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ANALOGUES OF THE ARTIN FACTORIZATION FORMULA FOR THE AUTOMORPHIC SCATTERING MATRIX AND SELBERG ZETA-FUNCTION ASSOCIATED TO A KLEINIAN GROUP

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ABSTRACT. For Kleinian groups acting on a hyperbolic three-space, we prove factorization formulas for both the Selberg zeta-function and the automorphic scattering matrix. We extend results of Venkov and Zograf from Fuchsian groups to Kleinian groups, and we give a proof that is simple and extendable to more general groups.

1. INTRODUCTION

In [VZ83] Venkov and Zograf gave an analogue of Artin's well-known factorization formula. More specifically, they gave factorization formulas for both the Selberg zeta-function and automorphic scattering matrix that are associated to a Fuchsian group (in the context of the Selberg spectral theory of automorphic functions).

Let $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ be a cofinite Kleinian group, and let Γ_1 be a finite-index normal subgroup of index n. Let $\chi \in \mathrm{Rep}(\Gamma_1, V)$ be a finite-dimensional unitary representation of Γ_1 in V, and let $\psi = U^\chi \in \mathrm{Rep}(\Gamma, V^n)$ be its induced representation to Γ.

Let $\mathfrak{S}(s,\Gamma_1,\chi), \mathfrak{S}(s,\Gamma,\psi)$ be automorphic scattering matrices (defined in §5), and set

$$\phi(s,\Gamma_1,\chi) \equiv \det \mathfrak{S}(s,\Gamma_1,\chi),$$
$$\phi(s,\Gamma,\psi) \equiv \det \mathfrak{S}(s,\Gamma,\psi).$$

Theorem. *Let Γ be a cofinite Kleinian group, and let Γ_1 be a finite-index normal subgroup of Γ. Let $\chi \in \mathrm{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for all regular $s \in \mathbb{C}$,*

$$\phi(s,\Gamma_1,\chi) = \phi(s,\Gamma,\psi).$$

In [VZ83] a slightly weaker version of this equation was derived. The authors showed that

$$\phi(s,\Gamma_1,\chi)\Omega(\Gamma_1,\chi)^{1-2s} = \phi(s,\Gamma,U^\chi)\Omega(\Gamma,U^\chi)^{1-2s},$$

where $\Omega(\,\cdot\,,\,\cdot\,)$ is a constant depending on the group and unitary representation.

We also have

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Theorem. *Let* Γ *be an arbitrary Kleinian group, and let* Γ_1 *be a finite-index normal subgroup of* Γ. *Let* $\chi \in \mathrm{Rep}(\Gamma_1, V)$, *and* $\psi = U^{\chi}$. *Then for* $\mathrm{Re}\, s > 1$,

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Here $Z(s, \cdot, \cdot)$ is the Selberg zeta-function (defined in §2.2).

Theorem. *Let* Γ *be an arbitrary Kleinian group, and let* Γ_1 *be a finite-index normal subgroup of* Γ. *Then for* $\mathrm{Re}\, s > 1$,

$$Z(s, \Gamma_1, \mathbf{1}) = Z(s, \Gamma, U^{\mathbf{1}}) = \prod_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_\vartheta}.$$

Here $(\Gamma_1 \setminus \Gamma)^*$ is the set of all pairwise inequivalent irreducible unitary representations of the group $\Gamma_1 \setminus \Gamma$; for $\vartheta \in (\Gamma_1 \setminus \Gamma)^*$, n_ϑ is the dimension of ϑ; $Z(s, \Gamma, \vartheta)$ is the Selberg zeta-function; $\mathbf{1}$ is the trivial representation of Γ_1, and $U^{\mathbf{1}}$ is its induced representation to Γ.

2. Preliminaries

In this section we state the preliminary results that will be needed. Our main references are [Fri05a, Fri05b], [EGM98], and [Ven82], and [Hej76, Hej83]. Unless stated otherwise, throughout this section $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is a cofinite Kleinian group and $\chi \in \mathrm{Rep}(\Gamma, V)$ is a finite-dimensional unitary representation of Γ in V.

