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# ON THE UPPER BOUND OF THE MULTIPLICITY CONJECTURE

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*Dedicated to Juergen Herzog on the occasion of his 65th birthday*

Abstract. Let $A = K[X_1, \dots, X_n]$ and let $I$ be a graded ideal in $A$. We show that the upper bound of the multiplicity conjecture of Herzog, Huneke and Srinivasan holds asymptotically (i.e., for $I^k$ and all $k \gg 0$) if $I$ belongs to any of the following large classes of ideals:
(1) radical ideals,
(2) monomial ideals with generators in different degrees,
(3) zero-dimensional ideals with generators in different degrees.
Surprisingly, our proof uses local techniques like analyticity, reductions, equimultiplicity and local results like Rees's theorem on multiplicities.

## 1. Introduction

Let $K$ be a field and $A = K[X_1, \dots, X_n]$ be a polynomial ring with standard grading. Let $I$ be a graded ideal of $A$. Let

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} A(-j)^{\beta_{p,j}(A/I)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} A(-j)^{\beta_{1,j}(A/I)} \longrightarrow A \longrightarrow 0$$

be a minimal graded free resolution of $A/I$. Set $p = \operatorname{projdim} A/I$ and $c = \operatorname{height} I$. Consider for $1 \le i \le p$ the numbers

$$M_i(A/I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(A/I) \neq 0\} \ \&\ m_i(A/I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(A/I) \neq 0\}.$$

Let $e(A/I)$ denote the multiplicity of $A/I$. Set

$$L(I) = \frac{1}{c!}\prod_{i=1}^{c} m_i(A/I) \quad \text{and} \quad U(I) = \frac{1}{c!}\prod_{i=1}^{c} M_i(A/I).$$

The conjecture of Herzog, Huneke and Srinivasan states that

**Conjecture 1.1.** *If $A/I$ is Cohen-Macaulay, then*

$$L(I) \le e(A/I) \le U(I).$$

If $A/I$ is not Cohen-Macaulay, then in [10] it is conjectured that

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**Conjecture 1.2.**

$$e(A/I) \leq U(I).$$

Both Conjectures 1.1 and 1.2 have been proved for many classes of ideals (see [6, 9, 10, 11, 14, 17, 20, 22, 24]). For extensions of this conjecture see [15, 16, 18, 23, 25]. For some new approaches to this problem see [1, 5, 7, 15, 16]. Our result is

**Theorem 1.3.** *Let $I$ be a graded ideal. If $I$ belongs to any of the following classes of ideals:*
(1) *radical ideals,*
(2) *monomial ideals with generators in different degrees,*
(3) *zero-dimensional ideals with generators in different degrees,*
*then $e(A/I^k) \leq U(I^k)$ for all $k \gg 0$.*

In [11, Theorem 2], the authors show $\lim_{k\to\infty} e(A/I^k)/U(I^k) \leq 1$. There are examples where $\lim_{k\to\infty} e(A/I^k)/U(I^k) = 1$; for instance see Section 4. In our proof we show that in the class of ideals of Theorem 1.3 we have that the limit on the left hand side is $< 1$. The surprising feature of our proof is the use of local techniques like equimultiplicity, reductions analyticity and local theorems like Rees multiplicity theorem (see subsection 2.2).

*Overview of the paper.* In section 2 we introduce notation and discuss a few preliminary facts that we need. In section 3 we prove Theorem 1.3. In section 4 we give an example of a class of ideals whose members satisfy $\lim_{k\to\infty} e(A/I^k)/U(I^k) = 1$.

## 2. Preliminaries

In this section we recall some notions in local algebra. We also discuss asymptotic behavior of regularity of ideals $I^k$ for $k \gg 0$. Finally we also recall that the function $k \mapsto e(A/I^k)$ is polynomial in $k$ for $k \gg 0$.

• *Some local notions:*

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and residue field $K = R/\mathfrak{m}$ which, for convenience, we assume is infinite. Let $\mathfrak{a}$ be an ideal in $R$. If $M$ is a finitely generated $R$-module, then $\mu(M)$ denotes its minimal number of generators and $\ell(M)$ denotes its length.

