results like Rees's theorem on multiplicities.

1. INTRODUCTION

Let K be a field and $A = K[X_1, \dots, X_n]$ be a polynomial ring with standard grading. Let I be a graded ideal of A . Let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{p,j}(A/I)} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1,j}(A/I)} \longrightarrow A \longrightarrow 0$$

be a minimal graded free resolution of A/I . Set $p = \text{projdim } A/I$ and $c = \text{height } I$. Consider for $1 \leq i \leq p$ the numbers

$$M_i(A/I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(A/I) \neq 0\} \text{ \& } m_i(A/I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(A/I) \neq 0\}.$$

Let $e(A/I)$ denote the multiplicity of A/I . Set

$$L(I) = \frac{1}{c!} \prod_{i=1}^c m_i(A/I) \quad \text{and} \quad U(I) = \frac{1}{c!} \prod_{i=1}^c M_i(A/I).$$

The conjecture of Herzog, Huneke and Srinivasan states that

Conjecture 1.1. *If A/I is Cohen-Macaulay, then*

$$L(I) \leq e(A/I) \leq U(I).$$

If A/I is not Cohen-Macaulay, then in [10] it is conjectured that

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Conjecture 1.2.

$$e(A/I) \leq U(I).$$

Both Conjectures 1.1 and 1.2 have been proved for many classes of ideals (see [6, 9, 10, 11, 14, 17, 20, 22, 24]). For extensions of this conjecture see [15, 16, 18, 23, 25]. For some new approaches to this problem see [1, 5, 7, 15, 16]. Our result is

Theorem 1.3. *Let I be a graded ideal. If I belongs to any of the following classes of ideals:*

- (1) *radical ideals,*
 - (2) *monomial ideals with generators in different degrees,*
 - (3) *zero-dimensional ideals with generators in different degrees,*
- then $e(A/I^k) \leq U(I^k)$ for all $k \gg 0$.

In [11, Theorem 2], the authors show $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) \leq 1$. There are examples where $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) = 1$; for instance see Section 4. In our proof we show that in the class of ideals of Theorem 1.3 we have that the limit on the left hand side is < 1 . The surprising feature of our proof is the use of local techniques like equimultiplicity, reductions analyticity and local theorems like Rees multiplicity theorem (see subsection 2.2).

Overview of the paper. In section 2 we introduce notation and discuss a few preliminary facts that we need. In section 3 we prove Theorem 1.3. In section 4 we give an example of a class of ideals whose members satisfy $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) = 1$.

2. PRELIMINARIES

In this section we recall some notions in local algebra. We also discuss asymptotic behavior of regularity of ideals I^k for $k \gg 0$. Finally we also recall that the function $k \mapsto e(A/I^k)$ is polynomial in k for $k \gg 0$.

- *Some local notions:*

Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d and residue field $K = R/\mathfrak{m}$ which, for convenience, we assume is infinite. Let \mathfrak{a} be an ideal in R . If M is a finitely generated R -module, then $\mu(M)$ denotes its minimal number of generators and $\ell(M)$ denotes its length.

2.1. The *analytic spread* of \mathfrak{a} is the Krull dimension of the fiber-cone $F(\mathfrak{a}) = \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{m} \mathfrak{a}^n$. We denote it by $s(\mathfrak{a})$. By [19, p. 150, Th. 1], $s(\mathfrak{a}) = \mu(\mathfrak{b})$, where \mathfrak{b} is a (any) minimal reduction of \mathfrak{a} . For the definition of reduction and minimal reduction; see [19, p. 146]. It can be shown that $\text{height}(\mathfrak{a}) \leq s(\mathfrak{a})$; see [19, p. 151, L. 4]. We say \mathfrak{a} is an *equimultiple* ideal if $\text{height}(\mathfrak{a}) = s(\mathfrak{a})$. If R is quasi-unmixed, then \mathfrak{a} is equimultiple if and only if $\text{gr}_{\mathfrak{a}} R = \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$, the associated graded ring of \mathfrak{a} , has a homogeneous system of parameters; see [8, 2.6].