2.1. **Cofinite Kleinian groups.** Let $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group acting on a hyperbolic three-space $\mathbb{H}^3$. Let V be a finite-dimensional complex inner product space with inner-product $\langle\ ,\ \rangle_V$, and let $\mathrm{Rep}(\Gamma, V)$ denote the set of finite-dimensional unitary representations of Γ in V. Let $\mathcal{F} \subset \Gamma$ be a fundamental domain for the action of Γ in $\mathbb{H}^3$.

Let $\chi \in \mathrm{Rep}(\Gamma, V)$. The Hilbert space of χ-*automorphic* functions is the set of measurable functions

$$\mathcal{H}(\Gamma, \chi) \equiv \{ f : \mathbb{H}^3 \to V \mid f(\gamma P) = \chi(\gamma) f(P)\ \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \\ \text{and } \langle f, f \rangle \equiv \textstyle\int_{\mathcal{F}} \langle f(P), f(P) \rangle_V \, dv(P) < \infty \}.$$

Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Next we briefly define the concept of a *singular* unitary representation. Let $\mathbb{P}$ be the the boundary of $\mathbb{H}^3$, the Riemann sphere. For every $\zeta \in \mathbb{P}^1$ let Γ_ζ denote the stabilizer subgroup of ζ in Γ,

$$\Gamma_\zeta \equiv \{\ \gamma \in \Gamma \mid \gamma\zeta = \zeta\ \},$$

and let Γ'_ζ be the maximal torsion-free parabolic subgroup of Γ_ζ (the maximal parabolic subgroup of Γ_ζ that does not contain elliptic elements). A point $\zeta \in \mathbb{P}^1$ is called a *cusp* of Γ if Γ'_ζ is a free abelian group of rank two. Two cusps ζ_1, ζ_2 are Γ-equivalent if $\zeta_1 \in \Gamma\zeta_2$, that is, if their Γ-orbits coincide.

Every cofinite Kleinian group has finitely many equivalence classes of cusps, so we fix a set $\{\zeta_\alpha\}_{\alpha=1}^{\kappa(\Gamma)}$ of representatives of these equivalence classes. For notational convenience we set $\Gamma_\alpha \equiv \Gamma_{\zeta_\alpha}$ and $\Gamma'_\alpha \equiv \Gamma'_{\zeta_\alpha}$.

Each cusp ζ_α has an associated lattice Λ_α (see [EGM98, Theorem 2.1.8]). For each cusp ζ_α of Γ, define the *singular* space $V_\alpha \equiv \{ v \in V \mid \chi(\gamma)v = v,\ \ \forall \gamma \in \Gamma_\alpha \}$, where $1 \le \alpha \le \kappa(\Gamma)$.

A representation $\chi \in \mathrm{Rep}$ is *singular* at the cusp ζ_α of Γ iff the subspace $V_\alpha \neq \{0\}$. If a cusp is not singular, it is called *regular.*

For each cusp ζ_α, set $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and $k(\Gamma,\chi) \equiv \sum_{\alpha=1}^{\kappa(\Gamma)} k_\alpha$.

2.2. **Selberg zeta-function.** Let $Z(s,\Gamma,\chi)$ denote the Selberg zeta-function associated to Γ and $\chi \in \mathrm{Rep}(\Gamma,\chi)$. We allow Γ to be an *arbitrary* Kleinian group. See [Fri05a] and [EGM98, Sections 5.2,5.4] for the details on its construction.

In [Fri05a], we gave the meromorphic continuation of $Z(s,\Gamma,\chi)$ to the left half plane under certain technical assumptions. A practical way of understanding the zeta-function is via its logarithmic derivative. For $\mathrm{Re}(s) > 1$,

$$\frac{d}{ds}\log Z(s,\Gamma,\chi) = \sum_{\{T\}_{\mathrm{lox}}} \frac{\mathrm{tr}(\chi(T))\log N(T_0)}{m(T)|a(T)-a(T)^{-1}|^2} N(T)^{-s}. \tag{2.1}$$

See [Fri05a] for the definition of the notation used above.

