ANALOGUES OF THE ARTIN FACTORIZATION FORMULA
FOR THE AUTOMORPHIC SCATTERING MATRIX
AND SELBERG ZETA-FUNCTION
ASSOCIATED TO A KLEINIAN GROUP

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(Communicated by Wen-Ching Winnie Li)

Abstract. For Kleinian groups acting on a hyperbolic three-space, we prove
factorization formulas for both the Selberg zeta-function and the automorphic
scattering matrix. We extend results of Venkov and Zograf from Fuchsian
groups to Kleinian groups, and we give a proof that is simple and extendable
to more general groups.

1. Introduction

In [VZ83] Venkov and Zograf gave an analogue of Artin’s well-known factor-
ization formula. More specifically, they gave factorization formulas for both the
Selberg zeta-function and automorphic scattering matrix that are associated to
a Fuchsian group (in the context of the Selberg spectral theory of automorphic
functions).

Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group, and let $\Gamma_1$ be a finite-index
normal subgroup of index $n$. Let $\chi \in \text{Rep}(\Gamma_1, V)$ be a finite-dimensional unitary
representation of $\Gamma_1$ in $V$, and let $\psi = U^\chi \in \text{Rep}(\Gamma, V^n)$ be its induced representation
to $\Gamma$.

Let $\mathcal{S}(s, \Gamma_1, \chi), \mathcal{S}(s, \Gamma, \psi)$ be automorphic scattering matrices (defined in §5),
and set
\[
\phi(s, \Gamma_1, \chi) \equiv \det \mathcal{S}(s, \Gamma_1, \chi), \\
\phi(s, \Gamma, \psi) \equiv \det \mathcal{S}(s, \Gamma, \psi).
\]

Theorem. Let $\Gamma$ be a cofinite Kleinian group, and let $\Gamma_1$ be a finite-index normal
subgroup of $\Gamma$. Let $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for all regular $s \in \mathbb{C}$,
\[
\phi(s, \Gamma_1, \chi) = \phi(s, \Gamma, \psi).
\]

In [VZ83] a slightly weaker version of this equation was derived. The authors
showed that
\[
\phi(s, \Gamma_1, \chi) \Omega(\Gamma_1, \chi)^{1-2s} = \phi(s, \Gamma, U^\chi) \Omega(\Gamma, U^\chi)^{1-2s},
\]
where $\Omega(\cdot, \cdot)$ is a constant depending on the group and unitary representation.

We also have

Received by the editors March 2, 2007.
2000 Mathematics Subject Classification. Primary 11F72.
Theorem. Let $\Gamma$ be an arbitrary Kleinian group, and let $\Gamma_1$ be a finite-index normal subgroup of $\Gamma$. Let $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for $\text{Re } s > 1$,

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Here $Z(s, \cdot, \cdot)$ is the Selberg zeta-function (defined in $\S$2.2).

Theorem. Let $\Gamma$ be an arbitrary Kleinian group, and let $\Gamma_1$ be a finite-index normal subgroup of $\Gamma$. Then for $\text{Re } s > 1$,

$$Z(s, \Gamma_1, 1) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_{\vartheta}}.$$

Here $(\Gamma_1 \setminus \Gamma)^*$ is the set of all pairwise inequivalent irreducible unitary representations of the group $\Gamma_1 \setminus \Gamma$; for $\vartheta \in (\Gamma_1 \setminus \Gamma)^*$, $n_\vartheta$ is the dimension of $\vartheta$; $Z(s, \Gamma, \vartheta)$ is the Selberg zeta-function; $1$ is the trivial representation of $\Gamma_1$, and $U^1$ is its induced representation to $\Gamma$.

2. Preliminaries

In this section we state the preliminary results that will be needed. Our main references are [Fri05a, Fri05b, EGM98], and [Ven82], and [Hej76, Hej83]. Unless stated otherwise, throughout this section $\Gamma < \text{PSL}(2, \mathbb{C})$ is a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$ is a finite-dimensional unitary representation of $\Gamma$ in $V$.

