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THE MOMENT AND ALMOST SURELY EXPONENTIAL STABILITY OF STOCHASTIC HEAT EQUATIONS

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Abstract. In this article, the p-th moment and almost surely exponential stability of the strong solution to a stochastic heat equation driven by an m-dimensional Brownian motion is investigated by a simple method. In particular, the sharp top Lyapunov exponents are explicitly calculated based on the representation of the strong solution.

1. Introduction

In recent years, the asymptotic behavior of the solutions to stochastic partial differential equations in separable Hilbert spaces, especially, stochastic heat equations, had been extensively studied by many authors because of its importance in applications; see [1, 2, 5, 7, 8, 9, 10, 11] and the references therein. In particular, in 2001, Caraballo et al. [2] observed that a white noise can be used to stabilize an unstable stochastic partial differential equation by using the classical and powerful method of the Lyapunov functional. Their observation is very interesting in probability theory, and it indicates that a multiplicative noise is extremely effective to stabilize an unstable stochastic system.

Motivated by their work, in this article, we principally consider the following stochastic heat equation disturbed by a Brownian motion with the homogeneous Dirichlet boundary:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \beta_0 u(t,x) + \beta_1 u(t,x)\dot{w}_1(t) + \beta_2 u(t,x)\dot{w}_2(t), \ x \in \mathcal{O} \tag{1.1}$$

with an initial value u^0 (deterministic or random), where $\mathcal{O} \in \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\mathcal{O}$ and $w(t) = (w_1(t), w_2(t))$ is a standard two-dimensional Brownian motion and $\beta_i, i = 0, 1, 2$, are arbitrary real numbers. We are especially interested in the top Lyapunov exponents of the unique strong solution, which express the sufficient and necessary conditions for the exponentially asymptotic stabilities and imply the stabilization of an unstable stochastic system by noises. Since the construction of Lyapunov functionals is, in general, very difficult, we attempt to find a simple way to investigate the exponential stability of stochastic dynamics. Roughly speaking, unlike the celebrated Lyapunov method,

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the Lyapunov exponent is calculated based on the explicit representation of the stochastic dynamics.

The main results will be described in the next section and a generalization will be formulated in the last section. Our conclusions greatly improve the existing work on stochastic heat equations of Caraballo et al. [2] and Ichikawa [6].

2. Main results and proofs

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and let $w(t) = (w_1(t), w_2(t))$ be a two-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Let $L^2(\mathcal{O})$ be the family of all the square integrable functions on $\mathcal{O}$ with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ and let $H_0^1(\mathcal{O})$ and $H^2(\mathcal{O})$ be the usual Sobolev spaces. Further, let $A = \Delta$ with the homogeneous Dirichlet boundary condition on $\partial\mathcal{O}$ with the domain $\mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. Assume $\{\lambda_n\}_{n\geq 1}$ and $\{e_n\}_{n\geq 1}$ are the eigenvalues and eigenfunctions for $-A$ such that $\{e_n\}_{n\geq 1}$ forms a canonical orthonormal basis of $L^2(\mathcal{O})$. Then, it is clear that λ_n are strictly positive and increasing and $e_n \in \mathcal{D}(A)$. Finally, we assume that if u^0 is random, then it is measurable with respect to $\mathcal{F}_0$, and if it is nonrandom, then $|u^0| \neq 0$ throughout this paper. Then u^0 has the unique representation $u^0(x) = \sum_{n=1}^{\infty} u_n^0 e_n(x)$, where $u_n^0 = \langle u^0, e_n \rangle$.

In the following, we will deal with the stability of the stochastic dynamic system determined by (1.1). More precisely, the stability relative to the following stochastic heat equation will be considered in the following:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \beta_0 u(t,x) + \beta_1 u(t,x)\dot{w}_1(t) + \beta_2 u(t,x)\dot{w}_2(t), & x \in \mathcal{O}, \\ u(t,x) = 0, & x \in \partial\mathcal{O}, \\ u(0,x) = u^0(x) \in \mathcal{D}(A), & x \in \partial\mathcal{O}. \end{cases} \tag{2.1}$$

