

SEMILATTICE STRUCTURES OF SPREADING MODELS

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ABSTRACT. Given a Banach space X , denote by $SP_w(X)$ the set of equivalence classes of spreading models of X generated by normalized weakly null sequences in X . It is known that $SP_w(X)$ is a semilattice, i.e., it is a partially ordered set in which every pair of elements has a least upper bound. We show that every countable semilattice that does not contain an infinite increasing sequence is order isomorphic to $SP_w(X)$ for some separable Banach space X .

Given a normalized basic sequence (y_i) in a Banach space and $\varepsilon_n \searrow 0$, using Ramsey's Theorem, one can find a subsequence (x_i) and a normalized basic sequence (\tilde{x}_i) such that for all $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subseteq [-1, 1]$,

$$\left| \left\| \sum a_i x_{k_i} \right\| - \left\| \sum a_i \tilde{x}_i \right\| \right| < \varepsilon_n$$

for all $n \leq k_1 < \dots < k_n$. The sequence (\tilde{x}_i) is called a spreading model of (x_i) . It is well-known that if (x_i) is in addition weakly null, then (\tilde{x}_i) is 1-spreading and suppression 1-unconditional. See [3], [5] for more about spreading models. A spreading model (\tilde{x}_i) is said to (C) -dominate another spreading model (\tilde{y}_i) if there is a $C < \infty$ such that for all $(a_i) \subseteq \mathbb{R}$,

$$\left\| \sum a_i \tilde{y}_i \right\| \leq C \left\| \sum a_i \tilde{x}_i \right\|.$$

The spreading models (\tilde{x}_i) and (\tilde{y}_i) are said to be *equivalent* if they dominate each other. Let $[(\tilde{x}_i)]$ denote the class of all spreading models which are equivalent to (\tilde{x}_i) . Let $SP_w(X)$ denote the set of all $[(\tilde{x}_i)]$ generated by normalized weakly null sequences in X . If $[(\tilde{x}_i)], [(\tilde{y}_i)] \in SP_w(X)$, we write $[(\tilde{x}_i)] \leq [(\tilde{y}_i)]$ if (\tilde{y}_i) dominates (\tilde{x}_i) . Hence, $(SP_w(X), \leq)$ is a partially ordered set. The paper [2] initiated the study of the order structures of $SP_w(X)$. It was established that every countable subset of $(SP_w(X), \leq)$ admits an upper bound ([2, Proposition 3.2]). Moreover, from the proof of this result, it follows that every pair of elements in $(SP_w(X), \leq)$ has a least upper bound. In other words, $(SP_w(X), \leq)$ is a *semilattice*. In [6], it was shown that if $SP_w(X)$ is countable, then it cannot admit a strictly increasing infinite sequence $(\tilde{x}_i^1) < (\tilde{x}_i^2) < \dots$. In [4], two methods of construction, utilizing Lorentz sequence spaces and Orlicz sequence spaces, respectively, were used to produce Banach spaces X so that $SP_w(X)$ has certain prescribed order structures. In the present paper, building on the techniques employed in [4, §2], we show that every

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countable semilattice that has no infinite increasing sequence is order isomorphic to $SP_w(X)$ for some Banach space X . This gives an affirmative answer to Problem 1.15 in [4]. (See, however, the remark at the end of the paper.)

1. A REPRESENTATION THEOREM FOR SEMILATTICES

Any collection of subsets of a set V that is closed under the taking of finite unions is a semilattice under the order of set inclusion. In this section, we show that any countable semilattice that does not admit an infinite increasing sequence may be represented in such a way using a countable set V . The result may be of independent interest.

Theorem 1. *Let L be a countable semilattice with no infinite increasing sequences. Then there exist a countable set V and an injective map $T : L \rightarrow 2^V \setminus \{\emptyset\}$ that preserves the semilattice structure of L , i.e., $T(x \vee y) = T(x) \cup T(y)$ for all $x, y \in L$.*

Suppose that L is a semilattice that satisfies the hypothesis of Theorem 1. Note that every nonempty subset of L has at least one maximal element; otherwise, it will admit an infinite increasing sequence. Set $L_0 = L$. If L_α is defined for some countable ordinal α and $L_\alpha \neq \emptyset$, let $L_{\alpha+1} = L_\alpha \setminus \{\text{maximal elements in } L_\alpha\}$. If α is a countable limit ordinal such that $L_{\alpha'} \neq \emptyset$ for all $\alpha' < \alpha$, let $L_\alpha = \bigcap_{\alpha' < \alpha} L_{\alpha'}$. Since (L_α) is a strictly decreasing transfinite sequence of subsets of the countable set L , $L_\alpha = \emptyset$ for some countable ordinal α . Let α_0 be the smallest ordinal such that $L_{\alpha_0} = \emptyset$. Enumerate L as a transfinite sequence $(e_\beta)_{\beta < \beta_0}$ so that if $e_{\beta_1} \in L_{\alpha_1} \setminus L_{\alpha_1+1}$ and $e_{\beta_2} \in L_{\alpha_2} \setminus L_{\alpha_2+1}$ for some $\alpha_1 < \alpha_2 < \alpha_0$, then $\beta_1 < \beta_2$. If $1 \leq \beta \leq \beta_0$, let $U_\beta = \{e_{\beta'} : \beta' < \beta\}$. Note that $L = U_{\beta_0}$.