2.1. Cofinite Kleinian groups. Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group acting on a hyperbolic three-space $\mathbb{H}^3$. Let $V$ be a finite-dimensional complex inner product space with inner-product $\langle , \rangle_V$, and let $\text{Rep}(\Gamma, V)$ denote the set of finite-dimensional unitary representations of $\Gamma$ in $V$. Let $\mathcal{F} \subset \Gamma$ be a fundamental domain for the action of $\Gamma$ in $\mathbb{H}^3$.

Let $\chi \in \text{Rep}(\Gamma, V)$. The Hilbert space of $\chi$-automorphic functions is the set of measurable functions

$$\mathcal{H}(\Gamma, \chi) = \{ f : \mathbb{H}^3 \to V \mid f(\gamma P) = \chi(\gamma) f(P) \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \text{ and } \langle f, f \rangle \equiv \int_\mathcal{F} \langle f(P), f(P) \rangle_V \, dv(P) < \infty \}. $$

Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Next we briefly define the concept of a singular unitary representation. Let $\mathbb{P}$ be the the boundary of $\mathbb{H}^3$, the Riemann sphere. For every $\zeta \in \mathbb{P}_1$ let $\Gamma_\zeta$ denote the stabilizer subgroup of $\zeta$ in $\Gamma$,

$$\Gamma_\zeta \equiv \{ \gamma \in \Gamma \mid \gamma \zeta = \zeta \},$$

and let $\Gamma'_\zeta$ be the maximal torsion-free parabolic subgroup of $\Gamma_\zeta$ (the maximal parabolic subgroup of $\Gamma_\zeta$ that does not contain elliptic elements). A point $\zeta \in \mathbb{P}_1$ is called a cusp of $\Gamma$ if $\Gamma'_\zeta$ is a free abelian group of rank two. Two cusps $\zeta_1, \zeta_2$ are $\Gamma$-equivalent if $\zeta_1 \in \Gamma \zeta_2$, that is, if their $\Gamma$-orbits coincide.

Every cofinite Kleinian group has finitely many equivalence classes of cusps, so we fix a set $\{ \zeta_\alpha \}_{\alpha = 1}^{\kappa(\Gamma)}$ of representatives of these equivalence classes. For notational convenience we set $\Gamma_{\alpha} \equiv \Gamma_{\zeta_\alpha}$ and $\Gamma'_\alpha \equiv \Gamma'_\zeta_\alpha$.

Each cusp $\zeta_\alpha$ has an associated lattice $\Lambda_\alpha$ (see [EGM98] Theorem 2.1.8). For each cusp $\zeta_\alpha$ of $\Gamma$, define the singular space $V_\alpha \equiv \{ v \in V \mid \chi(\gamma)v = v, \forall \gamma \in \Gamma_\alpha \}$, where $1 \leq \alpha \leq \kappa(\Gamma)$. 

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A representation $\chi \in \text{Rep}$ is singular at the cusp $\zeta_\alpha$ of $\Gamma$ iff the subspace $V_\alpha \neq \{0\}$. If a cusp is not singular, it is called regular.

For each cusp $\zeta_\alpha$, set $k_\alpha = \dim_\C V_\alpha$, and $k(\Gamma, \chi) \equiv \sum_{\alpha} = k_\alpha$.

2.2. Selberg zeta-function. Let $Z(s, \Gamma, \chi)$ denote the Selberg zeta-function associated to $\Gamma$ and $\chi \in \text{Rep}(\Gamma, \chi)$. We allow $\Gamma$ to be an arbitrary Kleinian group. See [Fri05a] and [EGM98, Sections 5.2, 5.4] for the details on its construction.

