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ABSOLUTELY BOUNDED MATRICES AND UNCONDITIONAL CONVERGENCE

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ABSTRACT. We characterize the so-called absolutely bounded matrices in terms of the (strong) unconditional convergence of their standard decompositions. There is a similar characterization of absolutely compact matrices, and both characterizations are closely related to some natural multiplication operators. Rudiments of the duality theory for the algebra of all absolutely bounded matrices are included.

1. INTRODUCTION

We shall consider infinite matrices as complex functions on $\mathbb{N} \times \mathbb{N}$. The commutative algebra of all bounded complex functions on $\mathbb{N} \times \mathbb{N}$, i.e. the algebra of all matrices with uniformly bounded entries equipped with Schur or the entrywise product, will be denoted by B_∞. Although this product is usually denoted by $x * y$ or $x \circ y$, we shall simply write xy, as is used for the product of functions and, when necessary, use the notation $x \circ y$ for the ordinary matricial product. Other operations on matrices will also be taken entrywise, e.g. $|x| = (|x_{ij}|)$ if $x = (x_{ij})$. Of course, $\|x\|_\infty = \sup\{|x_{ij}|;\ i, j \geq 1\}$ is the natural norm on B_∞.

We are interested in subalgebras of B_∞ connected with operators. Let us denote by B the set of all infinite complex matrices $x = (x_{ij})$ which represent bounded linear operators on l^2 via the usual matrix multiplication. We shall write $\|x\|$ for the operator norm of $x \in B$. Those elements in B which give compact operators on l^2 will be denoted by K and its Banach dual, the trace class, by T.

By a *component* of an infinite matrix x we mean the matrix with only one nonzero entry taken from x, i.e. $x_{(i,j)} = x_{ij} e_{(i,j)}$ for $i, j \in \mathbb{N}$ where $e_{(i,j)}$ is the usual matrix unit. Choose a square enumeration of the matrix units; that is, arrange them in the usual order: $e_{(1,1)},\ e_{(2,1)},\ e_{(2,2)},\ e_{(1,2)},\ e_{(3,1)},\ e_{(3,2)}, \ldots$. Always using this order we can write matrix units with a single index: $e_{(k)},\ k = 1, 2, \ldots$. Then the appropriate components of an arbitrary matrix x are denoted simply by $x_{(k)}$, $k = 1, 2, \ldots$. The series $x = \sum_k x_{(k)}$ will be called *the standard decomposition* of x.

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In the following three lemmas we rewrite known facts (folklore results) about infinite matrices in the new notation and give no proof, which can be found elsewhere.

Lemma 1.1. *Let x be an infinite matrix with components $x_{(k)}$, $k = 1, 2, \ldots$. Then the following assertions are equivalent:*
(i) $x \in B$;
(ii) $\sup_{n\geq 1} \|\sum_{k=1}^{n} x_{(k)}\| < \infty$;
(iii) *the series $x = \sum_k x_{(k)}$ converges in the strong operator topology in B.*

Moreover, if $x \in B$, then $\|x\| = \sup_n \|\sum_{k=1}^{n^2} x_{(k)}\|$.

Note that for $x \in K$ the series $x = \sum_k x_{(k)}$ is really a Schauder decomposition since it can easily be seen that in this case the series is norm convergent.

Lemma 1.2. *An infinite matrix x belongs to K if and only if the standard decomposition $x = \sum_k x_{(k)}$ converges in the operator norm $\|\cdot\|$.*

An analoguous result holds for the trace class operators.

Lemma 1.3. *An infinite matrix y belongs to T if and only if the standard decomposition $y = \sum_k y_{(k)}$ converges in the trace norm $\|\cdot\|_T$.*

Our main idea is to explore in a similar way the *unconditional* convergence of the standard decomposition in various operator topologies.