Lemma 2. (a) e_β is a minimal element in $U_{\beta+1}$.

(b) If $e_\beta = e_{\beta_1} \vee e_{\beta_2}$ (least upper bound taken in L), then $\beta \leq \min\{\beta_1, \beta_2\}$.

(c) If $e_{\beta_1}, e_{\beta_2} \in U_\beta$, then $e_{\beta_1} \vee e_{\beta_2}$ belongs to U_β .

Proof. (a) Suppose on the contrary that e_β is not a minimal element in $U_{\beta+1}$. Then there exists $e_{\beta'} \in U_{\beta+1}$ with $e_{\beta'} < e_\beta$. It follows from the definition of $U_{\beta+1}$ that $\beta' < \beta$. If $e_\beta \in L_\alpha \setminus L_{\alpha+1}$ and $e_{\beta'} \in L_{\alpha'} \setminus L_{\alpha'+1}$, then $\alpha' \leq \alpha$ and hence $L_\alpha \subseteq L_{\alpha'}$. Since $e_\beta, e_{\beta'} \in L_{\alpha'}$ and $e_{\beta'} < e_\beta$, $e_{\beta'}$ is not maximal in $L_{\alpha'}$. Thus $e_{\beta'} \in L_{\alpha'+1}$, a contradiction.

(b) Suppose that $\beta_1 < \beta$. Then $e_{\beta_1} \in U_{\beta+1}$ and $e_{\beta_1} < e_\beta$, contrary to the minimality of e_β in $U_{\beta+1}$. Thus $\beta \leq \beta_1$. Similarly, $\beta \leq \beta_2$.

(c) Follows immediately from (b). \square

If $1 \leq \beta < \omega_1$, write $\beta = \gamma + n$, where γ is a limit ordinal or 0, $n < \omega$, and let V_β denote the ordinal interval $[0, \gamma + 2n)$. We define a family of maps $T_\beta : U_\beta \rightarrow 2^{V_\beta} \setminus \{\emptyset\}$, $1 \leq \beta \leq \beta_0$, inductively so that $T = T_{\beta_0}$ is the map sought for in Theorem 1. Let $T_1 : U_1 = \{e_0\} \rightarrow 2^{V_1} \setminus \{\emptyset\}$ be defined by $T_1(e_0) = \{0, 1\}$. If T_β has been defined, $1 \leq \beta < \beta_0$, let

$$T_{\beta+1}(x) = \begin{cases} T_\beta(x) \cup \{\gamma + 2n, \gamma + 2n + 1\} & \text{if } x \in U_{\beta+1} \setminus \{e_\beta\}, \\ \bigcap_{e_\beta < z \in U_\beta} T_\beta(z) \cup \{\gamma + 2n + 1\} & \text{if } x = e_\beta. \end{cases}$$

When $\beta \leq \beta_0$ is a limit ordinal and $e_{\beta'} \in U_\beta$, let $T_\beta(e_{\beta'}) = \bigcup_{\beta' < \xi < \beta} T_\xi(e_{\beta'})$. The next result, which shows the compatibility of the definitions of T_β for different β 's, is the key to the subsequent arguments.

Lemma 3. *If $1 \leq \beta_1 < \beta_2 \leq \beta_0$ and $\beta_i = \gamma_i + n_i, i = 1, 2$, then*

$$T_{\beta_2}(e_{\beta_1}) = T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2).$$

Proof. If $\beta_2 = \beta_1 + 1$, the assertion clearly holds. Suppose that the assertion holds for some $\beta_2 > \beta_1$. By the definition of T_{β_2+1} ,

$$\begin{aligned} T_{\beta_2+1}(e_{\beta_1}) &= T_{\beta_2}(e_{\beta_1}) \cup \{\gamma_2 + 2n_2, \gamma_2 + 2n_2 + 1\} \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2) \\ &\quad \cup \{\gamma_2 + 2n_2, \gamma_2 + 2n_2 + 1\} \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2 + 2). \end{aligned}$$

Suppose that $\beta_2 \leq \beta_0$ is a limit ordinal and the assertion holds for all $\beta_1 < \xi < \beta_2$. For such ξ , let $\xi = \gamma_\xi + n_\xi$. By the inductive hypothesis,

$$T_\xi(e_{\beta_1}) = T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_\xi + 2n_\xi).$$