In [Fri05a], we gave the meromorphic continuation of $Z(s, \Gamma, \chi)$ to the left half plane under certain technical assumptions. A practical way of understanding the zeta-function is via its logarithmic derivative. For $\Re(s) > 1$,

$$
\frac{d}{ds} \log Z(s, \Gamma, \chi) = \sum_{\{P\}\in \infty} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T)|a(T) - a(T)^{-1}|^2} N(T)^{-s}. 
$$

See [Fri05a] for the definition of the notation used above.

Let $W(s, \Gamma, \chi) = \frac{d}{ds} \log Z(s, \Gamma, \chi)$. Then, for $\Re s > 1$,

$$
Z(s, \Gamma, \chi) = e^{\int W(s, \Gamma, \chi) \, ds + C},
$$

where $C$ is chosen so

$$
\lim_{s \to \infty} Z(s, \Gamma, \chi) = 1
$$

will be satisfied.

2.3. Selberg theory of $\Delta$. In this section we assume that $\Gamma$ is cofinite and we state some needed results concerning the Selberg trace formula associated to $\Gamma$ and $\chi \in \text{Rep}(\Gamma, V)$. We will not need the full trace formula found in [Fri05a, Fri05b]. Rather, only parts of its proof will be needed. More details can be found in [Fri05a, Fri05b, EGM98, and Ven82].

For $P = z + rj, \quad P' = z' + r'j \in \H^3$ set

$$
\delta(P, P') \equiv \cosh(d(P, P')) = \frac{|z - z'|^2 + r^2 + r'^2}{2rr'},
$$

where $d$ is the hyperbolic distance in $\H^3$.

For $k \in S([1, \infty))$, a Schwartz-class function, set $K(P, P') = k(\delta(P, P'))$. Note that for any $\gamma \in \text{PSL}(2, \C)$,

$$
K(\gamma P, P') = K(P, P') \quad \text{and} \quad K(P, \gamma P') = K(\gamma^{-1} P, P').
$$

For $\Theta \subset \Gamma, \quad \chi \in \text{Rep}(\Gamma, V)$, define

$$
K(P, P', \Theta, \psi) \equiv \sum_{\gamma \in \Theta} \chi(\gamma) K(P, \gamma P').
$$

The series above converges absolutely and uniformly on compact subsets of $\H^3 \times \H^3$.

For $\lambda \in \C, \quad \lambda = 1 - s^2$, the Selberg–Harish-Chandra transform of $k, h$ is defined by

$$
h(\lambda) = h(1-s^2) = \frac{\pi}{s} \int_1^\infty k \left( \frac{1}{2} \left( t + \frac{1}{t} \right) \right) \left( t^s - t^{-s} \right) \left( t - \frac{1}{t} \right) \frac{dt}{t}, \ \lambda = 1-s^2.
$$

In addition, let

$$
g(x) = \frac{1}{2\pi} \int_{\R} h(1+t^2)e^{-itx} \, dt.
$$
We can start with the function $h$ and work backwards to find $k$ (EGM98, Chapter 3). The pair $h, g$ is said to be admissible if $h$ is a holomorphic function on $\{ s \in \mathbb{C} \mid | \text{Im}(s) | < 2 + \delta \}$ for some $\delta > 0$, satisfying $h(1 + z^2) = O(1 + |z|^2)^{3/2-\epsilon}$ as $|z| \to \infty$.

For $v, w \in V$ let $v \otimes w$ be the linear operator in $V$, defined by $v \otimes w(x) = \langle x, w \rangle v$, where $x \in V$.