An infinite complex matrix (x_{ij}) such that $(|x_{ij}|)$ defines a bounded linear operator on l^2 is usually said to be absolutely bounded (see, e.g., [5]). Besides giving a general description of absolutely bounded matrices and introducing a related concept of absolutely compact matrices, in Section 2 we characterize both kinds of matrices in terms of Schur multiplication by matrices with uniformly bounded elements (see Theorems 2.5 and 2.7). Then we show in Section 3 that a matrix x is absolutely compact (respectively, absolutely bounded) if and only if the standard decomposition is norm (respectively, strongly) unconditionally convergent (see Theorems 3.1 and 3.2). This answers a question of Sunder [5] about the characterization of absolutely bounded matrices. We relate operator properties of infinite matrices to some of the well known notions from the theory of unconditional summability. Following the same analogy we give some natural characterizations of absolutely bounded and absolutely compact matrices in terms of various multiplication operators (Theorem 3.3). All these observations are quite elementary, but they shed, perhaps, a new light onto the otherwise classical material about infinite matrices. In Section 4 we briefly consider a little less familiar unconditional convergence of the standard decomposition in the trace norm (which is dual to the operator norm) and in a weaker norm (which is dual to the absolutely bounded norm).

2. Absolute boundedness

Let us call a matrix $x = (x_{ij})$ *absolutely bounded* if $|x| = (|x_{ij}|) \in B$, and let us denote by B_a the set of all absolutely bounded matrices. We shall write $|x| \leq |y|$ if $|x_{ij}| \leq |y_{ij}|$ for every $i, j \in \mathbb{N}$. Also, define $\|x\|_a = \|\,|x|\,\|$ for $x \in B_a$ and note that $\|x\|_a = \|x\|$ if $x \geq 0$ (i.e. if $x_{ij} \geq 0$ for every $i, j \in \mathbb{N}$).

Proposition 2.1. *Let $b \in B_\infty$, $y, z \in B_a$ and $|x| \leq |y|$. Then we have:*
(a) $x \in B_a$ *and* $\|x\| \leq \|x\|_a \leq \|y\|_a$;
(b) $y + z \in B_a$ *and* $\|y + z\|_a \leq \|y\|_a + \|z\|_a$;

(c) $yz \in B_a$ *and* $\|yz\|_a \leq \|y\|_a \, \|z\|_a$;
(d) $by \in B_a$ *and* $\|by\|_a \leq \|b\|_\infty \, \|y\|_a$.

Proof. First of all, since $|y| \in B$ and $|x| \leq |y|$, then (by definition of the operator norm of a matrix x acting in the usual way on a vector $\zeta \in l^2$) we have

$$\|x\zeta\|_2 \leq \| \, |x||\zeta| \, \|_2 \text{ and } \| \, |x|\zeta \, \|_2 \leq \| \, |x||\zeta| \, \|_2 \leq \| \, |y||\zeta| \, \|_2 \tag{1}$$

where $|\zeta| = (|\zeta_k|)$ if $\zeta = (\zeta_k) \in l^2$. Hence, (a) follows from (1) and (b) follows directly from (a) as $|y + z| \leq |y| + |z|$ holds for every y, z. The submultiplicativity of the norm in (c) is a well known inequality for Schur products (see, e.g., Satz III in [4]) and easily obtained directly. Finally, (d) can be immediately derived from the definition of the operator norm. □

By Proposition 2.1(a) every absolutely bounded matrix belongs to B, i.e. $B_a \subset B$, and by (b) and (c), B_a is a subalgebra in B. Moreover, by (d) B_a is an ideal in B_∞ with respect to Schur multiplication. Hence, B_a is also a Schur ideal in B (as $\|xy\|_a \leq \|x\|_\infty \|y\|_a \leq \|x\| \|y\|_a$ for $x \in B$, $y \in B_a$). By (a), (b), and (c) the function $\|\cdot\|_a$ defines an algebra norm, usually called the *absolute bounded norm*, on B_a, so that B_a is a normed algebra. It is useful to relate this norm to the standard decomposition.