2.2. If \mathfrak{a} is \mathfrak{m} -primary, then let $e(\mathfrak{a}, R)$ = multiplicity of R with respect to \mathfrak{a} ; i.e.,

$$e(\mathfrak{a}, R) = \lim_{n \rightarrow \infty} \frac{d!}{n^d} \ell \left(\frac{R}{\mathfrak{a}^n} \right).$$

Let $\mathfrak{b} \subseteq \mathfrak{a}$ be \mathfrak{m} -primary. Clearly $e(\mathfrak{b}, R) \geq e(\mathfrak{a}, R)$. It is easy to see that if \mathfrak{b} is a reduction of \mathfrak{a} , then $e(\mathfrak{b}, R) = e(\mathfrak{a}, R)$. A celebrated theorem due to Rees [21] shows that if R is quasi-unmixed and $e(\mathfrak{b}, R) = e(\mathfrak{a}, R)$, then \mathfrak{b} is a reduction of \mathfrak{a} .

- *Asymptotic behavior of regularity:*

Let $A = K[X_1, \dots, X_n]$. Let I be a graded ideal in A and let $p = \text{projdim } A/I$. Then

$$\text{reg}(I) = \max\{M_{i+1}(A/I) - i \mid i = 0, \dots, p - 1\}$$

is the regularity of I . Set $\text{reg}_i(I) = M_{i+1}(A/I) - i$ for $i = 0, \dots, p - 1$.

2.3. In [4, 2.4] and [12, 1] it is shown that $\text{reg}(I^k) = qk + r$ for $k \gg 0$. In [12, 5] it is shown that I has a reduction J such that $\text{reg}_0(J) = q$. In particular J is generated in degrees $\leq q$. We call such a reduction a *Kodiyalam reduction*.

2.4. In [11, 2.1(ii)] it is proved that for $i = 0, \dots, c - 1$,

$$\text{reg}_i(I^k) = qk + r_i \quad \text{for all } k \gg 0.$$

Therefore for $k \gg 0$,

$$U(I^k) = \frac{1}{c!} \prod_{i=1}^c M_i(A/I^k) = \frac{q^c}{c!} k^c + \dots + \text{lower terms in } k.$$

• The function $k \mapsto e(A/I^k)$.

2.5. Let $\text{Assh}(I) = \{\mathfrak{P} \in \text{Spec}(A) \mid \mathfrak{P} \supseteq I \text{ and } \dim A/\mathfrak{P} = \dim A/I\}$. Notice that all $\mathfrak{P} \in \text{Assh}(I)$ are graded ideals. The associativity formula of multiplicity ([2, 4.7.8]) then shows that

$$(2.1) \quad e(A/I) = \sum_{\mathfrak{P} \in \text{Assh}(I)} \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}})e(A/\mathfrak{P}).$$

Since $\text{Assh}(I^k) = \text{Assh}(I)$ for all $k \geq 1$ we have that

$$(2.2) \quad e(A/I^k) = \sum_{\mathfrak{P} \in \text{Assh}(I)} \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}^k)e(A/\mathfrak{P}).$$

Recall that $c = \text{height}(I)$. Since $k \mapsto \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}^k)$ is a polynomial function of degree c , it follows that $k \mapsto e(A/I^k)$ is a polynomial function of degree c . Furthermore, if $E(I)$ is the normalized leading coefficient of this function, then

$$(2.3) \quad E(I) = \sum_{\mathfrak{P} \in \text{Assh}(I)} e(I_{\mathfrak{P}}, A_{\mathfrak{P}})e(A/\mathfrak{P}).$$

Remark 2.6. Let $J \subseteq I$ be a graded ideal. If J is a reduction of I , then $J_{\mathfrak{P}}$ is a reduction of $I_{\mathfrak{P}}$ for all primes \mathfrak{P} . So $E(J) = E(I)$.