2.1. The *analytic spread* of $\mathfrak{a}$ is the Krull dimension of the fiber-cone $F(\mathfrak{a}) = \bigoplus_{n\geq 0} \mathfrak{a}^n/\mathfrak{m}\mathfrak{a}^n$. We denote it by $s(\mathfrak{a})$. By [19, p. 150, Th. 1], $s(\mathfrak{a}) = \mu(\mathfrak{b})$, where $\mathfrak{b}$ is a (any) minimal reduction of $\mathfrak{a}$. For the definition of reduction and minimal reduction; see [19, p. 146]. It can be shown that $\operatorname{height}(\mathfrak{a}) \leq s(\mathfrak{a})$; see [19, p. 151, L. 4]. We say $\mathfrak{a}$ is an *equimultiple* ideal if $\operatorname{height}(\mathfrak{a}) = s(\mathfrak{a})$. If $R$ is quasi-unmixed, then $\mathfrak{a}$ is equimultiple if and only if $\operatorname{gr}_{\mathfrak{a}} R = \bigoplus_{n\geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1}$, the associated graded ring of $\mathfrak{a}$, has a homogeneous system of parameters; see [8, 2.6].

2.2. If $\mathfrak{a}$ is $\mathfrak{m}$-primary, then let $e(\mathfrak{a}, R)$ = multiplicity of $R$ with respect to $\mathfrak{a}$; i.e.,

$$e(\mathfrak{a}, R) = \lim_{n\to\infty} \frac{d!}{n^d} \ell\left(\frac{R}{\mathfrak{a}^n}\right).$$

Let $\mathfrak{b} \subseteq \mathfrak{a}$ be $\mathfrak{m}$-primary. Clearly $e(\mathfrak{b}, R) \geq e(\mathfrak{a}, R)$. It is easy to see that if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $e(\mathfrak{b}, R) = e(\mathfrak{a}, R)$. A celebrated theorem due to Rees [21] shows that if $R$ is quasi-unmixed and $e(\mathfrak{b}, R) = e(\mathfrak{a}, R)$, then $\mathfrak{b}$ is a reduction of $\mathfrak{a}$.

• *Asymptotic behavior of regularity:*

Let $A = K[X_1, \dots, X_n]$. Let $I$ be a graded ideal in $A$ and let $p = \operatorname{projdim} A/I$. Then

$$\operatorname{reg}(I) = \max\{M_{i+1}(A/I) - i \mid i = 0, \dots, p-1\}$$

is the regularity of $I$. Set $\operatorname{reg}_i(I) = M_{i+1}(A/I) - i$ for $i = 0, \dots, p-1$.

2.3. In [4, 2.4] and [12, 1] it is shown that $\operatorname{reg}(I^k) = qk + r$ for $k \gg 0$. In [12, 5] it is shown that $I$ has a reduction $J$ such that $\operatorname{reg}_0(J) = q$. In particular $J$ is generated in degrees $\leq q$. We call such a reduction a *Kodiyalam reduction.*

2.4. In [11, 2.1(ii)] it is proved that for $i = 0, \dots, c-1$,

$$\operatorname{reg}_i(I^k) = qk + r_i \quad \text{for all } k \gg 0.$$

Therefore for $k \gg 0$,

$$U(I^k) = \frac{1}{c!} \prod_{i=1}^{c} M_i(A/I^k) = \frac{q^c}{c!} k^c + \cdots + \text{lower terms in } k.$$

- *The function* $k \mapsto e(A/I^k)$.

2.5. Let $\operatorname{Assh}(I) = \{\mathfrak{P} \in \operatorname{Spec}(A) \mid \mathfrak{P} \supseteq I \text{ and } \dim A/\mathfrak{P} = \dim A/I\}$. Notice that all $\mathfrak{P} \in \operatorname{Assh}(I)$ are graded ideals. The associativity formula of multiplicity ([2, 4.7.8]) then shows that

$$e(A/I) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}) e(A/\mathfrak{P}). \tag{2.1}$$

Since $\operatorname{Assh}(I^k) = \operatorname{Assh}(I)$ for all $k \geq 1$ we have that

$$e(A/I^k) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} \ell(A_{\mathfrak{P}}/I^k_{\mathfrak{P}}) e(A/\mathfrak{P}). \tag{2.2}$$

Recall that $c = \operatorname{height}(I)$. Since $k \mapsto \ell(A_{\mathfrak{P}}/I^k_{\mathfrak{P}})$ is a polynomial function of degree $c$, it follows that $k \mapsto e(A/I^k)$ is a polynomial function of degree $c$. Furthermore, if $E(I)$ is the normalized leading coefficient of this function, then

$$E(I) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) e(A/\mathfrak{P}). \tag{2.3}$$

*Remark* 2.6. Let $J \subseteq I$ be a graded ideal. If $J$ is a reduction of $I$, then $J_{\mathfrak{P}}$ is a reduction of $I_{\mathfrak{P}}$ for all primes $\mathfrak{P}$. So $E(J) = E(I)$.