Proposition 2.2. *Let* $x = \sum_k x_{(k)}$ *and* $y = \sum_k y_{(k)}$ *be the standard decompositions for* $x, y \in B_a$. *Then the following is true:*
(a) $|x| = \sum_k |x_{(k)}|$ *is the standard decomposition for* $|x|$;
(b) $\| \sum_{k=1}^n x_{(k)} \|_a \leq \|x\|_a$ *for every* $n \in \mathbb{N}$;
(c) $\|x - \sum_{k=1}^n x_{(k)}\|_a = \| \, |x| - \sum_{k=1}^n |x_{(k)}| \, \|$ *for every* $n \in \mathbb{N}$;
(d) $\|x - \sum_{k=1}^n x_{(k)}\|_a \leq \|y - \sum_{k=1}^n y_{(k)}\|_a$ *for every* $n \in \mathbb{N}$ *if* $|x| \leq |y|$.

Proof. All assertions follow easily from definitions or Proposition 2.1. Note, for example, that $|x_{(k)}| = |x|_{(k)}$ in (a), $| \sum_{k=1}^n x_{(k)}| = \sum_{k=1}^n |x|_{(k)} \leq |x|$ in (b), and $|x - \sum_{k=1}^n x_{(k)}| = | \sum_{k=n+1}^\infty x_{(k)}| = \sum_{k=n+1}^\infty |x|_{(k)} = |x| - \sum_{k=1}^n |x|_{(k)}$ in (c) and (d). □

Note that it follows from (c) that $x = \sum_k x_{(k)}$ is a Schauder decomposition of x in B_a if and only if $|x| = \sum_k |x|_{(k)}$ is a Schauder decomposition of $|x|$ in B (or in B_a).

The next result is proved in [1] in a more general situation. So we shall give no direct proof here.

Theorem 2.3. B_a *is a Banach algebra in the norm* $\| \cdot \|_a$.

Proof. One can produce the proof along the standard lines; however, see also [1], Theorem 3.2, the case $r = 1$. □

Remark 2.4. Note that B_a is not closed in the usual operator norm in B, as we see from the example $u = \bigoplus_{n=1}^\infty u_n / \sqrt[4]{n}$ where $u_n = \frac{1}{\sqrt{n}} (\omega^{jk})_{j,k=1}^n$, ω being a primitive n-th root of the unity (see Example 2.8 in [3]). Then u_n is a unitary matrix while $|u_n|$ is a positive matrix with all entries equal to $1/\sqrt{n}$, and so $\| \, |u_n| \, \| = \sqrt{n}$ for every n. It follows that $\| \, u - \bigoplus_{n=1}^m u_n / \sqrt[4]{n} \, \| \to 0$ as $m \to \infty$ and, hence, $u \in K \subset B$ while $|u| = \bigoplus_{n=1}^\infty |u_n| / \sqrt[4]{n} \notin B$, since $\| |u_n| \| / \sqrt[4]{n} = \sqrt[4]{n} \to \infty$ as $n \to \infty$ and, consequently, $u \notin B_a$.

Our first characterization of the elements in B_a is given in terms of the Schur product for various kinds of infinite matrices. By χ_S we denote the matrix which is the characteristic function of a subset $S \subset \mathbb{N} \times \mathbb{N}$.

Theorem 2.5. *Let x be an infinite matrix. The following assertions are equivalent:*
(i) $x \in B_a$*;*
(ii) $bx \in B$ for every $b \in B_\infty$*;*
(iii) $x\chi_S \in B$ for every subset $S \subset \mathbb{N} \times \mathbb{N}$*;*
(iv) $cx \in B$ for every $c = (c_{ij})$ with $c_{ij} \in \{-1, 1\}$ for each (i, j).

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 2.1(d), (ii) $\Longrightarrow$ (iii) is trivial and (iii) $\Longrightarrow$ (iv) is clear, since for every $c = (c_{ij}) \in B_\infty$ with $c_{ij} \in \{-1, 1\}$ for each (i, j), we have $x(c + e) \in B$ where e is the matrix with each entry equal to 1 (note that $xe = x \in B$). Finally, if (iv) holds, then not only $cx \in B$ but also $cy \in B$ and $cz \in B$ for every ± 1 matrix c, where y and z are the real and the imaginary parts of x. So, $|y|, |z| \in B$ and, since $|x| \le |y| + |z|$, (i) follows by Proposition 2.1(a). □

Note that in the case $x \in B_a$ we have in fact $bx \in B_a$ for every $b \in B_\infty$ and that both assertions are equivalent.