2.7. By subsections 2.4 and 2.5 we get that

$$\lim_{k \rightarrow \infty} \frac{e(A/I^k)}{U(I^k)} = \frac{E(I)}{q^c}.$$

Here q is as in subsection 2.3.

3. PROOF OF THEOREM 1.3

In this section we prove our result. We use [11, Theorem 2], where it is proved that $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) \leq 1$. In our proof we show that in the class of ideals of Theorem 1.3 we have that the limit on the left hand side is < 1 . Throughout this section q is as in section 2.3.

3.1. In [11, section 2] the authors assume K is infinite and then do the following:

- Let J be a Kodiyalam reduction of I and let $f_1, \dots, f_c \in J_q$ be c -generic q -forms. Set $L = (f_1, \dots, f_c)$.
- $e(A/L) = E(L) = q^c$.
- $E(I) \leq E(L)$.

The following observation is useful:

Observation 3.2. (1) In subsection 3.1, an ideal $L = (f_1, \dots, f_c)$, where $f_1, \dots, f_c \in J_q$ is a regular sequence, will do. In fact in [11, section 2] it is chosen generic just to ensure f_1, \dots, f_c is a regular sequence.

(2) We may choose $f_1 \in J_q$ to be any non-zero element.

To prove $E(I) < E(L)$ the following remark is useful:

Remark 3.3. L is unmixed. Also $\text{height } I = \text{height } L = c$. Thus $\text{Assh}(I) \subseteq \text{Assh}(A/L) = \text{Min}(A/L)$, the set of minimum primes of L . So to prove $E(I) < E(L)$, it suffices to show that there exists $\mathfrak{P} \in \text{Assh}(I)$ such that $e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) < e(L_{\mathfrak{P}}, A_{\mathfrak{P}})$.

We now give

Proof of Theorem 1.3. We prove that for each class of ideals considered we have $E(I) < E(L) = q^c$. We also assume K is infinite. This follows from the usual standard trick in the case when K is finite.

Case 1. I is a radical ideal.

In this case we have the following:

Claim. $E(I) = e(A/I)$.

Let $I = Q_1 \cap \dots \cap Q_s$ be a minimal irredundant primary decomposition of I . Set $\mathfrak{P}_i = \sqrt{Q_i}$ for $i = 1, \dots, s$. Let $\mathfrak{P} \in \text{Assh}(I)$. Then $\mathfrak{P} = \mathfrak{P}_i$ for some i . As I is a radical ideal we have $I_{\mathfrak{P}} = \mathfrak{P}A_{\mathfrak{P}}$. Notice $A_{\mathfrak{P}}$ is a regular local ring of dimension c . So

$$e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) = e(\mathfrak{P}A_{\mathfrak{P}}, A_{\mathfrak{P}}) = 1,$$

$$\ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}) = \ell(A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}) = 1.$$

Therefore by (2.1) and (2.3) we get

$$E(I) = \sum_{\mathfrak{P} \in \text{Assh}(I)} e(I_{\mathfrak{P}}, A_{\mathfrak{P}})e(A/\mathfrak{P}) = \sum_{\mathfrak{P} \in \text{Assh}(I)} e(A/\mathfrak{P}),$$

$$e(A/I) = \sum_{\mathfrak{P} \in \text{Assh}(I)} \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}})e(A/\mathfrak{P}) = \sum_{\mathfrak{P} \in \text{Assh}(I)} e(A/\mathfrak{P}).$$

Thus $E(I) = e(A/I)$.

Set $\mathfrak{m} = (X_1, \dots, X_n)$. Notice that $I_{\mathfrak{m}}$ is a radical ideal of $A_{\mathfrak{m}}$.

Subcase 1. The ideal $I_{\mathfrak{m}}$ is equimultiple. Then by a result due to Cowsik and Nori [3] we have that $I_{\mathfrak{m}}$ is generated by a regular sequence. Since I is graded it follows that I is also generated by a regular sequence. In this case by [9] we have that $e(A/I^k) \leq U(I^k)$ for all $k \geq 1$.