2.7. By subsections 2.4 and 2.5 we get that

$$\lim_{k \mapsto \infty} \frac{e(A/I^k)}{U(I^k)} = \frac{E(I)}{q^c}.$$

Here $q$ is as in subsection 2.3.

## 3. Proof of Theorem 1.3

In this section we prove our result. We use [11, Theorem 2], where it is proved that $\lim_{k \to \infty} e(A/I^k)/U(I^k) \leq 1$. In our proof we show that in the class of ideals of Theorem 1.3 we have that the limit on the left hand side is $< 1$. Throughout this section $q$ is as in section 2.3.

3.1. In [11, section 2] the authors assume $K$ is infinite and then do the following:

- Let $J$ be a Kodiyalam reduction of $I$ and let $f_1, \ldots, f_c \in J_q$ be $c$-generic $q$-forms. Set $L = (f_1, \ldots, f_c)$.
- $e(A/L) = E(L) = q^c$.
- $E(I) \leq E(L)$.

The following observation is useful:

**Observation 3.2.** (1) In subsection 3.1, an ideal $L = (f_1, \ldots, f_c)$, where $f_1, \ldots, f_c \in J_q$ is a regular sequence, will do. In fact in [11, section 2] it is chosen generic just to ensure $f_1, \ldots, f_c$ is a regular sequence.
(2) We may choose $f_1 \in J_q$ to be any non-zero element.

To prove $E(I) < E(L)$ the following remark is useful:

*Remark* 3.3. $L$ is unmixed. Also $\operatorname{height} I = \operatorname{height} L = c$. Thus $\operatorname{Assh}(I) \subseteq \operatorname{Assh}(A/L) = \operatorname{Min}(A/L)$, the set of minimum primes of $L$. So to prove $E(I) < E(L)$, it suffices to show that there exists $\mathfrak{P} \in \operatorname{Assh}(I)$ such that $e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) < e(L_{\mathfrak{P}}, A_{\mathfrak{P}})$.

We now give

*Proof of Theorem* 1.3. We prove that for each class of ideals considered we have $E(I) < E(L) = q^c$. We also assume $K$ is infinite. This follows from the usual standard trick in the case when $K$ is finite.

*Case* 1. *$I$ is a radical ideal.*

In this case we have the following:

*Claim.* $E(I) = e(A/I)$.

Let $I = Q_1 \cap \cdots \cap Q_s$ be a minimal irredundant primary decomposition of $I$. Set $\mathfrak{P}_i = \sqrt{Q_i}$ for $i = 1, \ldots, s$.. Let $\mathfrak{P} \in \operatorname{Assh}(I)$. Then $\mathfrak{P} = \mathfrak{P}_i$ for some $i$. *As $I$ is a radical ideal* we have $I_{\mathfrak{P}} = \mathfrak{P}A_{\mathfrak{P}}$. Notice $A_{\mathfrak{P}}$ is a regular local ring of dimension $c$. So

$$e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) = e(\mathfrak{P}A_{\mathfrak{P}}, A_{\mathfrak{P}}) = 1,$$
$$\ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}) = \ell(A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}) = 1.$$

Therefore by (2.1) and (2.3) we get

$$E(I) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} e(I_{\mathfrak{P}}, A_{\mathfrak{P}}) e(A/\mathfrak{P}) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} e(A/\mathfrak{P}),$$
$$e(A/I) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} \ell(A_{\mathfrak{P}}/I_{\mathfrak{P}}) e(A/\mathfrak{P}) = \sum_{\mathfrak{P} \in \operatorname{Assh}(I)} e(A/\mathfrak{P}).$$

Thus $E(I) = e(A/I)$.

Set $\mathfrak{m} = (X_1, \ldots, X_n)$. Notice that $I_{\mathfrak{m}}$ is a radical ideal of $A_{\mathfrak{m}}$.

*Subcase* 1. *The ideal $I_{\mathfrak{m}}$ is equimultiple.* Then by a result due to Cowsik and Nori [3] we have that $I_{\mathfrak{m}}$ is generated by a regular sequence. Since $I$ is graded it follows that $I$ is also generated by a regular sequence. In this case by [9] we have that $e(A/I^k) \leq U(I^k)$ for all $k \geq 1$.