Since β_2 is a limit ordinal, we have

$$\begin{aligned} T_{\beta_2}(e_{\beta_1}) &= \bigcup_{\beta_1 < \xi < \beta_2} T_\xi(e_{\beta_1}) \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \beta_2) \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2), \end{aligned}$$

as required. (Note that $n_2 = 0$ since β_2 is a limit ordinal.) □

Lemma 4. *The map $T_\beta : U_\beta \rightarrow 2^{V_\beta} \setminus \{\emptyset\}$ is injective if $1 \leq \beta \leq \beta_0$.*

Proof. Suppose that e_{β_1} and e_{β_2} are distinct elements in U_β , with $\beta_1 < \beta_2 < \beta$. Write $\beta_2 = \gamma_2 + n_2$. It follows from Lemma 3 that $\gamma_2 + 2n_2 \in T_\beta(e_{\beta_1}) \setminus T_\beta(e_{\beta_2})$. □

Proposition 5. *If $1 \leq \beta \leq \beta_0$, then $T_\beta(x \vee y) = T_\beta(x) \cup T_\beta(y)$ for all $x, y \in U_\beta$. In particular, $T_\beta(x) \subseteq T_\beta(y)$ if $x \leq y$.*

Proof. The second statement follows easily from the first. We prove the first statement by induction on β . The result is clear if $\beta = 1$. Suppose that the assertion is true for some $\beta, 1 \leq \beta < \beta_0$. Let $x = e_{\beta_1}, y = e_{\beta_2} \in U_{\beta+1}$. We may assume that $\beta_1 < \beta_2 < \beta + 1$. Write $\beta = \gamma + n$, and $\beta_i = \gamma_i + n_i, i = 1, 2$, and consider two cases.

Case 1. $\beta_1 < \beta_2 < \beta$.

By Lemma 3 and the inductive hypothesis,

$$\begin{aligned} T_{\beta+1}(e_{\beta_1}) \cup T_{\beta+1}(e_{\beta_2}) &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma + 2n) \\ &\quad \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \gamma + 2n) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\ &= T_\beta(e_{\beta_1}) \cup T_\beta(e_{\beta_2}) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\ &= T_\beta(e_{\beta_1} \vee e_{\beta_2}) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\ &= T_{\beta+1}(e_{\beta_1} \vee e_{\beta_2}), \end{aligned}$$

by definition of $T_{\beta+1}$, since $e_{\beta_1} \vee e_{\beta_2} \neq e_\beta$ by part (b) of Lemma 2.

Case 2. $\beta_1 < \beta_2 = \beta$.

In this case,

$$T_{\beta+1}(x) \cup T_{\beta+1}(y) = \bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] \cup \{\gamma + 2n, \gamma + 2n + 1\}.$$

Note that by part (b) of Lemma 2, $x \vee e_\beta = e_\xi$ for some $\xi \leq \beta_1$. Hence, $x \vee e_\beta \in U_{\beta+1} \setminus \{e_\beta\} = U_\beta$. Thus, it suffices to show that

$$\bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] = T_\beta(x \vee e_\beta) = T_\beta(x \vee y).$$

Since $e_\beta < x \vee e_\beta \in U_\beta$, $\bigcap_{e_\beta < z \in U_\beta} T_\beta(z) \subseteq T_\beta(x \vee e_\beta)$. By the inductive hypothesis, $T_\beta(x) \subseteq T_\beta(x \vee e_\beta)$. It follows that $\bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] \subseteq T_\beta(x \vee e_\beta)$. On the other hand, if $e_\beta < z \in U_\beta$, then $x \vee e_\beta \leq x \vee z \in U_\beta$. By the inductive hypothesis, $T_\beta(x \vee e_\beta) \subseteq T_\beta(x \vee z) = T_\beta(x) \cup T_\beta(z)$. Therefore, $T_\beta(x \vee e_\beta) \subseteq \bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)]$.

Suppose that β is a limit ordinal and the proposition holds for all $\beta' < \beta$. Let $x, y \in U_\beta$. We may assume that $x = e_{\beta_1}$ and $y = e_{\beta_2}$ for some $\beta_1 < \beta_2 < \beta$. Let $\beta_i = \gamma_i + n_i$, $i = 1, 2$. Using Lemma 3 and the inductive hypothesis,

$$\begin{aligned} T_\beta(x) \cup T_\beta(y) &= T_{\beta_1+1}(e_{\beta_1}) \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_1 + 2n_1 + 2, \beta) \\ &= T_{\beta_2+1}(e_{\beta_1}) \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \beta) \\ &= T_{\beta_2+1}(e_{\beta_1} \vee e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \beta). \end{aligned}$$