In the same way as we introduced absolutely bounded matrices we can also define $K_a = \{x; |x| \in K\}$. Elements in K_a will be called *absolutely compact* matrices. Obviously, $K_a \subset B_a$. Let F be the set of all matrices of finite rank and F_0 the set of all matrices with finitely many nonzero components. We shall denote the closure of a subset $M \subset B_a$ in B or B_a by $\overline{M}$ and $\overline{M}^a$, respectively.

The following proposition is almost entirely proven in [3] in slightly different notation (see Theorem 2.6 and Example 2.8).

Proposition 2.6. *Let F_0, F and K_a be as above. Then we have:*
(a) *K_a is a closed Schur ideal in B_a;*
(b) *$F_0 \subset F \subset K_a \subset K \cap B_a \subset K$ with all inclusions proper;*
(c) *$\overline{F_0} = \overline{F} = K$ and $\overline{F_0}^a = \overline{F}^a = K_a$.*

Proof. Only (a) needs a justification. If $x \in B_a$ and $y \in K_a$, then $xy \in B_a$. By Proposition 2.2(c) we have $\||xy| - \sum_{k=1}^n |x_{(k)}y_{(k)}|\| = \|xy - \sum_{k=1}^n x_{(k)}y_{(k)}\|_a$ $= \|x(y - \sum_{k=1}^n y_{(k)})\|_a \le \|x\|_a \|y - \sum_{k=1}^n y_{(k)}\|_a = \|x\|_a \||y| - \sum_{k=1}^n |y_{(k)}|\|$. Since the right-hand side converges to 0 by Lemma 1.2, we see, by Lemma 1.2 again, that $|xy| \in K$ or $xy \in K_a$, and the set K_a is a Schur ideal in B_a. It is closed in B_a as we see by using the inequality $\||x| - |y|\| \le \|x - y\|_a$ for $x, y \in B_a$. □

Note that $K \cap B_a$ is also a Schur ideal in B_a and that it is closed in B_a. It would be nice to have an independent description of its elements. We now give a characterization of elements in K_a.

Theorem 2.7. *Let x be an infinite matrix. The following assertions are equivalent:*
(i) $x \in K_a$*;*
(ii) *$bx \in K$ for every $b \in B_\infty$;*
(iii) *$x\chi_S \in K$ for every subset $S \subset \mathbb{N} \times \mathbb{N}$;*
(iv) *$cx \in K$ for every $c = (c_{ij})$ with $c_{ij} \in \{-1, 1\}$ for each (i, j).*

Proof. If $x \in K_a$, we have $\|x - \sum_{k=1}^n x_{(k)}\|_a = \| \, |x| - \sum_{k=1}^n |x|_{(k)}\| \to 0$ for $n \to \infty$ by Lemma 1.2 and Proposition 2.2(c) and also $\|b(x - \sum_{k=1}^n x_{(k)})\| \to 0$ for every

$b \in B_\infty$ by Proposition 2.1. But $b\sum_{k=1}^n x_{(k)} = \sum_{k=1}^n (bx)_{(k)} \in F_0$ and, hence, $bx \in \overline{F_0} = K$. Thus, (i) implies (ii). Other implications follow in the same way as in the proof of Theorem 2.5. □

Again, $x \in K_a$ if and only if $bx \in K_a$ for every $b \in B_\infty$. Also, since $bx \in K$ for every $b \in B_\infty$ only if $x \in K_a$, and since $K_a \neq K$ by Proposition 2.6(b), we see that K is not a Schur ideal in B_∞.

3. Unconditional convergence

Recall that the unconditional convergence of a series $x = \sum_k x_k$ in a Banach space means that for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$ the series $x = \sum_k x_{\sigma(k)}$ converges in the norm of the given space. There are several other characterizations of this notion including subseries convergence, sign convergence, bounded multiplier convergence, etc., and also some of the appropriate weak notions, such as weak subseries convergence or weak sign convergence, are equivalent to the unconditional convergence in norm (the proof is nontrivial; see [2], Chapter 1).

The following simple theorem gives a characterization of the elements of K_a in terms of norm convergence of the standard decomposition.