Subcase 2. I_m is not equimultiple. Let L be as in subsection 3.1. Then L is not a reduction of I . (Otherwise L_m will be a reduction of I_m and this will imply that I_m is equimultiple.)

In particular $L \neq I$. Consider the exact sequence

$$0 \longrightarrow \frac{I}{L} \longrightarrow \frac{A}{L} \longrightarrow \frac{A}{I} \longrightarrow 0.$$

Since $I \neq L$ we have that $(I/L)_{\mathfrak{P}} \neq 0$ for some $\mathfrak{P} \in \text{Ass}(A/L) = \text{Min}(A/L)$. In particular $\dim I/L = \dim A/L = \dim A/I$. It follows that

$$E(L) = e(A/L) = e(A/I) + e(I/L) > e(A/I) = E(I).$$

This implies the result in this case.

Case 2. I is a monomial ideal with generators in different degrees.

Let $\mathfrak{P} \in \text{Assh}(I)$. As I is a monomial ideal, \mathfrak{P} is generated by a subset of the variables [2, 4.4.15]. Say $\mathfrak{P} = (X_{i_1}, \dots, X_{i_s})$. Let $G(I) = \{u_1, \dots, u_a\}$ be the unique set of minimal monomial generators of I . Assume $\deg u_1 < q$. Set $\alpha = q - \deg u_1$. Set $f_1 = X_{i_1}^\alpha u_1$ and let $L = (f_1, f_2, \dots, f_c)$ (see Observation 3.2(2)). As $f_1 \in \mathfrak{P}I_{\mathfrak{P}}$, it follows that $L_{\mathfrak{P}}$ is not a minimal reduction of $I_{\mathfrak{P}}$ [19, Lemma 2]. Therefore by Rees's theorem $e(L_{\mathfrak{P}}, A_{\mathfrak{P}}) > e(I_{\mathfrak{P}}, A_{\mathfrak{P}})$. So by Remark 3.3 we get $E(L) > E(I)$.

Case 3. I is a zero-dimensional ideal with generators in different degrees.

Notice that in this case $\text{Assh}(I) = \{(X_1, \dots, X_n)\}$. The proof is similar to Case 2. □

Remark 3.4. For Cases 2 and 3 in our theorem note that the ideal can never have a pure resolution. Notice also that $\lim_{k \rightarrow \infty} \{U(I^k) - e(A/I^k)\} = \infty$. This gives further evidence of the “improved” multiplicity conjectures that suggest that Cohen-Macaulay ideals with pure resolutions are the only ones for which the bounds are sharp.

4. AN EXAMPLE

In [11], the authors state that its easy to construct examples of ideals with $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) = 1$. For the sake of completeness we give a large class of ideals where $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) = 1$. The notation will be as in subsection 3. Set $\mathfrak{m} = (X_1, \dots, X_n)$.

4.1. Let $q \geq 2$ and let $I \subseteq \mathfrak{m}^q$ be a zero-dimensional ideal generated by q -forms. It is easily verified that $\text{reg}(I^k) = qk + r$ for $k \gg 0$ (use [4, 3.2]). Let $f_1, \dots, f_n \in I$ be any regular sequence of q -forms. Set $L = (f_1, \dots, f_n)$. Notice that $e(L_m, A_m) = q^n = e(\mathfrak{m}^q, A_m)$. So by a theorem of Rees (see section 2.2), L_m is a reduction of $\mathfrak{m}^q A_m$. It follows that L is a reduction of \mathfrak{m}^q . Therefore L is also a reduction of I . By Remark 2.6 we get that $E(I) = E(L) = q^c$. So by section 2.7 we get $\lim_{k \rightarrow \infty} e(A/I^k)/U(I^k) = 1$.

Remark 4.2. We do not know as yet whether the upper bound of the multiplicity conjecture holds asymptotically for all ideals in the class described in subsection 4.1.

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