**Lemma 2.1** (Fri05b). Let $k \in \mathcal{S}([1, \infty))$ and $h : \mathbb{C} \to \mathbb{C}$ be the Selberg–Harish-Chandra Transform of $k$. Then

$$K(P, Q, \Gamma, \chi) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)}$$

$$+ \frac{1}{4\pi} \sum_{\alpha = 1}^{\kappa} \sum_{l = 1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma_\alpha']}{|A_{\alpha}|} \int_{\mathbb{R}} h(1 + t^2) E_{al}(P, it) \otimes \overline{E_{al}(Q, it)} \, dt.$$  

The sum and integrals converge on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$. Here $D$ is an indexing set of the eigenfunctions $e_m$ of $\Delta$ with corresponding eigenvalues $\lambda_m$, $E_{al}(P, s)$ are the Eisenstein series associated to the singular cusps of $\Gamma$, $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and $|A_{\alpha}|$ is the Euclidean area of a fundamental domain for the lattice $A_{\alpha}$. If a cusp is regular, it is omitted from the sum in (2.6).

Next, we split up $K_{\Gamma}$ as a sum of two kernels. The first kernel,

$$H_{\Gamma}(P, Q) = \frac{1}{4\pi} \sum_{\alpha = 1}^{\kappa} \sum_{l = 1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma_\alpha']}{|A_{\alpha}|} \int_{\mathbb{R}} h(1 + t^2) E_{al}(P, it) \otimes \overline{E_{al}(Q, it)} \, dt,$$

is not of Hilbert-Schmidt class, while the second kernel,

$$L_{\Gamma}(P, Q) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)},$$

is of trace class.

Suppose $Y > 0$ is sufficiently large. Then for all $A > Y$, there exists a compact set $\mathcal{F}_A \subset \mathbb{H}^3$ such that

$$(2.7) \quad \mathcal{F} = \mathcal{F}_A \cup \mathcal{F}_1(A) \cup \cdots \cup \mathcal{F}_A(A)$$

is a fundamental domain for $\Gamma$. The sets $\mathcal{F}_m(A)$ are cusp sectors (see EGM98, Proposition 2.3.9). It follows that

$$(2.8) \quad \lim_{A \to \infty} \left( \int_{\mathcal{F}_A} \text{tr}_V(K_{\Gamma}(P, P)) \, dv(P) - \int_{\mathcal{F}_A} \text{tr}_V(H_{\Gamma}(P, P)) \, dv(P) \right) = \int_{\mathcal{F}} \text{tr}_V(L_{\Gamma}(P, P)) \, dv(P) = \sum_{m \in D} h(\lambda_m) < \infty.$$  

The infinite sum is absolutely convergent.

Let $\mathcal{S}(s)$ denote the automorphic scattering matrix associated to $\Gamma$ and $\chi$ (see Fri05b), and let

$$\phi(s) = \det \mathcal{S}(s).$$
Upon applying the vector form of the Maaß-Selberg relations (see [Roe66], [Ven82], [Fri05b]), we obtain

\[
\int_{\mathcal{F}_A} \text{tr}_V(H_T(P,P)) \, dv(P) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'(it)}{\phi} h(1 + t^2) \, dt + \frac{h(1)}{4} \log(A) - 1 + 4\pi \int_{\mathbb{R}} \phi'(it) \, h(1 + t^2) \, dt + o(1) .
\]

The integral on the right-hand side converges absolutely.

3. INDUCED REPRESENTATIONS

Let \( \Gamma \) be an arbitrary Kleinian group, and let \( \Gamma_1 \triangleleft \Gamma \) be a finite-index normal subgroup of index \( n \). Let \( \mathcal{F}, \mathfrak{F} \) be fundamental domains of \( \Gamma, \Gamma_1 \) respectively, with \([\alpha_i]_{i=1}^n \) a complete set of representatives for the right-cosets of \( \Gamma_1 \setminus \Gamma \), satisfying

\[
\mathfrak{F} = \bigcup_{i=1}^n \alpha_i(\mathcal{F}) .
\]

Let \( V \) be a finite-dimensional hermitian vector space, and let \( \chi \) be a finite-dimensional unitary representation of \( \Gamma_1 \) in \( V \). Set

\[
\mathfrak{X}(\gamma) = \begin{cases} 
\chi(\gamma), & \gamma \in \Gamma_1, \\
0, & \gamma \notin \Gamma_1 .
\end{cases}
\]