Theorem 3.1. *For a matrix x with the standard decomposition $x = \sum_k x_{(k)}$ the following assertions are equivalent:*
(i) *$x \in K_a$;*
(ii) *$x = \sum_k x_{(k)}$ is unconditionally convergent in the operator norm;*
(iii) *$|x| = \sum_k |x|_{(k)}$ is convergent in the operator norm.*

Proof. If $x \in K_a$, then $bx \in K$ for every $b \in B_\infty$ by Theorem 2.7. Hence, the series $bx = \sum_k b_{(k)} x_{(k)}$ converges in K for every $b \in B_\infty$. This is equivalent to the unconditional convergence of the series $x = \sum_k x_{(k)}$ in K by Theorem 1.6 in [2]. Further, (iii) follows immediately from (ii) since there is $b \in B_\infty$ such that $|x| = bx$. Finally, the convergence in (iii) obviously implies $|x| \in K$ by Lemma 1.2. □

With Theorem 3.1 in mind, we can easily explain Theorem 2.7 in terms of the unconditional convergence: (ii) corresponds to bounded multiplier convergence of the standard decomposition in norm, (iii) means norm subseries convergence, and (iv) is related to sign convergence (see Theorem 1.9 in [2]).

Replacing norm convergence with convergence in the strong operator topology one obtains an analoguous characterization for B_a (instead of K_a). Moreover, this is still true if one replaces the strong operator topology with the weak, or even the ultraweak, topology on B.

Before stating the theorem note that in [3] a matrix $x = (x_{ij})$ is called *absolutely summable* if $\sum_{i,j} |x_{ij}| < \infty$ and that the set of all absolutely summable matrices is denoted by AS. It follows that for $x \in AS$ the standard decomposition is absolutely convergent in the operator norm, and hence, for example, $AS \subset K_a$. The space AS is a Banach algebra in the norm $\|x\|_{AS} = \sum_{i,j} |x_{ij}|$; in fact, $AS = l^1(\mathbb{N} \times \mathbb{N})$. Note also that AS is a predual of B_∞ and the dual of $B_0 = \{x = (x_{ij});\ \lim_{n\to\infty} \|x - \sum_{i,j=1}^n x_{(i,j)}\|_\infty = 0\}$. For later use we define the summing functional t on AS by $t(z) = \sum_{i,j} z_{ij}$, $z \in AS$.

Theorem 3.2. *For a matrix x with the standard decomposition $x = \sum_k x_{(k)}$ the following assertions are equivalent:*

(i) $x \in B_a$;
(ii) $x = \sum_k x_{(k)}$ *is unconditionally convergent in the strong (resp. weak, ultraweak) operator topology;*
(iii) $|x| = \sum_k |x|_{(k)}$ *is convergent in the strong (resp. weak, ultraweak) operator topology;*
(iv) $\sum_{i,j} |\eta_i x_{ij} \xi_j| < \infty$ *for every* $\xi = (\xi_i), \eta = (\eta_j) \in l^2$;
(v) $\sum_{i,j} |x_{ij} y_{ij}| < \infty$ *for every* $y = (y_{ij}) \in T$.