Then Proposition 6.29 in [3] claims that there exists a unique strong solution $u(t,x)$ to (2.1). Here for the strong solution, we mean that there exists a predictable process $u(t) \in \mathcal{D}(A)$ such that $u \in C([0,\infty); L^2(\mathcal{O}))$ and the stochastic integral equation below holds:

$$\begin{aligned} u(t,x) &= u^0(x) + \int_0^t (\Delta u(s,x) + \beta_0 u(s,x))\, ds \\ &\quad + \int_0^t \beta_1 u(s,x) dw_1(s) + \int_0^t \beta_2 u(s,x) dw_2(s), \text{ a.e. } x \in \mathcal{O}. \end{aligned} \tag{2.2}$$

In this article, the (asymptotic) stabilities both in the p-th moment and in the almost sure sense as below will be discussed. In addition, without essential change, our main results in this paper can be generalized to a wide class of strongly elliptic operators A with compact resolvent; for example, see [3, 4]. To illustrate the methods, we will only investigate the simple operator A defined in the above.

Definition 2.1. We say the solution $u(t)$ to (2.1) is almost surely and p-th moment, respectively, exponentially stable if we have

$$\alpha(u^0) := \limsup_{t\to\infty} \log \frac{|u(t)|}{t} < 0, \text{ a.s.}$$

and

$$\gamma(u^0) := \limsup_{t\to\infty} \log \frac{\mathbb{E}[|u(t)|^p]}{t} < 0,$$

respectively.

For simplicity, we will call $\alpha(u^0)$ and $\gamma(u^0)$ in the definition above the Lyapunov exponents in the almost surely sense and in the p-th moment respectively. It is clear that $\alpha(u^0)$ is generally random. For more knowledge about the stability, we refer the readers to the monograph [8].

To formulate the main results in this article, we will first discuss the stability of the following heat equation:

$$\frac{\partial v}{\partial t}(t,x) = \Delta v(t,x) + \beta_0 v(t,x),\ x \in \mathcal{O}, \tag{2.3}$$

with the Dirichlet condition $v(t,x) = 0,\ x \in \partial\mathcal{O}$ and the initial value $u^0 \in \mathcal{D}(A)$ (may be random). It is well known that there exists a unique strong solution to (2.3) in a similar sense as in the above, which will be denoted by $v(t,x)$ in the sequel. The p-th moment and the almost surely exponential stabilities of $v(t)$ can be defined similarly as in Definition 2.1.

Define

$$n_0 := \inf\{n;\ u_n^0 \neq 0\}.$$

For simplicity of description, we state the following hypothesis:

H: We assume that the p-th moment of $|u^0|$ exists for some $p > 0$ and that it is strictly positive, i.e.,

$$0 < \mathbb{E}[|u^0|^p] < \infty.$$

Lemma 2.1. *For the strong solution $v(t,x)$ to* (2.3)*, the following holds.*
(1) *Assume that u^0 is not random. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log|v(t)| = \beta_0 - \lambda_{n_0}.$$

In particular, for any given u^0, $v(t)$ is exponentially stable in the almost sure sense if and only if $\lambda_{n_0} > \beta_0$.
(2) *Assume that u^0 is random. Then under* ***H****, we have*

$$\limsup_{t\to\infty} \frac{1}{t}\log|v(t)| \le \beta_0 - \lambda_1 \ \text{ and } \ \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|v(t)|^p] \le (\beta_0 - \lambda_1)p. \tag{2.4}$$

In particular, for any $p > 0$ and initial value u^0, the solution $v(t)$ is exponentially stable in p-th moment if $\lambda_1 > \beta_0$.

Proof. The first claim is proved in [7] for $d \le 3$, and similar arguments are feasible for the general case. Here we only prove (2). From the uniqueness of the solution, it follows that

$$v(t,x) = \sum_{n=1}^{\infty} \exp\{(-\lambda_n + \beta_0)t\} u_n^0 e_n(x).$$

Therefore, we have

$$\begin{aligned}\mathbb{E}[|v(t)|^p] &= \mathbb{E}\left[\left(\sum_{n=1}^{\infty}\left(\exp\{(-\lambda_n+\beta_0)t\}u_n^0\right)^2\right)^{p/2}\right]\\ &\le \exp\{(-\lambda_1+\beta_0)pt\}\mathbb{E}\left[\left(\sum_{n=1}^{\infty}(u_n^0)^2\right)^{p/2}\right]\\ &= \exp\{(-\lambda_1+\beta_0)pt\}\mathbb{E}[|u^0|^p],\end{aligned}$$

which completes the proof of the second inequality in (2.4). Additionally, the first inequality in (2.4) can be shown in a similar way. □

Remark 2.1. For any initial value u^0 and each p, $\lambda_1 > \beta_0$ is sufficient for both p-th moment and almost surely exponential stability. This is completely different from the disturbed system (2.1) as we will see below.