Let $W(s,\Gamma,\chi) = \frac{d}{ds}\log Z(s,\Gamma,\chi)$. Then, for $\mathrm{Re}\, s > 1$,

$$Z(s,\Gamma,\chi) = e^{\int W(s,\Gamma,\chi)\,ds + C}, \tag{2.2}$$

where C is chosen so

$$\lim_{s\to\infty} Z(s,\Gamma,\chi) = 1$$

will be satisfied.

2.3. **Selberg theory of Δ.** In this section we assume that Γ is cofinite and we state some needed results concerning the Selberg trace formula associated to Γ and $\chi \in \mathrm{Rep}(\Gamma,V)$. We will not need the full trace formula found in [Fri05a, Fri05b]. Rather, only parts of its proof will be needed. More details can be found in [Fri05a, Fri05b], [EGM98], and [Ven82].

For $P = z + rj,\ P' = z' + r'j \in \mathbb{H}^3$ set

$$\delta(P,P') \equiv \cosh(d(P,P')) = \frac{|z-z'|^2 + r^2 + r'^2}{2rr'},$$

where d is the hyperbolic distance in $\mathbb{H}^3$.

For $k \in \mathcal{S}([1,\infty))$, a Schwartz-class function, set $K(P,P') = k(\delta(P,P'))$. Note that for any $\gamma \in \mathrm{PSL}(2,\mathbb{C})$,

$$K(\gamma P,\gamma P') = K(P,P') \quad \text{and} \quad K(P,\gamma P') = K(\gamma^{-1}P,P'). \tag{2.3}$$

For $\Theta \subset \Gamma$, $\chi \in \mathrm{Rep}(\Gamma,V)$, define

$$K(P,P',\Theta,\psi) \equiv \sum_{\gamma\in\Theta} \chi(\gamma)K(P,\gamma P'). \tag{2.4}$$

The series above converges absolutely and uniformly on compact subsets of $\mathbb{H}^3\times\mathbb{H}^3$.

For $\lambda \in \mathbb{C}$, $\lambda = 1 - s^2$, the Selberg–Harish-Chandra transform of k, h is defined by

$$h(\lambda) = h(1-s^2) \equiv \frac{\pi}{s}\int_1^\infty k\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right)(t^s - t^{-s})\left(t - \frac{1}{t}\right)\frac{dt}{t},\ \lambda = 1 - s^2. \tag{2.5}$$

In addition, let

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} h(1+t^2)e^{-itx}\,dt.$$

We can start with the function h and work backwards to find k [EGM98, Chapter 3]. The pair h, g is said to be *admissible* if h is a holomorphic function on $\{s \in \mathbb{C} \,|\, |\operatorname{Im}(s)| < 2+\delta\}$ for some $\delta > 0$, satisfying $h(1+z^2) = O(1+|z|^2)^{3/2-\epsilon})$ as $|z| \to \infty$.

For $v, w \in V$ let $v \otimes \overline{w}$ be the linear operator in V, defined by $v \otimes \overline{w}(x) = \langle x, w\rangle v$, where $x \in V$.

Lemma 2.1 ([Fri05b])**.** *Let $k \in \mathcal{S}([1,\infty))$ and $h : \mathbb{C} \to \mathbb{C}$ be the Selberg–Harish-Chandra Transform of k. Then*

$$K(P,Q,\Gamma,\chi) = \sum_{m\in\mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)} + \frac{1}{4\pi}\sum_{\alpha=1}^{\kappa(\Gamma)}\sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h\left(1+t^2\right) E_{\alpha l}(P,it) \otimes \overline{E_{\alpha l}(Q,it)}\, dt. \tag{2.6}$$

The sum and integrals converge on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$. Here $\mathcal{D}$ is an indexing set of the eigenfunctions e_m of Δ with corresponding eigenvalues λ_m, $E_{\alpha l}(P,s)$ are the Eisenstein series associated to the singular cusps of Γ, $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and $|\Lambda_\alpha|$ is the Euclidean area of a fundamental domain for the lattice Λ_α. If a cusp is regular, it is omitted from the sum in (2.6).