Let \( \psi \equiv U^x \) be the induced representation of \( \chi \) from \( \Gamma_1 \) to \( \Gamma \). More explicitly,

\[
\psi : \Gamma \mapsto \text{GL}(V^n) ,
\]

and for \( \gamma \in \Gamma \) and \( v_i \in V \),

\[
\psi(\gamma) \left( \sum_{i=1}^n \oplus v_i \right) = \sum_{i=1}^n \oplus \sum_{j=1}^n \mathfrak{X}(\alpha_i \gamma \alpha_i^{-1}) v_j .
\]

It follows from (3.2) that for \( \gamma \in \Gamma \),

\[
\text{tr}_{V^n} \psi(\gamma) = \sum_{i=1}^n \text{tr}_V \mathfrak{X}(\alpha_i \gamma \alpha_i^{-1}) .
\]

We will need the following result (see [RS71] and [VZ83, Theorem 2.1]):

**Lemma 3.1.** There exists an isometry between the Hilbert spaces \( \mathcal{H}(\Gamma_1, \chi) \) and \( \mathcal{H}(\Gamma, \psi) \), which takes the operator \( \Delta(\Gamma_1, \chi) \) to \( \Delta(\Gamma, \psi) \).

We abuse notation and call a set \( \Theta \subset \Gamma \) normal if for all \( \gamma \in \Gamma \), we have \( \gamma \Theta \gamma^{-1} = \Theta \).

**Lemma 3.2.** Let \( \Gamma_1 \triangleleft \Gamma \) be a finite-index, normal subgroup of \( \Gamma \) of index \( n \), \( \chi \in \text{Rep}(\Gamma_1, V) \), \( \psi \equiv U^x \in \text{Rep}(\Gamma, V^n) \), and let \( \Theta \subset \Gamma \) be a normal subset. Suppose that

\[
\int_{\mathcal{F}} \text{tr}_{V^n} K(P,P,\Theta,\psi) \, dv(P)
\]

converges absolutely. Then

\[
\int_{\mathcal{F}} \text{tr}_{V^n} K(P,P,\Theta,\psi) \, dv(P) = \int_{\mathfrak{F}} \text{tr}_V K(P,P,\Theta \cap \Gamma_1, \chi) \, dv(P) .
\]
Proof. By (3.3),
\[
\int_{\mathcal{F}} \text{tr}_{V^n} K(P, P, \Theta, \psi) \, dv(P)
\]
\[= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta} \text{tr}_{V} (\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P) \]
\[= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\alpha_i \gamma \alpha_i^{-1} \in \Gamma_1} \text{tr}_{V} (\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P) \]
\[= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \text{tr}_{V} (\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P) \quad \text{(by normality).} \]

Now, since \( \Theta \) and \( \Gamma_1 \) are normal, \( \gamma \in \Theta \cap \Gamma_1 \) implies that \( \alpha_i \gamma \alpha_i^{-1} \in \Theta \cap \Gamma_1 \). Also, as \( \gamma \) goes through each element of \( \Theta \cap \Gamma_1 \), so does \( \alpha_i \gamma \alpha_i^{-1} \). So, for each \( i \), setting \( \beta = \alpha_i \gamma \alpha_i^{-1} \) we obtain
\[\sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \text{tr}_{V} (\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P) \]
\[= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_{V} (\beta) K(P, \alpha_i^{-1} \beta \alpha_i P) \, dv(P) \]
(3.4)
\[= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_{V} (\beta) K(P, \beta \alpha_i P) \, dv(P) \]
(3.5)
\[= \sum_{i=1}^{n} \int_{\alpha_i(\mathcal{F})} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_{V} (\beta) K(Q, \beta Q) \, dv(Q) \]
(3.6)
\[= \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_{V} (\beta) K(Q, \beta Q) \, dv(Q) \]
\[= \int_{\mathcal{F}} \text{tr}_{V} K(P, P, \Theta \cap \Gamma_1, \chi) \, dv(P), \]
where in (3.5) we set \( Q = \alpha_i P \). In (3.4) we used Equation (2.3), and we used the fact that \( dv(\alpha_i Q) = dv(Q) \); and in (3.6) we tiled the fundamental domain \( \mathcal{F} \) according to (3.1).
\[\Box\]