Lemma 2.2. *The unique strong solution $u(t,x)$ can be written as a Fourier series:*

$$u(t,x) = \sum_{n=1}^{\infty} z_n(t)e_n(x), \tag{2.5}$$

where $z_n(t)$ is the unique strong solution of the stochastic Ito equation with initial value u_n^0:

$$dz_n(t) = (-\lambda_n+\beta_0)z_n(t)dt + \beta_1 z_n(t)dw_1(t) + \beta_2 z_n(t)dw_2(t). \tag{2.6}$$

Proof. Denote $\sum_{n=1}^{\infty} z_n(t)e_n(x)$ by $\bar{u}(t,x)$ for convenience. We first note that $\bar{u}(t,x)$ is a strong solution to the stochastic heat equation (2.1). Indeed, it is obvious that $\sum_{n=1}^{\infty} z_n(t)e_n \in \mathcal{D}(A)$ and

$$\Delta\bar{u}(t,x) = -\sum_{n=1}^{\infty}\lambda_n z_n(t)e_n(x). \tag{2.7}$$

On the other hand, using integration by parts and the stochastic Fubini theorem, we deduce that

$$\begin{aligned}\bar{u}(t,x) &= \sum_{n=1}^{\infty}\Big\{u_n^0 e_n(x) + \int_0^t(-\lambda_n+\beta_0)z_n(s)e_n(x)ds\\ &\quad + \beta_1\int_0^t z_n(s)e_n(x)dw_1(s) + \beta_2\int_0^t z_n(s)e_n(x)dw_2(s)\Big\}\\ &= \sum_{n=1}^{\infty}u_n^0 e_n(x) + \int_0^t\sum_{n=1}^{\infty}(-\lambda_n+\beta_0)z_n(s)e_n(x)ds\\ &\quad +\beta_1\int_0^t\sum_{n=1}^{\infty}z_n(s)e_n(x)dw_1(s) + \beta_2\int_0^t\sum_{n=1}^{\infty}z_n(s)e_n(x)dw_2(s).\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8) we see that $\bar{u}(t,x)$ satisfies the integral equation (2.2). As a consequence, the conclusion follows from the uniqueness of the strong solution to (2.1). □

Lemma 2.3. *For each n, the unique solution $z_n(t)$ to the stochastic Ito equation* (2.6) *equals*

$$u_n^0 \exp\left\{\left(-\lambda_n + \beta_0 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2\right)t + \beta_1 w_1(t) + \beta_2 w_2(t)\right\}.$$

Proof. This follows easily from the Ito formula. □

We first study the exponential stability in the p-th moment sense. As in Lemma 2.1, the results depend on the randomness of u_0. We are mainly interested in calculating the sharp top Lyapunov exponent.

Theorem 2.4. *Assume $u(t,x)$ is the unique strong solution to* (2.1). *Then we have*
(1) *If u_0 is deterministic, then*

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}[|u(t)|^p] = (\beta_0 - \lambda_{n_0})p + \frac{1}{2}(\beta_1^2+\beta_2^2)(p^2-p).$$

(2) *Suppose u_0 is random and independent of $w(t)$. If, in addition,* ***H*** *holds, then*

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}[|u(t)|^p] \le (\beta_0 - \lambda_1)p + \frac{1}{2}(\beta_1^2+\beta_2^2)(p^2-p).$$

Proof. From Lemma 2.2 and Lemma 2.3, it follows that

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} u_n^0 \exp\left\{\left(-\lambda_n + \beta_0 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2\right)t + \beta_1 w_1(t) + \beta_2 w_2(t)\right\} e_n(x) \\ &= \sum_{n=1}^{\infty} u_n^0 e_n(x) \exp\{(\beta_0-\lambda_n)t\} \exp\left\{-\frac{1}{2}(\beta_1^2+\beta_2^2)t + \beta_1 w_1(t) + \beta_2 w_2(t)\right\} \\ &= v(t,x)\exp\left\{\beta_1 w_1(t) - \frac{1}{2}\beta_1^2 t + \beta_2 w_2(t) - \frac{1}{2}\beta_2^2 t\right\}. \end{aligned}$$