By (b) of Lemma 2, $e_{\beta_1} \vee e_{\beta_2} = e_\eta$ for some $\eta \leq \beta_1$. By Lemma 3,

$$\begin{aligned} T_{\beta_2+1}(e_{\beta_1} \vee e_{\beta_2}) &= T_{\eta+1}(e_\eta) \cup [\gamma_\eta + 2n_\eta + 2, \gamma_2 + 2n_2 + 2), \\ \text{and } T_\beta(e_{\beta_1} \vee e_{\beta_2}) &= T_{\eta+1}(e_\eta) \cup [\gamma_\eta + 2n_\eta + 2, \beta), \end{aligned}$$

where $\eta = \gamma_\eta + n_\eta$. Combining the three preceding equations gives $T_\beta(x) \cup T_\beta(y) = T_\beta(x \vee y)$. □

Proof of Theorem 1. Since $L = U_{\beta_0}$, Theorem 1 follows immediately from Lemma 4 and Proposition 5 by taking $\beta = \beta_0$ in each instance. □

2. GOOD LORENTZ FUNCTIONS

A *Lorentz sequence* is a nonincreasing sequence $(w(n))_{n=1}^\infty$ of positive numbers such that $w(1) = 1$, $\lim_n w(n) = 0$ and $\sum_{n=1}^\infty w(n) = \infty$. A Lorentz sequence is *C-submultiplicative* if $S(mn) \leq CS(m)S(n)$ for all $m, n \in \mathbb{N}$, where $S(n) = \sum_{k=1}^n w(k)$. In [4, §2], an infinite sequence of 1-submultiplicative Lorentz sequences is constructed so that the maxima of any two incomparable *finite* subsets are incomparable (see [4, Proposition 2.6]). For our purpose, we require an infinite sequence of *C-submultiplicative* Lorentz sequences so that the supremum of any (finite or infinite) subset remains a *C-submultiplicative* Lorentz function, and that the suprema of any two incomparable (finite or infinite) subsets are incomparable (Proposition 10). This is done by tweaking the arguments in [4, §2]. Following [4], we will find it more convenient to work with functions defined on real intervals. If $2 \leq N < \infty$, a *good Lorentz function* (GLF) on $(0, N]$ is a function $w : (0, N] \rightarrow (0, \infty)$ such that

- (1) $w(x) = 1$, $x \in (0, 2]$,
- (2) w is nonincreasing, and
- (3) if $1 \leq x, y \leq xy \leq N$, then $\int_0^{xy} w \leq \int_0^x w \cdot \int_0^y w$.

A *GLF* on $(0, \infty)$ (or simply a *GLF*) is a function $w : (0, \infty) \rightarrow (0, \infty)$ such that $w|_{(0, N]}$ is a *GLF* on $(0, N]$ for any $N \geq 2$, $\lim_{x \rightarrow \infty} w(x) = 0$ and $\int_0^\infty w = \infty$. It is an easy exercise to verify that if w is a *GLF*, then $(w(n))_{n=1}^\infty$ is a 4-submultiplicative Lorentz sequence.

If (u_i) is a finite or infinite sequence of real-valued functions with pairwise disjoint domains, let $\bigoplus_i u_i$ denote the set theoretic union. The constant 1 function with domain I is denoted by 1_I . We now recall the relevant facts from [4]. Note that the quantity $S(x)$ there corresponds to $\int_0^x w$ in our notation.

Lemma 6 ([4, Lemma 2.2]). *Let w be a *GLF* on $(0, N]$, $N \geq 2$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $w \oplus \varepsilon 1_{(N, N^2]}$ is a *GLF* on $(0, N^2]$.*

Repeated applications of Lemma 6 yield

Lemma 7. *Let G be a finite set of *GLF*'s on $(0, N]$, $N \geq 2$. For any $N' > N$ and any $\varepsilon > 0$, there is a function $v : (N, N'] \rightarrow (0, \infty)$ such that $w \oplus v$ is a *GLF* on $(0, N']$ for all $w \in G$, $v(x) \leq \varepsilon$, $x \in (N, N']$, and $\int_N^{N'} v < \varepsilon$.*

On the other hand, the proof of [4, Lemma 2.4] allows us to obtain *GLF* extensions with large total weight.

Lemma 8. *Let G be a finite set of *GLF*'s on $(0, N]$, $N \geq 2$, and set*

$$K = \min_{w \in G} \int_0^N w.$$

For any $\varepsilon > 0$, there is a function $v : (N, N'] \rightarrow (0, \infty)$, $N' > N$, such that

- (1) *for all $w \in G$, $w \oplus v$ is a *GLF* on $(0, N']$,*
- (2) *$v(x) \leq \varepsilon$, $x \in (N, N']$,*
- (3) *$\int_N^{N'} v \geq \frac{K}{2}$.*