Lemma 3.2 works well when the point-pair-invariant \( k \) gives rise to an integral kernel of trace-class. However, a careful look at the proof shows that we could replace \( \mathcal{F} \) by a truncated fundamental domain \( \mathcal{F}_A \).

For the rest of this section, we assume that \( \Gamma \) is cofinite. Let \( Y > 0 \) be sufficiently large so that for \( A > Y \), \( \mathcal{F} \) decomposes into \( \mathcal{F} = \mathcal{F}_A \cup \mathcal{F}^A \), where \( \mathcal{F}_A \) is compact and \( \mathcal{F}^A \) is a union of cusp sectors (2.7). Since
\[\mathcal{F} = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F}) \]
it follows that \( \mathcal{F}_A = \mathcal{F}_A \cup \mathcal{F}^A \) with \( \mathcal{F}_A = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F}_A) \) and \( \mathcal{F}^A = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F}^A) \); \( \mathcal{F}^A \) is also a union of cusp sectors.
Lemma 3.3. Let $\Gamma_1 \triangleleft \Gamma$ be a finite-index, normal subgroup of $\Gamma$ of index $n$, $\chi \in \text{Rep}(\Gamma_1, V)$, $\psi \equiv U^\chi \in \text{Rep}(\Gamma, V^n)$, and let $\Theta \subset \Gamma$ be a normal subset. Suppose that
\[ \int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Theta, \psi) \, dv(P) \]
converges absolutely. Then
\[ \int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Theta, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_V K(P, P \cap \Gamma_1, \chi) \, dv(P). \]

4. Factorization formula of the Selberg zeta-function

In this section we prove the analogue of the Venkov-Zograf factorization formula (\cite{VZ83}) for arbitrary Kleinian groups.

Throughout this section, $\Gamma$ is an arbitrary Kleinian group and $\Gamma_1 \triangleleft \Gamma$ is a finite-index normal subgroup of index $n$.

Theorem 4.1. Suppose that $\Gamma_1 \triangleleft \Gamma$, $\chi \in \text{Rep}(\Gamma_1, V)$, and that $\psi = U^\chi \in \text{Rep}(\Gamma, V^n)$. Then for $\text{Re}(s) > 1$,
\[ Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi). \]

Proof. By (2.2), it suffices to show that
\[ W(s, \Gamma_1, \chi) = W(s, \Gamma, \psi). \]
As $\Theta = \Gamma_{\text{lox}} \subset \Gamma$ be the set of all loxodromic elements of $\Gamma$. Note that $\Theta$ is normal since the conjugate of a loxodromic element is loxodromic. For $\text{Re} s > 1$, $\delta > 1$, set
\[ k_s(\delta) \equiv \frac{1}{4\pi} \frac{(\delta + \sqrt{\delta^2 - 1})^{-s}}{\sqrt{\delta^2 - 1}}. \]
For $P, P' \in \mathbb{H}^3$, set $K_s(P, P') = k_s(\delta(P, P'))$, and, using (2.3), define $K_s(P, P', \Theta, \psi)$ and $K_s(P, P', \Theta \cap \Gamma_1, \chi)$. By \cite{EGM98} pages 185-198 and \cite{Fri05} Lemma 6.3, it follows that
\[ W(s, \Gamma_1, \chi) = \int_{\mathcal{F}} \text{tr}_V K_s(P, P \cap \Gamma_1, \chi) \, dv(P) \]
and
\[ W(s, \Gamma, \psi) = \int_{\mathcal{F}} \text{tr}_{V^n} K_s(P, P, \Theta, \psi) \, dv(P). \]
The result now follows from Lemma 3.2. \qed

Compare our proof with the proof given in \cite{VZ83} Theorem 3.1.