Hence, if u_0 is not random, we see that

$$\begin{aligned} \mathbb{E}[|u(t)|^p] &= |v(t)|^p \mathbb{E}\left[\exp\left\{\beta_1 p w_1(t) - \frac{1}{2}\beta_1^2 pt + \beta_2 p w_2(t) - \frac{1}{2}\beta_2^2 pt\right\}\right] \\ &= |v(t)|^p \mathbb{E}[\exp\{\beta_1 p w_1(t)\}]\mathbb{E}[\exp\{\beta_2 p w_2(t)\}]\exp\left\{-\frac{1}{2}(\beta_1^2+\beta_2^2)pt\right\} \\ &= |v(t)|^p \exp\left\{\frac{t}{2}(\beta_1^2+\beta_2^2)(p^2-p)\right\}, \end{aligned} \tag{2.9}$$

where the independency of $w_1(t)$ and $w_2(t)$ and the exponential martingale property with respect to Brownian motion have been used for the second and third equations respectively. Then, combining (1) in Lemma 2.1 and (2.9), we obtain the first assertion.

If u_0 is random, then so is n_0. Therefore, we cannot obtain the sharp top Lyapunov exponent as in (1). However, by (2) in Lemma 2.1, we deduce easily that

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}[|u(t)|^p] &= \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}[|v(t)|^p] + \frac{1}{2}(\beta_1^2+\beta_2^2)(p^2-p) \\ &\le (\beta_0-\lambda_1)p + \frac{1}{2}(\beta_1^2+\beta_2^2)(p^2-p). \end{aligned}$$

Then the proof of the theorem is completed. □

From the above theorem, it is obvious that the following corollary holds, which expresses the sufficient and necessary conditions for moment exponential stability.

Corollary 2.5. *For any initial value $u^0 \in \mathcal{D}(A)$, the solution $u(t)$ is p-th moment exponentially stable for any p satisfying*

$$p < 1 + \frac{2(\lambda_1 - \beta_0)}{(\beta_1^2 + \beta_2^2)}. \tag{2.10}$$

In particular, if u^0 is nonrandom, the p-th moment exponential stability is satisfied if and only if $\lambda_{n_0} > \beta_0 + \frac{1}{2}(\beta_1^2 + \beta_2^2)(p-1)$.

Remark 2.2. (1) Here we restrict ourselves to the one-dimensional case. Plainly, we suppose $\mathcal{O} = (0,1)$. If $\beta_1 = 1$ and $\beta_0 = \beta_2 = 0$, then it was shown that the solution is p-th moment exponentially stable up to $p = 19$; see Example 1 in [6]. However, by Corollary 2.5, we see that it is p-th moment exponentially stable for all $p < 2\pi^2 + 1$ $(= 20.739\cdots)$.

(2) Assume that $\beta_1 \neq 0$. The relation (2.10) indicates that the solution will become unstable in p-th moment if $|\beta_2|$ is large enough for $p > 1$. This is exactly opposite to the almost surely stability, as we will see below.

(3) It is worth mentioning that for any $\beta_0 \in \mathbb{R}$, the unstable deterministic system can be stabilized in p-th moment if either $|\beta_1|$ or $|\beta_2|$ is large enough for all small enough $p < 1$. In addition, if $p = 1$, the moment stability is independent of the disturbance term $w(t)$; see Lemma 2.1.

In the following theorem, we consider the almost surely exponential stability (i.e., pathwise stability) for $u(t)$.

Theorem 2.6. (1) *Assume u^0 is deterministic. Then the top Lyapunov exponent of $u(t)$ equals $\beta_0 - \lambda_{n_0} - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2$. More precisely, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log |u(t)| = \beta_0 - \lambda_{n_0} - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2.$$

In particular, the solution is almost surely exponentially stable if and only if

$$\lambda_{n_0} > \beta_0 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2.$$

(2) *Assume u^0 is a random variable independent of $w(t)$. Then the following holds:*

$$\limsup_{t\to\infty} \frac{1}{t} \log |u(t)| \leq \beta_0 - \lambda_1 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\beta_2^2.$$

Proof. Since the proof is similar to that of Theorem 2.4, we will only state the outline of the proof here. By the Fourier expansion 2.4 of $u(t,x)$, we have

$$|u(t)| = |v(t)| \exp\left\{\beta_1 w_1(t) - \frac{1}{2}\beta_1^2 t + \beta_2 w_2(t) - \frac{1}{2}\beta_2^2 t\right\}.$$