Proof. By Lemma 7, there is a function $v_1 : (N, N^2] \rightarrow (0, \infty)$ such that $w \oplus v_1$ is a *GLF* on $(0, N^2]$ for all $w \in G$ and $v_1(x) \leq \varepsilon$ whenever $x \in (N, N^2]$. If $\int_N^{N^2} v_1 \geq \frac{K}{2}$, then we are done. Otherwise, from the proof of [4, Lemma 2.4], for each $w \in G$, there exists $\varepsilon_w > 0$ so that for any $0 < \varepsilon' \leq \varepsilon_w$, $w \oplus v_1 \oplus \varepsilon' 1_{[N^2, N(\varepsilon')]}$ is a *GLF* on $[0, N(\varepsilon')]$, where $N(\varepsilon')$ satisfies

$$\int_N^{N(\varepsilon')} v_1 \oplus \varepsilon' 1_{[N^2, N(\varepsilon')]} = \frac{1}{2} \int_0^N w \geq \frac{K}{2}.$$

Choose $w_0 \in G$ so that $\varepsilon_{w_0} \leq \varepsilon_w$ for all $w \in G$. Let $\varepsilon' = \varepsilon_{w_0}$ and $N' = N(\varepsilon_{w_0})$. The function $v = v_1 \oplus \varepsilon' 1_{[N^2, N']}$ has the desired properties. \square

We may repeat the preceding lemma to obtain

Lemma 9. *Let G be a finite set of *GLF*'s on $(0, N]$, $N \geq 2$. For any $K < \infty$ and any $\varepsilon > 0$, there is a function $v : (N, N'] \rightarrow (0, \infty)$, $N' > N$, such that*

- (1) *for all $w \in G$, $w \oplus v$ is a *GLF* on $(0, N']$,*
- (2) *$v(x) \leq \varepsilon$, $x \in (N, N']$,*
- (3) *$\int_N^{N'} v \geq K$.*

Proposition 10. *There exists an infinite sequence $(w_p)_{p=1}^\infty$ of *GLF*'s on $(0, \infty)$ such that for every nonempty $M \subseteq \mathbb{N}$ and every $p' \notin M$,*

- (1) *$w_M = \sup_{p \in M} w_p$ is a *GLF* on $(0, \infty)$,*

(2)

$$\sup_n \frac{\int_0^n w_{p'}}{\int_0^n w_M} = \infty.$$

Proof. The desired family of incomparable GLF’s is constructed by defining its elements inductively on successive intervals. On each of the segments, each of the w_p ’s is chosen to be either “high” or “low”.

Let $((p_i, q_i))_{i=1}^\infty$ be an enumeration of $\{(p, q) : p < q, p, q \in \mathbb{N}\}$ and fix a positive sequence (ε_i) decreasing to 0. For all $p \in \mathbb{N}$, define $w_p^0 : (0, 2] \rightarrow (0, \infty)$ by $w_p^0(x) = 1$. Set $G_0 = \{w_p^0 : p \in \mathbb{N}\}$.

Assume that for some $i \in \mathbb{N}$, functions $w_p^j : (N_{j-1}, N_j] \rightarrow (0, \infty)$, $0 \leq j < i$ ($N_{-1} = 0, N_0 = 2$), $p \in \mathbb{N}$, have been defined so that $G_{i-1} = \{w_{r_0}^0 \oplus \dots \oplus w_{r_{i-1}}^{i-1} : r_0, \dots, r_{i-1} \in \mathbb{N}\}$ is a finite set of GLF’s on $(0, N_{i-1}]$ and that $\{w_p^j : p \in \mathbb{N}\}$ is a totally ordered set of functions (in the pointwise order) for each $j \in [0, i)$. Set $K_{i-1} = \int_0^{N_{i-1}} \max G_{i-1}$, where by $\max G_{i-1}$ we mean the pointwise maximum of the set of functions G_{i-1} . By Lemma 9, choose a function $w_{p_i}^i$ on $(N_{i-1}, N_i]$, $N_i > N_{i-1}$, such that $w \oplus w_{p_i}^i$ is a GLF on $(0, N_i]$ for all $w \in G_{i-1}$, that $w_{p_i}^i(x) \leq \varepsilon_i$ for all $x \in (N_{i-1}, N_i]$ and that $\int_{N_{i-1}}^{N_i} w_{p_i}^i \geq q_i K_{i-1}$. On the other hand, by Lemma 7, there exists v on $(N_{i-1}, N_i]$ such that $w \oplus v$ is a GLF on $(0, N_i]$ for all $w \in G_{i-1}$, that $v(x) \leq w_{p_i}^i(N_i)$ for all $x \in (N_{i-1}, N_i]$ and that $\int_{N_{i-1}}^{N_i} v \leq 1$. Define $w_p^i = v$ for all $p \neq p_i$. Note that $G_i = \{w \oplus w_p^i : w \in G_{i-1}, p \in \mathbb{N}\}$ is a finite set of GLF’s on $(0, N_i]$. Obviously, the set $\{w_p^i : p \in \mathbb{N}\} = \{w_{p_i}^i, v\}$ is totally ordered. This completes the inductive construction. Define $w_p = \bigoplus_i w_p^i$, $p \in \mathbb{N}$. Observe that $K_0 = 2$ and $K_i \geq K_{i-1} + q_i K_{i-1} \geq 3K_{i-1}$. Hence $K_i \rightarrow \infty$. Thus

$$N_i \geq N_i - N_{i-1} \geq \int_{N_{i-1}}^{N_i} w_{p_i}^i \geq q_i K_{i-1} \rightarrow \infty.$$