Next, we split up K_Γ as a sum of two kernels. The first kernel,

$$H_\Gamma(P,Q) = \frac{1}{4\pi}\sum_{\alpha=1}^{\kappa}\sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h\left(1+t^2\right) E_{\alpha l}(P,it) \otimes \overline{E_{\alpha l}(Q,it)}\, dt,$$

is not of Hilbert-Schmidt class, while the second kernel,

$$L_\Gamma(P,Q) = \sum_{m\in\mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)},$$

is of trace class.

Suppose $Y > 0$ is sufficiently large. Then for all $A > Y$, there exists a compact set $\mathcal{F}_A \subset \mathbb{H}^3$ such that

$$\mathcal{F} \equiv \mathcal{F}_A \cup \mathcal{F}_1(A) \cup \cdots \cup \mathcal{F}_\kappa(A) \tag{2.7}$$

is a fundamental domain for Γ. The sets $\mathcal{F}_\alpha(A)$ are cusp sectors (see [EGM98], Proposition 2.3.9). It follows that

$$\lim_{A\to\infty}\left(\int_{\mathcal{F}_A} \operatorname{tr}_V(K_\Gamma(P,P))\, dv(P) - \int_{\mathcal{F}_A} \operatorname{tr}_V(H_\Gamma(P,P))\, dv(P)\right) = \int_{\mathcal{F}} \operatorname{tr}_V(L_\Gamma(P,P))\, dv(P) = \sum_{m\in\mathcal{D}} h(\lambda_m) < \infty. \tag{2.8}$$

The infinite sum is absolutely convergent.

Let $\mathfrak{S}(s)$ denote the *automorphic scattering matrix* associated to Γ and χ (see [Fri05b]), and let

$$\phi(s) = \det \mathfrak{S}(s).$$

Upon applying the vector form of the Maaß-Selberg relations (see [Roe66], [Ven82], [Fri05b]), we obtain

$$\begin{aligned}(2.9)\quad &\int_{\mathcal{F}_A} \operatorname{tr}_V(H_\Gamma(P,P))\,dv(P)\\ &= g(0)k(\Gamma,\chi)\log(A) - \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\phi'}{\phi}(it)h(1+t^2)\,dt + \frac{h(1)\operatorname{tr}\mathfrak{S}(0)}{4} + \underset{A\to\infty}{o(1)}\,.\end{aligned}$$

The integral on the right-hand side converges absolutely.

3. Induced representations

Let Γ be an *arbitrary* Kleinian group, and let $\Gamma_1 \lhd \Gamma$ be a finite-index normal subgroup of index n. Let $\mathcal{F}, \mathfrak{F}$ be fundamental domains of Γ, Γ_1 respectively, with $[\alpha_i]_{i=1}^n$ a complete set of representatives for the right-cosets of $\Gamma_1 \setminus \Gamma$, satisfying

$$\mathfrak{F} = \bigcup_{i=1}^n \alpha_i(\mathcal{F}). \tag{3.1}$$

Let V be a finite-dimensional hermitian vector space, and let χ be a finite-dimensional unitary representation of Γ_1 in V. Set

$$\overline{\chi}(\gamma) = \begin{cases} \chi(\gamma), & \gamma \in \Gamma_1, \\ 0, & \gamma \notin \Gamma_1. \end{cases}$$

Let $\psi \equiv U^\chi$ be the induced representation of χ from Γ_1 to Γ. More explicitly,

$$\psi : \Gamma \mapsto \mathrm{GL}(V^n),$$

and for $\gamma \in \Gamma$ and $v_i \in V$,

$$\psi(\gamma)\left(\sum_{i=1}^n \oplus v_i\right) = \sum_{i=1}^n \oplus \sum_{j=1}^n \overline{\chi}(\alpha_i\gamma\alpha_j^{-1})v_j. \tag{3.2}$$