Then, by the law of large numbers relative to Brownian motion, we see that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log |u(t)| &= \lim_{t\to\infty} \frac{1}{t} \log |v(t)| + \lim_{t\to\infty} \left(\frac{w_1(t)}{t} + \frac{w_2(t)}{t}\right) - \frac{1}{2}(\beta_1^2 + \beta_2^2) \\ &= \beta_0 - \lambda_{n_0} - \frac{1}{2}(\beta_1^2 + \beta_2^2). \qquad (2.11)\end{aligned}$$

Therefore, (1) is obtained. On the other hand, by similar arguments as the above and (2) in Lemma 2.1, one can easily prove the second claim. Thus the proof is completed. □

Remark 2.3. Assuming $\mathcal{O} = (0, 1)$ and $\beta_1 = 0$, Caraballo et al. [2] found that if $\beta_2^2 > 2(\beta_0 - \pi^2)$, then the stochastic system becomes exponentially stable (see Subsection 3.1 [2]). But by virtue of the above theorem, we know that $\beta_2^2 > 2(\beta_0 - \pi^2)$ is also necessary for the stability if $n_0 = 1$.

3. Generalization

In this section, we are interested in the stochastic heat equation driven by an m-dimensional Brownian motion $w(t) = (w_1(t), \dots, w_m(t))$:

$$(3.1)\qquad \begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \beta_0 u(t,x) + \sum_{i=1}^m \beta_i u(t,x)\dot{w}_i(t), & x \in \mathcal{O}, t > 0, \\ u(t,x) = 0, & x \in \partial\mathcal{O}, t > 0, \\ u(0,x) = u^0(x) \in \mathcal{D}(A), & x \in \partial\mathcal{O}, \end{cases}$$

where $\beta_i, i = 0, 1, \dots, m$ are arbitrary real numbers and $m \geq 3$. Then the unique strong solution $u(t,x)$ is also assured by Proposition 6.29 in [3]. Then with the obvious modification of the proofs in Section 2, we can extend the main theorems in the last section to the stochastic dynamics determined by (3.1). We will only state the theorems and omit the detailed proofs. In addition, we will assume u^0 is deterministic in the sequel for simplicity.

Theorem 3.1. *Suppose that $u(t,x)$ is the unique strong solution to* (3.1)*. Then we have*

- *The top Lyapunov exponent of $u(t)$ in p-th moment equals*
$$(\beta_0 - \lambda_{n_0})p + \frac{1}{2}(p^2 - p)\sum_{i=1}^m \beta_i^2.$$
In particular, the sufficient and necessary condition of the p-th moment exponential stability of $u(t)$ is $\lambda_{n_0} > \beta_0 + \frac{1}{2}\sum_{i=1}^m \beta_i^2(p-1)$.
- *The almost surely exponential stability holds if and only if $\beta_i, i = 0, 1, \dots, m$ satisfy the relation*
$$\lambda_{n_0} > \beta_0 - \frac{1}{2}\sum_{i=1}^m \beta_i^2.$$
More precisely, we have $\lim_{t\to\infty} \frac{1}{t}\log|u(t)| = -\lambda_{n_0} + \beta_0 - \frac{1}{2}\sum_{i=1}^m \beta_i^2$.

We will conclude this article with the following remark.

Remark 3.1. (1) As we have seen, if $m = 2$, then the almost surely exponential stability holds only for $\beta_1^2 + \beta_2^2 > 2(\beta_0 - \lambda_{n_0})$. In other words, the system will be unstable if $\beta_1^2 + \beta_2^2 \leq 2(\beta_0 - \lambda_{n_0})$. However, the above theorem shows that any unstable stochastic system can be stabilized by additional multiplicative noise $\beta_3 u(t)\dot{w}(t)$ by choosing large enough β_3. Generally, Theorem 3.1 implies that additional noise can be used to stabilize any almost surely unstable system, while it does not hold for moment stability. Precisely, p is decreasing as m increases for any β_i.

(2) For a random initial value u^0, the following relations are satisfied:

$$\alpha(u^0) \leq \beta_0 - \lambda_1 - \frac{1}{2}\sum_{i=1}^m \beta_i^2$$

and

$$\gamma(u^0) \le (\beta_0 - \lambda_1)p + \frac{1}{2}(p^2 - p)\sum_{i=1}^{m}\beta_i^2.$$

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