Hence w_p is defined on $(0, \infty)$ for all $p \in \mathbb{N}$. If $\emptyset \neq M \subseteq \mathbb{N}$, let $w_M = \sup_{p \in M} w_p$. We claim that w_M is a GLF on $(0, \infty)$. By definition, $w_M|_{(0, N_i]} \in G_i$ for all $i \in \mathbb{N}$. Thus w_M is a GLF on $(0, N_i]$ for all $i \in \mathbb{N}$. Also note that $w_M(x) \leq \varepsilon_i$ for all $x \in (N_{i-1}, N_i]$. Therefore, $\lim_{x \rightarrow \infty} w_M(x) = 0$. Furthermore, since $w_M = w_{p_i}^i$ on $(N_{i-1}, N_i]$, if $p_i \in M$,

$$\int_0^\infty w_M > \sup_{\{i:p_i \in M\}} \int_{N_{i-1}}^{N_i} w_{p_i}^i \geq \sup_{\{i:p_i \in M\}} q_i K_{i-1}.$$

Because of the enumeration, $p_i \in M$ holds for infinitely many i . It follows that $\int_0^\infty w_M = \infty$. This shows that w_M is a GLF on $(0, \infty)$.

Finally, note that for all i such that $p_i \notin M$, $\int_{N_{i-1}}^{N_i} w_M \leq 1$ by construction. In particular, if $p' \notin M$, then for all i such that $p_i = p'$,

$$\begin{aligned} \int_0^{N_i} w_M &\leq \int_0^{N_{i-1}} w_M + \int_{N_{i-1}}^{N_i} w_M \\ &\leq \int_0^{N_{i-1}} \max G_{i-1} + \max_{p \in M} \int_{N_{i-1}}^{N_i} w_p \leq K_{i-1} + 1. \end{aligned}$$

On the other hand, for all such i ,

$$\int_0^{N_i} w_{p'} \geq \int_{N_{i-1}}^{N_i} w_{p'} = \int_{N_{i-1}}^{N_i} w_{p_i}^i \geq q_i K_{i-1}.$$

Hence

$$\sup_n \frac{\int_0^n w_{p'}}{\int_0^n w_M} = \infty. \quad \square$$

Given a Lorentz sequence $(w(n))_{n=1}^\infty$ and $1 \leq p < \infty$, the Lorentz sequence space $d(w, p)$ consists of all real sequences (a_n) such that $\sum (a_n^*)^p w_n < \infty$, where (a_n^*) denotes the nonincreasing rearrangement of $(|a_n|)$.

Corollary 11. *Let $(w_p)_{p=1}^\infty$ be as above. For every $M \subseteq \mathbb{N}$ and $p \notin M$, the unit vector basis of $d(w_M, 1)$ does not dominate that of $d(w_p, 1)$.*

Proof. Let (v_i) and (u_i) denote the respective unit vector bases of $d(w_p, 1)$ and $d(w_M, 1)$. According to Proposition 10, for any $K < \infty$, there exists $N \in \mathbb{N}$ such that $\int_0^{N+1} w_p \geq K \int_0^{N+1} w_M$. Then

$$\begin{aligned} \left\| \sum_{i=1}^N v_i \right\| &= \sum_{i=1}^N w_p(i) \geq \int_1^{N+1} w_p = \int_0^{N+1} w_p - 1 \\ &\geq K \int_0^{N+1} w_M - 1 \geq K \sum_{i=1}^N w_M(i) - 1 = K \left\| \sum_{i=1}^N u_i \right\| - 1. \end{aligned}$$

The result follows since K is arbitrary. □

3. COUNTABLE SEMILATTICES OF SPREADING MODELS

In this section, we show that every countable semilattice without an infinite increasing sequence is order isomorphic to some $SP_w(X)$. If (x_i) and (y_i) are sequences in the Banach spaces X and Y , respectively, let $(x_i) \oplus (y_i)$ denote the sequence $(z_i) = (x_i, y_i)$ in the direct sum $X \oplus Y$. The ℓ^p -sum of an infinite sequence (X_j) of Banach spaces is denoted by $(\sum_{j=1}^\infty \oplus X_j)_p$. We omit the easy proof of the next lemma.