Proof. If $x \in B_a$, then $bx \in B$ for every $b \in B_\infty$ by Theorem 2.5. Hence, for each $\zeta \in l^2$ the series $bx\zeta = \sum_k b_{(k)} x_{(k)} \zeta$ converges in l^2 for every $b \in B_\infty$. This is equivalent to the unconditional convergence of the series $x\zeta = \sum_k x_{(k)}\zeta$ in l^2 by Theorem 1.6 in [2]. Further, (iii) follows immediately from (ii) since there is $b \in B_\infty$ such that $|x| = bx$, and the convergence in (iii) implies $|x| \in B$ by Lemma 1.1. Hence, (i) and the "strong" parts of (ii) and (iii) are equivalent. Moreover, (iv) can be interpreted as the weak unconditional convergence of the series $\sum_k x_{(k)}\xi$ for every $\xi \in l^2$, i.e. the unconditional convergence of the scalar series $\sum_k \langle x_{(k)}\xi, \eta\rangle$ for every pair $\xi, \eta \in l^2$, which is of course the same as the absolute convergence of this series. Thus, (iv) is a consequence of (ii). On the other hand, (iv) is the same as $\sum_{i,j} \langle |x|_{(i,j)} |\xi|, |\eta|\rangle = \sum_k \langle |x|_{(k)} |\xi|, |\eta|\rangle < \infty$ for every $\xi, \eta \in l^2$. Since this means that we have the (unconditional) weak operator convergence in (iii), it follows that $|x| \in B$, i.e. (i) holds. Finally, note that if $x \in B_a$, $y \in T$ and if we take $b \in B_\infty$ such that $by = |y|$, then we have $b|x| \in B$ and $\sum_{i,j} |x_{ij} y_{ij}| = \sum_{i,j} b_{ij} |x|_{ij} y_{ij} = \mathrm{trace}(b|x| \circ y^T) < \infty$ where $\circ$ means the usual matricial product. Hence, (v) follows from (i). Conversely, suppose that (v) holds. This is the same as $\sum_{i,j} |t(x_{(i,j)} y)| < \infty$ or $\sum_k |t(x_{(k)} y)| < \infty$. Note that the duality between B and T is given by the bilinear functional $\mathrm{trace}(x \circ y^T)$ for $x \in B, y \in T$. Since $\mathrm{trace}(x \circ y^T) = t(xy)$, the condition $\sum_k |t(x_{(k)} y)| < \infty$ is equivalent to the unconditional convergence of the standard decomposition $x = \sum_k x_{(k)}$ in the weak-* topology, i.e. the ultraweak operator topology, on B. A similar conclusion holds for $|x|$ instead of x. Therefore, also the "ultraweak" parts of (ii) and (iii) are equivalent to (i). $\square$

Note that (v) means that $xy \in AS$ for every $y \in T$, while (iv) means that $xy \in AS$ for every rank-one matrix $y = (\eta_i \xi_j) \in B$ with $(\xi_i), (\eta_j) \in l^2$.

Having the above characterization we see that for $x \in K_a$ (or $x \in B_a$) the exact order of taking components does not matter, and we can simply write $x = \sum_{i,j} x_{(i,j)}$ with the convergence in the norm (or the strong) operator sense.

For appropriate matrices x define the multiplication operators $u_x : T \to AS$, $v_x : B_\infty \to B$ and $v_x^0 : B_0 \to K$ by $u_x(y) = xy$ for $y \in T$, $v_x(b) = bx$ for $b \in B_\infty$ and $v_x^0 = v_x|_{B_0}$, the restriction of v_x to B_0, respectively. Then the following theorem gives a characterization of B_a and K_a in terms of these operators.

Theorem 3.3. *Let x be an infinite matrix and u_x, v_x and v_x^0 the multiplication operators as above. Then we have:*
(a) $x \in B_a$ *if and only if any one of these operators is defined and bounded;*
(b) $x \in K_a$ *if and only if any one of these operators is defined and compact.*
Moreover, if $x \in B_a$, then $\|u_x\| = \|v_x\| = \|v_x^0\| = \|x\|_a$.

Proof. Let $x \in B_a$. Then the mapping $v_x : b \mapsto bx$ is bounded as an operator from B_∞ to B by Proposition 2.1, and consequently $v_x^0 = v_x|_{B_0}$ is a bounded operator from B_0 to B. We now show that in fact $v_x^0(B_0) \subset K$. Let $b \in B_0$ and $\epsilon > 0$. Then, for n big enough, we have by Proposition 2.1

$$\begin{aligned}\|b(x - \sum_{k=1}^{n} x_{(k)})\| &= \|(b - \sum_{k=1}^{n} b_{(k)})x\| \leq \|(b - \sum_{k=1}^{n} b_{(k)})x\|_a \\ &\leq \|b - \sum_{k=1}^{n} b_{(k)}\|_\infty \|x\|_a \leq \epsilon \|x\|_a.\end{aligned}$$

It follows that $\|bx - \sum_{k=1}^{n}(bx)_{(k)}\| \to 0$ as $n \to \infty$ and, hence, $bx \in K$. So, $v_x^0 : B_0 \to K$ is bounded, and then also its Banach adjoint $(v_x^0)^*$ is bounded as an operator from $T = K^*$ to $AS = B_0^*$. But now it is straightforward to prove (from the definition) that $(v_x^0)^*$ is again a multiplication operator (from T to AS), equal to u_x. In the same way we find that $(u_x)^* = v_x$.