It follows from (3.2) that for $\gamma \in \Gamma$,

$$\operatorname{tr}_{V^n}\psi(\gamma) = \sum_{i=1}^n \operatorname{tr}_V \overline{\chi}(\alpha_i\gamma\alpha_i^{-1}). \tag{3.3}$$

We will need the following result (see [RS71] and [VZ83, Theorem 2.1]):

Lemma 3.1. *There exists an isometry between the Hilbert spaces $\mathcal{H}(\Gamma_1,\chi)$ and $\mathcal{H}(\Gamma,\psi)$, which takes the operator $\Delta(\Gamma_1,\chi)$ to $\Delta(\Gamma,\psi)$.*

We abuse notation and call a set $\Theta \subset \Gamma$ *normal* if for all $\gamma \in \Gamma$, we have $\gamma\Theta\gamma^{-1} = \Theta$.

Lemma 3.2. *Let $\Gamma_1 \lhd \Gamma$ be a finite-index, normal subgroup of Γ of index n, $\chi \in \mathrm{Rep}(\Gamma_1, V)$, $\psi \equiv U^\chi \in \mathrm{Rep}(\Gamma, V^n)$, and let $\Theta \subset \Gamma$ be a normal subset. Suppose that*

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} K(P,P,\Theta,\psi)\,dv(P)$$

converges absolutely. Then

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} K(P,P,\Theta,\psi)\,dv(P) = \int_{\mathfrak{F}} \operatorname{tr}_V K(P,P,\Theta\cap\Gamma_1,\chi)\,dv(P).$$

Proof. By (3.3),

$$\begin{aligned}
&\int_{\mathcal{F}} \mathrm{tr}_{V^n} K(P,P,\Theta,\psi)\, dv(P) \\
&= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma\in\Theta} \mathrm{tr}_V\, \overline{\chi}(\alpha_i\gamma\alpha_i^{-1}) K(P,\gamma P)\, dv(P) \\
&= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\substack{\gamma\in\Theta \\ \alpha_i\gamma\alpha_i^{-1}\in\Gamma_1}} \mathrm{tr}_V\, \chi(\alpha_i\gamma\alpha_i^{-1}) K(P,\gamma P)\, dv(P) \\
&= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\alpha_i\gamma\alpha_i^{-1}) K(P,\gamma P)\, dv(P) \quad \text{(by normality).}
\end{aligned}$$

Now, since Θ and Γ_1 are normal, $\gamma \in \Theta\cap\Gamma_1$ implies that $\alpha_i\gamma\alpha_i^{-1} \in \Theta\cap\Gamma_1$. Also, as γ goes through each element of $\Theta\cap\Gamma_1$, so does $\alpha_i\gamma\alpha_i^{-1}$. So, for each i, setting $\beta = \alpha_i\gamma\alpha_i^{-1}$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\alpha_i\gamma\alpha_i^{-1}) K(P,\gamma P)\, dv(P) \\
&\quad= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\beta) K(P,\alpha_i^{-1}\beta\alpha_i P)\, dv(P) \\
&\quad= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\beta) K(\alpha_i P,\beta\alpha_i P)\, dv(P) \qquad (3.4) \\
&\quad= \sum_{i=1}^{n} \int_{\alpha_i(\mathcal{F})} \sum_{\beta\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\beta) K(Q,\beta Q)\, dv(Q) \qquad (3.5) \\
&\quad= \int_{\mathfrak{F}} \sum_{\beta\in\Theta\cap\Gamma_1} \mathrm{tr}_V\, \chi(\beta) K(Q,\beta Q)\, dv(Q) \qquad (3.6) \\
&\quad= \int_{\mathfrak{F}} \mathrm{tr}_V\, K(P,P,\Theta\cap\Gamma_1,\chi)\, dv(P),
\end{aligned}$$

where in (3.5) we set $Q = \alpha_i P$. In (3.4) we used Equation (2.3), and we used the fact that $dv(\alpha_i Q) = dv(Q)$; and in (3.6) we *tiled* the fundamental domain $\mathcal{F}$ according to (3.1). □

Lemma 3.2 works well when the point-pair-invariant k gives rise to an integral kernal of trace-class. However, a careful look at the proof shows that we could replace $\mathcal{F}$ by a *truncated* fundamental domain $\mathcal{F}_A$.