If we let $\chi = 1$, the trivial one-dimensional representation, it follows that $\psi = U^1$ can be decomposed into irreducible sub-representations, explicitly:
\[ \psi = \bigoplus_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} n_\vartheta \vartheta, \]
where $(\Gamma_1 \backslash \Gamma)^*$ denotes the set of all pairwise inequivalent irreducible unitary representations of the group $\Gamma_1 \backslash \Gamma$, and $n_\vartheta$ is the dimension of $\vartheta$. For more details see \cite{Kir76}.

Next, by (2.4) and (2.2) it follows that for $\vartheta_1, \vartheta_2 \in \text{Rep}(\Gamma, V_i)$ ($i = 1, 2$),
\[ Z(s, \Gamma, \vartheta_1 \oplus \vartheta_2) = Z(s, \Gamma, \vartheta_1) Z(s, \Gamma, \vartheta_2). \]

\[1\] Of course, each $\chi$ must be extended from $\Gamma_1 \backslash \Gamma$ to $\Gamma$. 

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We have

**Theorem 4.2.** Let $\Gamma_1 \subset \Gamma$. Then for $\text{Re} s > 1$,

$$Z(s, \Gamma_1, 1) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_{\vartheta}}.$$ 

5. **Factorization formula for the determinant of the scattering matrix**

In this section we prove the analogous result to Theorem 4.1 for the automorphic scattering matrix. Throughout this section, $\Gamma$ is a cofinite Kleinian group.

Let $\Gamma_1 \subset \Gamma$, $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^{\chi}$. Let $k \in \mathcal{S}([1, \infty))$, and let $h$ be given by (2.5). It follows form Lemma 3.1 that $\Delta(\Gamma_1, \chi)$ and $\Delta(\Gamma, \psi)$ have the same eigenvalues. Hence (from (2.3))

$$\int_{\mathcal{F}} \text{tr}_{V^n} L(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}} \text{tr}_{V} L(P, P, \Gamma_1, \chi) \, dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m).$$

Now, since $\int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_{V} K(P, P, \Gamma_1, \chi) \, dv(P)$, it follows from Equation (2.3) that

$$\int_{\mathcal{F}_A} \text{tr}_{V^n} H(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_{V} H(P, P, \Gamma_1, \chi) \, dv(P) + o(1).$$

However, from Equation (2.9), we obtain

$$\int_{\mathcal{F}_A} \text{tr}_{V^n} H(P, P, \Gamma_1, \chi) \, dv(P) = g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathcal{R}} \frac{\phi_{\Gamma_1}(it)}{\bar{\phi}_{\Gamma_1}} h(1 + t^2) \, dt + \frac{h(1)\text{tr} \mathcal{S}_{\Gamma_1}(0)}{4} + o(1)_{A \to \infty}.$$

and

$$\int_{\mathcal{F}_A} \text{tr}_{V} H(P, P, \Gamma_1, \chi) \, dv(P) = g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathcal{R}} \frac{\phi_{\Gamma_1}(it)}{\bar{\phi}_{\Gamma_1}} h(1 + t^2) \, dt + \frac{h(1)\text{tr} \mathcal{S}_{\Gamma_1}(0)}{4} + o(1)_{A \to \infty}.$$

Hence, we have

$$g(0)k(\Gamma, \psi) \log(A) = g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathcal{R}} \frac{\phi_{\Gamma}(it)}{\bar{\phi}_{\Gamma}} h(1 + t^2) \, dt + \frac{h(1)\text{tr} \mathcal{S}_{\Gamma}(0)}{4} + o(1)_{A \to \infty}.$$