Lemma 12. *Let $w_1 = (w_1(n))$ and $w_2 = (w_2(n))$ be Lorentz sequences. Then $w = w_1 \vee w_2 = (w_1(n) \vee w_2(n))$ is a Lorentz sequence. Moreover, if (u_n^1) and (u_n^2) are the respective unit vector bases of $d(w_1, 1)$ and $d(w_2, 1)$, then $(u_n^1) \oplus (u_n^2)$ is equivalent to (u_n) , the unit vector basis of $d(w, 1)$.*

Lemma 13 ([4, Lemma 3.6]). *Let $X = (\sum_{j=1}^\infty \oplus X_j)_p$, where $1 \leq p < \infty$ and each X_j is an infinite-dimensional Banach space, and let (\tilde{x}_i) be a spreading model generated by a normalized weakly null sequence in X . Then there exist a nonnegative $(c_j)_{j=0}^\infty$ with $\sum_{j=0}^\infty c_j^p = 1$ and normalized spreading models $(\tilde{x}_i^j)_i$ in X_j generated by weakly null sequences such that for all scalars (a_i) ,*

$$(1) \quad \left\| \sum_i a_i \tilde{x}_i \right\| = \left[\sum_{j=1}^\infty c_j^p \left\| \sum_i a_i \tilde{x}_i^j \right\|^p + c_0^p \sum_i |a_i|^p \right]^{1/p}.$$

Remark. If $p = 1$, the final term on the right of equation (1) may be omitted, i.e., $c_0 = 0$. In fact, according to the proof of Lemma 13 in [4, Lemma 3.6], the spreading model (\tilde{x}_i) is generated by a weakly null sequence (x_i) in X in such a way that $c_0 = \lim \|x_i - P_i(x_i)\|$, where $P_i(x_i) = (x_i^1, x_i^2, \dots, x_i^i, 0, 0, 0, \dots)$. However,

since ℓ^1 has the Schur property (weakly null sequences are norm null), it is easy to see that $\lim \|x_i - P_i(x_i)\| = 0$ for any weakly null sequence (x_i) in $(\sum_{j=1}^\infty \oplus X_j)_1$.

The following is the crucial property of Lorentz sequence spaces that we require. It can be deduced from the arguments in [1, §4]:

Theorem 14 ([1]). *Let $w = (w(n))$ be a C -submultiplicative Lorentz sequence and let (u_n) be the unit vector basis of $d(w, 1)$. For any $\varepsilon > 0$, every normalized block basis in $d(w, 1)$ has a subsequence (x_n) such that either*

- (a) (x_n) is equivalent to the unit vector basis of ℓ^1 , or
- (b) there exists $c > 0$ such that for all $(a_n) \in c_{00}$,

$$(2) \quad c \left\| \sum a_n u_n \right\| \leq \left\| \sum a_n x_n \right\| \leq (C + \varepsilon) \left\| \sum a_n u_n \right\|.$$

In particular, if (\tilde{x}_n) is a spreading model generated by a normalized weakly null sequence, then (\tilde{x}_n) satisfies (2) in place of (x_n) .

Theorem 15. *Given a countable semilattice L with no infinite increasing sequence, there is a Banach space X_L such that $SP_w(X_L)$ is order isomorphic to L .*

Proof. By Theorem 1, there exists a countable set V and an injective map $T : L \rightarrow 2^V \setminus \{\emptyset\}$ such that $T(e \vee f) = T(e) \cup T(f)$ for all $e, f \in L$. Since V is countable, by Proposition 10 (and Corollary 11), there is a family $(w_v)_{v \in V}$ of 4-submultiplicative Lorentz sequences such that for each nonempty subset M of V , $w_M = \sup_{v \in M} w_v$ is again a (4-submultiplicative) Lorentz sequence. Moreover, if $p \notin M$, the unit vector basis of $d(w_M, 1)$ does not dominate that of $d(w_p, 1)$. Set $X_L = (\bigoplus_{e \in L} d(w_{T_e}, 1))_1$. For any $e \in L$, let (u_i^e) be the unit vector basis of $d(w_{T_e}, 1)$. Hence, (u_i^e) may be regarded in an obvious way as a normalized weakly null sequence in X_L which generates a spreading model equivalent to itself. Thus $[(u_i^e)]$, the equivalence class containing (u_i^e) , is an element of $SP_w(X_L)$. Define a map $\Theta : L \rightarrow SP_w(X_L)$ by $\Theta e = [(u_i^e)]$. We will show that Θ is a bijection such that $\Theta e_1 \leq \Theta e_2$ if and only if $e_1 \leq e_2$. Hence $SP_w(X_L)$ is order isomorphic to L .