On the other hand, if any of the operators in (i), (ii) or (iii) is bounded, the same is true for the others (by duality), and all operators have the same norm. In particular, we have $bx \in B$ for every $b \in B_\infty$, and this is equivalent to $x \in B_a$ (by Theorem 2.5). Therefore, (a) is proved.

To prove (b), suppose v_x is a compact operator. Then for every $b \in B_\infty$ the sequence $\{\sum_{k=1}^{n} b_{(k)}\}$ is bounded in B_∞, and it is possible to choose a sequence $\{\sum_{k=1}^{n_j} b_{(k)}x_{(k)}\}$ which is convergent in B (necessarily towards bx because of its strong convergence to bx). So, $bx = \lim_j \sum_{k=1}^{n_j}(bx)_{(k)} \in K$ for every $b \in B_\infty$, and hence $x \in K_a$. The converse is also true by Proposition 2.1(d), since in this case v_x can be approximated by finite rank operators. Namely, $\|v_x b - \sum_{k=1}^{n} v_{x_{(k)}} b\| \leq \|b\|_\infty \|x - \sum_{k=1}^{n} x_{(k)}\|_a$ for every $b \in B_\infty$, and hence $\|v_x - \sum_{k=1}^{n} v_{x_{(k)}}\| \to 0$ as $n \to \infty$. So, v_x is compact if and only if $x \in K_a$. By using the theorem of Schauder it follows that u_x and v_x^0 are also compact if and only if $x \in K_a$. □

Since the map $b \mapsto bx$ from B_∞ to B is closed, the boundedness of v_x follows merely from the fact that $bx \in B$ for every $b \in B_\infty$ (by the Closed Graph Theorem). The same is true for the map $b \mapsto bx$ from B_0 to B (or to K) or for the map $y \mapsto xy$ from T to AS. Hence, using the general notation $M(X, Y)$ for the space of all matrices multiplying pointwise X into Y and the notation $CM(X, Y)$ for the space of all matrices representing compact multiplication operators from X to Y, we have the following consequence.

Corollary 3.4. *Using the above notation we have:*
(a) $B_a = M(B_\infty, B) = M(B_0, B) = M(B_0, K) = M(T, AS)$;
(b) $K_a = CM(B_\infty, B) = CM(B_0, B) = CM(B_0, K) = CM(T, AS)$.

Proof. We shall only show that $M(B_0, B)$ is equal to the rest in (a) (and similary for $CM(B_0, B)$ in (b)); other equalities are the ingredient of Theorem 3.3. We obviously have $M(B_\infty, B) \subset M(B_0, B)$. On the other hand, we can prove as before that $bx \in K$ for every $x \in M(B_0, B)$. Hence, $(v_x^0)^* = u_x$ is well defined and bounded as an operator from T to AS, and the equality follows from Theorem 3.3. □

Remark 3.5. It is well known that an operator between two Banach spaces X and Y is weakly compact if and only if its second adjoint maps X^{**} into (the canonical image of) Y in Y^{**}. As $x \in K_a$ is equivalent to $bx \in K$ for every $b \in B_\infty$ by Theorem 2.7, this means that $v_x^0 : B_0 \to K$ is weakly compact. Thus, in light of Theorem 3.3(b), we see that the multiplication operator v_x^0 (or $v_x = (v_x^0)^{**}$) is weakly compact if and only if it is compact, and this happens if and only if $x \in K_a$.

4. Other norms and duality

The Banach algebra $M(B_a, AS)$, introduced in [3] (see Definition 3.1), turns out to play a similar role in the duality of absolutely bounded and absolutely compact matrices as T plays in the duality of bounded and compact matrices. A natural algebra norm on $M(B_a, AS)$ is given by $\|y\|_{M(B_a,AS)} = \sup\{\|xy\|_{AS};\ x \in B_a,\ \|x\|_a \le 1\}$. It is straightforward to prove that $M(B_a, AS)$ is a Banach algebra in the norm $\|\cdot\|_{M(B_a,AS)}$, but we omit the proof since it is given in [3] for a larger class of algebras (see Theorem 3.2).