Equation (5.3) is true for all admissible pairs $h, g$. Hence we must have

$$k(\Gamma, \psi) = k(\Gamma_1, \chi),$$

$$\text{tr} \mathcal{S}_{\Gamma}(0) = \text{tr} \mathcal{S}_{\Gamma_1}(0),$$

and

$$\frac{\phi_{\Gamma}(z)}{\phi_{\Gamma_i}(z)} = \frac{\phi_{\Gamma_1}(z)}{\phi_{\Gamma_1}(z)}.$$

We will prove (5.4) shortly.
Theorem 5.1. Let $\Gamma_1 \lhd \Gamma$, $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^x \in \text{Rep}(\Gamma, V^n)$. Let
\[ \phi_{\Gamma_1}(s) \equiv \det S_{\Gamma_1}(s) \equiv \det S(s, \Gamma_1, \chi), \]
\[ \phi_{\Gamma}(s) \equiv \det S_{\Gamma}(s) \equiv \det S(s, \Gamma, \psi). \]
Then for all regular $s$,
\[ \phi_{\Gamma_1}(s) = \phi_{\Gamma}(s). \]

Proof. For $r > 0$, $z \in \mathbb{C}$, let $h(z) = e^{-rz^2}$ (and it follows that)
\[ g(x) = \frac{e^{-r}}{\sqrt{4\pi r}} e^{-x^2/(4r)}. \]
By considering the asymptotics of (5.3), it follows that
\[ k(\Gamma, \psi) = k(\Gamma_1, \chi) \]
and
\[ \text{tr } S_{\Gamma}(0) = \text{tr } S_{\Gamma_1}(0). \]
Hence
\[ \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}} h(1 + t^2) \, dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)h}{\phi_{\Gamma}}(1 + t^2) \, dt, \]
which implies that
\[ \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}} e^{-rt^2} \, dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}} e^{-rt^2} \, dt \quad (r > 0). \]
Now, by the functional equation for $S_{\Gamma}(s)$ (see [Fri05a])
\[ \mathcal{S}_\Gamma(s)\mathcal{S}_\Gamma(-s) = I. \]
So
\[ \phi_{\Gamma}(s)\phi_{\Gamma}(-s) = 1. \]
Hence $\frac{\phi_{\Gamma}'(it)}{\phi_{\Gamma}}(it)$ is an even function of $t$; so is $\frac{\phi_{\Gamma}'_{\Gamma}(it)}{\phi_{\Gamma}}_{\Gamma}(it)$. Thus
\[ \int_{0}^{\infty} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}} e^{-rt^2} \, dt = \int_{0}^{\infty} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}} e^{-rt^2} \, dt \quad (r > 0). \]
Next, the substitution $u = t^2$ allows us to rewrite the above integral as a Laplace transform, and by uniqueness (the Laplace transform is invertible), it follows that
\[ \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}}(it) = \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}}(it), \]
and (by analytic continuation)
\[ \frac{\phi'_{\Gamma}(s)}{\phi_{\Gamma}}(s) = \frac{\phi'_{\Gamma}(s)}{\phi_{\Gamma}}(s). \]
Integrating and exponentiating give us a constant $C_1$ so that
\[ \phi_{\Gamma} = C_1 \cdot \phi_{\Gamma_1}. \]
However, the functional equation (for $\mathcal{S}_\Gamma(s)$) implies that
\[ \phi_{\Gamma}(0) = (-1)^{(k_\Gamma - \text{tr } \mathcal{S}_\Gamma(0))/2} (-1)^{(k_{\Gamma_1} - \text{tr } \mathcal{S}_{\Gamma_1}(0))/2} = \phi_{\Gamma_1}(0) \neq 0. \]
Hence $C_1 = 1$. \qed
Acknowledgements

I would like to thank Professor Leon Takhtajan for originally suggesting this problem to me and for reading this paper. I would also like to thank Peter Zograf for reading this paper and for useful discussions, and the anonymous referee for correcting some errors.

References