We first show that Θ is onto. Let $[(\tilde{x}_i)]$ be an element in $SP_w(X_L)$. By Lemma 13 and the subsequent Remark, there exist a nonnegative sequence $(c_e)_{e \in L}$ with $\sum c_e = 1$ and normalized spreading models (\tilde{x}_i^e) in $d(w_{T_e}, 1)$ such that

$$(3) \quad \left\| \sum_i a_i \tilde{x}_i \right\| = \sum_{e \in L} c_e \left\| \sum_i a_i \tilde{x}_i^e \right\|.$$

Since each w_{T_e} is 4-submultiplicative, according to Theorem 14, for each $e \in L$, there exists $b_e > 0$ such that

$$(4) \quad b_e \left\| \sum_i a_i u_i^e \right\| \leq \left\| \sum_i a_i \tilde{x}_i^e \right\| \leq 5 \left\| \sum_i a_i u_i^e \right\|.$$

Let $I = \{e \in L : c_e > 0\}$. If I is infinite, write its elements in a sequence $(e_i)_{i=1}^\infty$. Since the sequence $(\bigvee_{i=1}^n e_i)_{n=1}^\infty$ has no strictly increasing infinite subsequence, there is a finite subset J of I such that $\bigvee_{e \in J} e \geq e'$ for all $e' \in I$. If I is finite, take $J = I$. Let $f = \bigvee_{e \in J} e$. We claim that (\tilde{x}_i) is equivalent to (u_i^f) . Observe that $e \leq f$ for all $e \in I$. Hence $T_e \subseteq T_f$ and thus $w_{T_e} \leq w_{T_f}$. Therefore, (u_i^e) is

1-dominated by (u_i^f) . By (3) and (4),

$$\begin{aligned} \left\| \sum_i a_i \tilde{x}_i \right\| &= \sum_{e \in L} c_e \left\| \sum_i a_i \tilde{x}_i^e \right\| = \sum_{e \in I} c_e \left\| \sum_i a_i \tilde{x}_i^e \right\| \\ &\leq 5 \sum_{e \in I} c_e \left\| \sum_i a_i u_i^e \right\| \leq 5 \sum_{e \in I} c_e \left\| \sum_i a_i u_i^f \right\| = 5 \left\| \sum_i a_i u_i^f \right\|. \end{aligned}$$

On the other hand, by Lemma 12, $\bigoplus_{e \in J} (u_i^e)$ is equivalent to (u_i^f) . Using (3) and (4) again,

$$\begin{aligned} \left\| \sum_i a_i \tilde{x}_i \right\| &= \sum_{e \in I} c_e \left\| \sum_i a_i \tilde{x}_i^e \right\| \geq \sum_{e \in I} c_e b_e \left\| \sum_i a_i u_i^e \right\| \\ &\geq \sum_{e \in J} c_e b_e \left\| \sum_i a_i u_i^e \right\| \geq \min_{e \in J} \{c_e b_e\} \sum_{e \in J} \left\| \sum_i a_i u_i^e \right\| \\ &\geq K \left\| \sum_i a_i u_i^f \right\| \text{ for some } K > 0. \end{aligned}$$

This shows that (\tilde{x}_i) is equivalent to (u_i^f) . Hence $\Theta f = [(u_i^f)] = [(\tilde{x}_i)]$.

Next we show that

$$(5) \quad e_1 \leq e_2 \Leftrightarrow \Theta e_1 \leq \Theta e_2.$$

If $e_1 \leq e_2$, then $T e_1 \subseteq T e_2$ and hence $w_{T e_1} \leq w_{T e_2}$. It follows that $[(u_i^{e_1})] \leq [(u_i^{e_2})]$. On the other hand, if $e_1 \not\leq e_2$, then $T(e_1) \not\subseteq T(e_2)$. Choose $p \in T(e_1) \setminus T(e_2)$. By Corollary 11, $(u_i^{e_2})$ does not dominate (v_i) , the unit vector basis of $d(w_p, 1)$. But obviously $(u_i^{e_1})$ dominates (v_i) . Hence $[(u_i^{e_1})] \not\leq [(u_i^{e_2})]$. Note that (5) also implies that Θ is injective. Hence $\Theta : L \rightarrow SP_w(X_L)$ is an order isomorphism. \square

Remark. The example given here is nonreflexive. Given a countable semilattice L without an infinite increasing sequence, the ℓ^p ($1 < p < \infty$) version of the space defined above, i.e., $X_p = \left(\bigoplus_{e \in L} d(w_{T e}, p) \right)_p$, which is a reflexive space, has the property that $SP_w(X_p)$ is order isomorphic to the semilattice $\hat{L} = \{a\} \cup L$, $a > e$, for all $e \in L$. We do not know how to obtain a reflexive example for general semilattices. In fact, according to the authors of [4], it is not known if there is a reflexive space X such that $SP_w(X)$ is order isomorphic to $(\{\{1, 2\}, \{1\}, \{2\}\}, \subseteq)$.

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