For the rest of this section, we assume that Γ is cofinite. Let $Y > 0$ be sufficiently large so that for $A > Y$, $\mathcal{F}$ decomposes into $\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}^A$, where $\mathcal{F}_A$ is compact and $\mathcal{F}^A$ is a union of cusp sectors (2.7). Since

$$\mathfrak{F} = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F})$$

it follows that $\mathfrak{F} = \mathfrak{F}_A \cup \mathfrak{F}^A$ with $\mathfrak{F}_A = \bigcup_{i=1}^n \alpha_i(\mathcal{F}_A)$ and $\mathfrak{F}^A = \bigcup_{i=1}^n \alpha_i(\mathcal{F}^A)$; $\mathfrak{F}^A$ is also a union of cusp sectors.

Lemma 3.3. *Let $\Gamma_1 \lhd \Gamma$ be a finite-index, normal subgroup of Γ of index n, $\chi \in \mathrm{Rep}(\Gamma_1, V)$, $\psi \equiv U^\chi \in \mathrm{Rep}(\Gamma, V^n)$, and let $\Theta \subset \Gamma$ be a normal subset. Suppose that*

$$\int_{\mathcal{F}_A} \mathrm{tr}_{V^n} K(P, P, \Theta, \psi)\, dv(P)$$

converges absolutely. Then

$$\int_{\mathcal{F}_A} \mathrm{tr}_{V^n} K(P, P, \Theta, \psi)\, dv(P) = \int_{\mathfrak{F}_A} \mathrm{tr}_V K(P, P, \Theta \cap \Gamma_1, \chi)\, dv(P).$$

4. Factorization formula of the Selberg zeta-function

In this section we prove the analogue of the Venkov-Zograf factorization formula ([VZ83]) for *arbitrary* Kleinian groups.

Throughout this section, Γ is an arbitrary Kleinian group and $\Gamma_1 \lhd \Gamma$ is a finite-index normal subgroup of index n.

Theorem 4.1. *Suppose that $\Gamma_1 \lhd \Gamma$, $\chi \in \mathrm{Rep}(\Gamma_1, V)$, and that $\psi = U^\chi \in \mathrm{Rep}(\Gamma, V^n)$. Then for $\mathrm{Re}(s) > 1$,*

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Proof. By (2.2), it suffices to show that

$$W(s, \Gamma_1, \chi) = W(s, \Gamma, \psi). \tag{4.1}$$

Let $\Theta = \Gamma^{\mathrm{lox}} \subset \Gamma$ be the set of *all* loxodromic elements of Γ. Note that Θ is *normal* since the conjugate of a loxodromic element is loxodromic. For $\mathrm{Re}\, s > 1$, $\delta > 1$, set

$$k_s(\delta) \equiv \frac{1}{4\pi} \frac{\left(\delta + \sqrt{\delta^2 - 1}\right)^{-s}}{\sqrt{\delta^2 - 1}}.$$

For $P, P' \in \mathbb{H}^3$, set $K_s(P, P') = k_s(\delta(P, P'))$, and, using (2.4), define $K_s(P, P', \Theta, \psi)$ and $K_s(P, P', \Theta \cap \Gamma_1, \chi)$. By [EGM98, pages 185-198] and [Fri05a, Lemma 6.3], it follows that

$$W(s, \Gamma_1, \chi) = \int_{\mathfrak{F}} \mathrm{tr}_V K_s(P, P, \Theta \cap \Gamma_1, \chi)\, dv(P)$$

and

$$W(s, \Gamma, \psi) = \int_{\mathcal{F}} \mathrm{tr}_{V^n} K_s(P, P, \Theta, \psi)\, dv(P).$$

The result now follows from Lemma 3.2. □

Compare our proof with the proof given in [VZ83, Theorem 3.1].