*Subcase* 2. *$I_{\mathfrak{m}}$ is not equimultiple.* Let $L$ be as in subsection 3.1. Then $L$ is not a reduction of $I$. (Otherwise $L_{\mathfrak{m}}$ will be a reduction of $I_{\mathfrak{m}}$ and this will imply that $I_{\mathfrak{m}}$ is equimultiple.)

In particular $L \neq I$. Consider the exact sequence

$$0 \longrightarrow \frac{I}{L} \longrightarrow \frac{A}{L} \longrightarrow \frac{A}{I} \longrightarrow 0.$$

Since $I \neq L$ we have that $(I/L)_{\mathfrak{P}} \neq 0$ for some $\mathfrak{P} \in \operatorname{Ass}(A/L) = \operatorname{Min}(A/L)$. In particular $\dim I/L = \dim A/L = \dim A/I$. It follows that

$$E(L) = e(A/L) = e(A/I) + e(I/L) > e(A/I) = E(I).$$

This implies the result in this case.

*Case* 2. *$I$ is a monomial ideal with generators in different degrees.*

Let $\mathfrak{P} \in \operatorname{Assh}(I)$. As $I$ is a monomial ideal, $\mathfrak{P}$ is generated by a subset of the variables [2, 4.4.15]. Say $\mathfrak{P} = (X_{i_1}, \dots, X_{i_s})$. Let $G(I) = \{u_1, \dots, u_a\}$ be the unique set of minimal monomial generators of $I$. Assume $\deg u_1 < q$. Set $\alpha = q - \deg u_1$. Set $f_1 = X_{i_1}^{\alpha} u_1$ and let $L = (f_1, f_2, \dots, f_c)$ (see Observation 3.2(2)). As $f_1 \in \mathfrak{P} I_{\mathfrak{P}}$, it follows that $L_{\mathfrak{P}}$ is not a minimal reduction of $I_{\mathfrak{P}}$ [19, Lemma 2]. Therefore by Rees's theorem $e(L_{\mathfrak{P}}, A_{\mathfrak{P}}) > e(I_{\mathfrak{P}}, A_{\mathfrak{P}})$. So by Remark 3.3 we get $E(L) > E(I)$.

*Case* 3. *$I$ is a zero-dimensional ideal with generators in different degrees.*

Notice that in this case $\operatorname{Assh}(I) = \{(X_1, \dots, X_n)\}$. The proof is similar to Case 2. □

*Remark* 3.4. For Cases 2 and 3 in our theorem note that the ideal can never have a pure resolution. Notice also that $\lim_{k\to\infty}\{U(I^k) - e(A/I^k)\} = \infty$. This gives further evidence of the "improved" multiplicity conjectures that suggest that Cohen-Macaulay ideals with pure resolutions are the only ones for which the bounds are sharp.

## 4. An example

In [11], the authors state that its easy to construct examples of ideals with $\lim_{k\to\infty} e(A/I^k)/U(I^k) = 1$. For the sake of completeness we give a large class of ideals where $\lim_{k\to\infty} e(A/I^k)/U(I^k) = 1$. The notation will be as in subsection 3. Set $\mathfrak{m} = (X_1, \dots, X_n)$.

4.1. Let $q \geq 2$ and let $I \subseteq \mathfrak{m}^q$ be a zero-dimensional ideal generated by $q$-forms. It is easily verified that $\operatorname{reg}(I^k) = qk + r$ for $k \gg 0$ (use [4, 3.2]). Let $f_1, \dots, f_n \in I$ be any regular sequence of $q$-forms. Set $L = (f_1, \dots, f_n)$. Notice that $e(L_{\mathfrak{m}}, A_{\mathfrak{m}}) = q^n = e(\mathfrak{m}^q, A_{\mathfrak{m}})$. So by a theorem of Rees (see section 2.2), $L_{\mathfrak{m}}$ is a reduction of $\mathfrak{m}^q A_{\mathfrak{m}}$. It follows that $L$ is a reduction of $\mathfrak{m}^q$. Therefore $L$ is also a reduction of $I$. By Remark 2.6 we get that $E(I) = E(L) = q^c$. So by section 2.7 we get $\lim_{k\to\infty} e(A/I^k)/U(I^k) = 1$.