The following duality theorem is also proven in [3] in a more general situation (see [3], Theorem 3.10 and Theorem 3.14).

Theorem 4.1. *We have $K_a^* = M(B_a, AS)$ and $M(B_a, AS)^* = B_a$.*

Corollary 4.2. *$B_a^* = M(B_a, AS) \oplus (K_a)^\perp$, where $(K_a)^\perp$ is the annihilator in B_a^* of $K_a \subset B_a$.*

Another nice algebra, also introduced in [3], is $M(B, AS)$. We relate both algebras to the trace class T.

Proposition 4.3. *$M(B, AS) \subset T \subset M(B_a, AS)$ with both inclusions proper.*

Proof. The proper inclusion $M(B, AS) \subset T$ is proven in [3] (see Corollary 2.4 and Example 2.5). The inclusion $T \subset M(B_a, AS)$ is clear from Theorem 3.3(a). Here the equality cannot hold, as can be seen from the counterexample $y = \bigoplus_{n=1}^\infty u_n/n^2$ with the usual unitary matrices u_n defined in Remark 2.4. In this case $y \notin T$ while $|y| \in T$, so that $|y| \in M(B_a, AS)$ and, hence, also $y \in M(B_a, AS)$. □

We conclude with two theorems giving a characterization of elements in $M(B_a, AS)$ and $M(B, AS)$ in terms of the unconditional convergence of the standard decomposition, similar to the characterization of absolutely compact matrices in Theorem 3.1.

Theorem 4.4. *For a matrix y with the standard decomposition $y = \sum_k y_{(k)}$, the following assertions are equivalent:*
(i) *$y \in M(B_a, AS)$;*
(ii) *$y = \sum_k y_{(k)}$ is unconditionally convergent in the norm $\|\cdot\|_{M(B_a,AS)}$;*
(iii) *$|y| = \sum_k |y|_{(k)}$ is convergent in the norm $\|\cdot\|_{M(B_a,AS)}$.*

Proof. Note first that $by \in M(B_a, AS)$ for every $b \in B_\infty$ and $y \in M(B_a, AS)$ and that $\|by\|_{M(B_a,AS)} \le \|b\|_\infty \|y\|_{M(B_a,AS)}$. From this we can immediately derive that $y \in M(B_a, AS)$ if and only if $|y| \in M(B_a, AS)$.

If $y \in M(B_a, AS)$, it follows from the proofs of Lemma 3.6 and Lemma 3.7 in [3] that the series $y = \sum_k y_{(k)}$ converges in the norm $\|\cdot\|_{M(B_a,AS)}$. Replacing y with by where $b \in B_\infty$ and using the known characterization of the unconditional summability from [2] we see that the convergence of the series $y = \sum_k y_{(k)}$ is unconditional. The implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are then trivial by the remark in the first paragraph. □

Note that by Theorem 4.4(ii), the ideal of all matrices with finite support is dense in $M(B_a, AS)$ in the norm of $M(B_a, AS)$ and, hence, also $M(B, AS)$ and T are dense in $M(B_a, AS)$.

Theorem 4.5. *For a matrix y with the standard decomposition $y = \sum_k y_{(k)}$, the following assertions are equivalent:*
(i) $y \in M(B, AS)$;
(ii) $y \in M(B_\infty, T)$;
(iii) $y = \sum_k y_{(k)}$ *is (weakly) unconditionally convergent in the trace class T.*

Proof. The equivalence (i) $\iff$ (ii) can be obtained along the same lines as in proving previous theorems by the use of elementary duality theory. Note that (i) is equivalent to $\sum_{i,j} |x_{ij} y_{ij}| < \infty$ for every $x \in B$, and this is essentially the weak unconditional convergence of the series in (iii). On the other hand, (ii) means the bounded multiplier convergence of the standard decomposition for $y \in T$ and is equivalent to its unconditional convergence by the general theory (see [2]). □

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