If we let $\chi = \mathbf{1}$, the trivial one-dimensional representation, it follows that $\psi = U^{\mathbf{1}}$ can be decomposed into irreducible sub-representations, explicitly:

$$\psi = \bigoplus_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} n_\vartheta \vartheta,$$

where $(\Gamma_1 \setminus \Gamma)^*$ denotes the set of all pairwise inequivalent irreducible unitary representations of the group[1] $\Gamma_1 \setminus \Gamma$, and n_ϑ is the dimension of ϑ. For more details see [Kir76].

Next, by (2.1) and (2.2) it follows that for $\vartheta_1, \vartheta_2 \in \mathrm{Rep}(\Gamma, V_i)$ $(i = 1, 2)$,

$$Z(s, \Gamma, \vartheta_1 \oplus \vartheta_2) = Z(s, \Gamma, \vartheta_1) Z(s, \Gamma, \vartheta_2).$$

[1] Of course, each χ must be extended from $\Gamma_1 \setminus \Gamma$ to Γ.

We have

Theorem 4.2. *Let* $\Gamma_1 \lhd \Gamma$. *Then for* $\operatorname{Re} s > 1$,

$$Z(s, \Gamma_1, \mathbf{1}) = Z(s, \Gamma, U^{\mathbf{1}}) = \prod_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_\vartheta}.$$

5. Factorization formula for the determinant of the scattering matrix

In this section we prove the analogous result to Theorem 4.1 for the automorphic scattering matrix. Throughout this section, Γ is a cofinite Kleinian group.

Let $\Gamma_1 \lhd \Gamma$, $\chi \in \operatorname{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Let $k \in \mathcal{S}([1,\infty))$, and let h be given by (2.5). It follows form Lemma 3.1 that $\Delta(\Gamma_1, \chi)$ and $\Delta(\Gamma, \psi)$ have the same eigenvalues. Hence (from §2.3)

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} L(P, P, \Gamma, \psi)\, dv(P) = \int_{\mathfrak{F}} \operatorname{tr}_V L(P, P, \Gamma_1, \chi)\, dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m).$$

Now, since $\int_{\mathcal{F}_A} \operatorname{tr}_{V^n} K(P, P, \Gamma, \psi)\, dv(P) = \int_{\mathfrak{F}_A} \operatorname{tr}_V K(P, P, \Gamma_1, \chi)\, dv(P)$, it follows from Equation (2.8) that

$$\int_{\mathcal{F}_A} \operatorname{tr}_{V^n} H(P, P, \Gamma, \psi)\, dv(P) = \int_{\mathfrak{F}_A} \operatorname{tr}_V H(P, P, \Gamma_1, \chi)\, dv(P) + \underset{A \to \infty}{o(1)}.$$

However, from Equation (2.9), we obtain

$$\begin{aligned} (5.1)\quad & \int_{\mathcal{F}_A} \operatorname{tr}_{V^n} H(P, P, \Gamma, \psi)\, dv(P) \\ &= g(0) k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi_\Gamma'}{\phi_\Gamma}(it) h(1+t^2)\, dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_\Gamma(0)}{4} + \underset{A \to \infty}{o(1)} \end{aligned}$$

and

$$\begin{aligned} (5.2)\quad & \int_{\mathcal{F}_A} \operatorname{tr}_V H(P, P, \Gamma_1, \chi)\, dv(P) \\ &= g(0) k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi_{\Gamma_1}'}{\phi_{\Gamma_1}}(it) h(1+t^2)\, dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0)}{4} + \underset{A \to \infty}{o(1)}. \end{aligned}$$