*Remark* 4.2. We do not know as yet whether the upper bound of the multiplicity conjecture holds asymptotically for all ideals in the class described in subsection 4.1.

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## References


1. M. Boij and J. Söderberg, *Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture*. Preprint, 2006.
2. W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Revised edition, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge (1998). MR1251956 (95h:13020)
3. R.C. Cowsik and M.V. Nori, *On the fibres of blowing up*. J. Indian Math. Soc. **40**, no. 1-4, 217–222 (1976). MR0572990 (58:28011)
4. S.D. Cutkosky, J. Herzog and N.V. Trung, Asymptotic behavior of the Castelnuovo-Mumford regularity, Compositio Math. **118**, no. 3, 243–261 (1999). MR1711319 (2000f:13037)
5. C.A. Francisco, *New approaches to bounding the multiplicity of an ideal*. J. Algebra **299**, no. 1, 309–328 (2006). MR2225778 (2007b:13043)
6. L.H. Gold, *A degree bound for codimension two lattice ideals*. J. Pure Appl. Algebra **182**, no. 2–3, 201–207 (2003). MR1903053 (2004i:13023)
7. L. Gold, H. Schenck and H. Srinivasan, *Betti numbers and degree bounds for some linked zero-schemes*. J. Pure Appl. Algebra **210**, 481–491 (2007). MR2320011
8. U. Grothe, M. Herrmann and U. Orbanz, *Graded Cohen-Macaulay rings associated to equimultiple ideals*. Math. Z. **186**, no. 4, 531–556 (1984). MR744964 (86c:13019)
9. E. Guardo and A. Van Tuyl, *Powers of complete intersections: Graded Betti numbers and applications*. Ill. J. Math. **49**, no. 1, 265–279 (2005). MR2157379 (2006k:13035)
10. J. Herzog and H. Srinivasan, *Bounds for multiplicities*. Trans. Am. Math. Soc. **350**, no. 7, 2879–2902 (1998). MR1458304 (99g:13033)
11. J. Herzog and X. Zheng, *Notes on the multiplicity conjecture*. Collect. Math. **57**, no. 2, 211–226 (2006). MR2223853 (2007a:13029)
12. V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity, Proc. Amer. Math. Soc. **128**, no. 2, 407–411 (2000). MR1621961 (2000c:13027)
13. H. Matsumura, *Comutative ring theory*. Cambridge Studies in Advanced Math. **8**, Cambridge University Press, Cambridge (1986). MR879273 (88h:13001)
14. J. Migliore, U. Nagel and T. Römer, *The multiplicity conjecture in low codimensions*. Math. Res. Lett. **12**, no. 5–6, 731–747 (2005). MR2189234 (2006i:13042)
15. J. Migliore, U. Nagel and T. Römer, *Extensions of the multiplicity conjecture*. Trans. Am. Math. Soc. **360**, 2965–2985 (2008). MR2379783
16. J. Migliore, U. Nagel and F. Zanello, *An improved multiplicity conjecture for codimension three Gorenstein algebras*. Preprint, 2006.
17. R.M. Miró-Roig, *A note on the multiplicity of determinantal ideals*. J. Algebra **299**, no. 2, 714–724 (2006). MR2228336 (2007a:13030)
18. R.M. Miró-Roig, *Betti numbers of determinantal ideals*. Preprint, 2007.
19. D.G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge. Philos. Soc. **50**, 145–158 (1954) MR0059889 (15:596a)
20. I. Novik and E. Swartz, *Face ring multiplicity via CM-connectivity sequences*. Preprint, 2006.
21. D. Rees, *$\mathfrak{a}$-transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. (2) **57**, 8-17 (1961). MR0118750 (22:9521)
22. T. Römer, *Note on bounds for multiplicities*. J. Pure Appl. Algebra **195**, no. 1, 113–123 (2005). MR2100313 (2005g:13041)
23. T. Römer, *Betti numbers and shifts in minimal graded free resolutions*. Preprint, 2007.
24. H. Srinivasan, *A note on the multiplicities of Gorenstein algebras*. J. Algebra **208**, no. 2, 425–443 (1998). MR1655460 (99m:13046)
25. F. Zanello, *Improving the bounds in the multiplicity conjecture: the codimension 3 level case*. J. Pure Appl. Algebra **209**, no. 1, 79–89 (2007). MR2292118 (2008a:13034)


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