Hence, we have

$$\begin{aligned} (5.3)\quad & g(0) k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi_\Gamma'}{\phi_\Gamma}(it) h(1+t^2)\, dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_\Gamma(0)}{4} \\ &= g(0) k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi_{\Gamma_1}'}{\phi_{\Gamma_1}}(it) h(1+t^2)\, dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0)}{4} + \underset{A \to \infty}{o(1)}. \end{aligned}$$

Equation (5.3) is true for *all* admissible pairs h, g. Hence we must have

$$k(\Gamma, \psi) = k(\Gamma_1, \chi),$$
$$\operatorname{tr} \mathfrak{S}_\Gamma(0) = \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0),$$

and

$$(5.4)\qquad \frac{\phi_\Gamma'}{\phi_\Gamma}(z) = \frac{\phi_{\Gamma_1}'}{\phi_{\Gamma_1}}(z).$$

We will prove (5.4) shortly.

Theorem 5.1. *Let* $\Gamma_1 \lhd \Gamma$, $\chi \in \operatorname{Rep}(\Gamma_1, V)$, *and* $\psi = U^\chi \in \operatorname{Rep}(\Gamma, V^n)$. *Let*

$$\phi_{\Gamma_1}(s) \equiv \det \mathfrak{S}_{\Gamma_1}(s) \equiv \det \mathfrak{S}(s, \Gamma_1, \chi),$$
$$\phi_{\Gamma}(s) \equiv \det \mathfrak{S}_{\Gamma}(s) \equiv \det \mathfrak{S}(s, \Gamma, \psi).$$

Then for all regular s,

$$\phi_{\Gamma_1}(s) = \phi_\Gamma(s).$$

Proof. For $r > 0, z \in \mathbb{C}$, let $h(z) = e^{-rz^2}$ (and it follows that)

$$g(x) = \frac{e^{-r}}{\sqrt{4\pi r}} e^{-x^2/(4r)}.$$

By considering the asymptotics of (5.3), it follows that

$$k(\Gamma, \psi) = k(\Gamma_1, \chi)$$

and

$$\operatorname{tr} \mathfrak{S}_\Gamma(0) = \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0).$$

Hence

$$\int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) h(1+t^2)\, dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) h(1+t^2)\, dt,$$

which implies that

$$\int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) e^{-rt^2}\, dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) e^{-rt^2}\, dt \quad (r > 0).$$

Now, by the functional equation for $\mathfrak{S}_\Gamma(s)$ (see [Fri05a])

$$\mathfrak{S}_\Gamma(s)\mathfrak{S}_\Gamma(-s) = I.$$

So

$$\phi_\Gamma(s)\phi_\Gamma(-s) = 1.$$

Hence $\frac{\phi'_\Gamma}{\phi_\Gamma}(it)$ is an even function of t; so is $\frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it)$. Thus

$$\int_0^\infty \frac{\phi'_\Gamma}{\phi_\Gamma}(it) e^{-rt^2}\, dt = \int_0^\infty \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) e^{-rt^2}\, dt \quad (r > 0).$$

Next, the substitution $u = t^2$ allows us to rewrite the above integral as a Laplace transform, and by uniqueness (the Laplace transform is invertible), it follows that $\frac{\phi'_\Gamma}{\phi_\Gamma}(it) = \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it)$, and (by analytic continuation)

$$\frac{\phi'_\Gamma}{\phi_\Gamma}(s) = \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(s).$$

Integrating and exponentiating give us a constant C_1 so that

$$\phi_\Gamma = C_1 \cdot \phi_{\Gamma_1}.$$

However, the functional equation (for $\mathfrak{S}_\Gamma(s)$) implies that

$$\phi_\Gamma(0) = (-1)^{(k_\Gamma - \operatorname{tr} \mathfrak{S}_\Gamma(0))/2} = (-1)^{(k_{\Gamma_1} - \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0))/2} = \phi_{\Gamma_1}(0) \neq 0.$$

Hence $C_1 = 1$. □

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