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DOWN-UP ALGEBRAS AT ROOTS OF UNITY

RAJESH S. KULKARNI

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ABSTRACT. In this note, we show that down-up algebras at roots of unity are maximal orders over their centers.

1. INTRODUCTION

Down-up algebras were defined by Benkart and Roby ([2]) for combinatorial reasons. They are associative algebras $A = A(\alpha, \beta, \gamma)$ generated by u, d over $\mathbb{C}$ and satisfy the relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma d, \tag{1.1}$$

$$du^2 = \alpha udu + \beta u^2d + \gamma u, \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. These have been studied by various authors, for example, in [2, 3, 5, 6, 7]. We consider the case when $\beta \neq 0$, so that the algebra A is Noetherian ([6]). These algebras have a rich representation theory; see [2, 7]. We recall some basic structural properties.

Let $\xi = ud$, $\eta = du$. Then $\xi\eta = \eta\xi$. Consider the automorphism θ of $\mathbb{C}[\xi, \eta]$ given by

$$\theta(\xi) = \frac{\eta - \alpha\xi - \gamma}{\beta}, \qquad \theta(\eta) = \xi.$$

Then $ur = \theta(r)u$, $dr = \theta^{-1}(r)d$ for any $r \in \mathbb{C}[\xi, \eta]$. This observation makes A into a hyperbolic ring ([7]) or a generalized Weyl algebra ([6]). Also, it is easy to see that $\mathbb{C}[\xi, \eta]$ embeds into A. We recall two other basic facts about this family of algebras: A has a PBW-basis $\{u^i(du)^jd^k\}$ ([2, 7]), and A is a domain if and only if $\beta \neq 0$. ([6, 7]). The first property has been exploited in studying representations when the parameters are generic. We will use both of these properties in later sections.

In [7], we also studied the case when the automorphism θ has finite order. We call these algebras *down-up algebras at roots of unity*; see Section 2 below for details. They have been of recent interest; see, for example, [3, 5]. In [7], we showed that certain localizations of the algebra A are crossed product algebras. We also showed that away from a subvariety of the variety defined by the center, the algebra is Azumaya. This together with other results from [7] raises a natural question: is the algebra A at roots of unity a maximal order over its center? In this article, we prove that the answer is affirmative. In fact, first we prove that the down-up algebra

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A is a reflexive order in the central simple algebra $A \otimes K$ where K is the fraction field of the center of A (which is a domain). This result is then used in proving the main result. The technique for this main result should be useful in other situations as well. We use some old theorems of M. Harada and S. Williamson ([4, 10]) which give a criterion for a crossed product algebra over a discrete valuation ring to be a maximal order. These theorems together with characterization of maximal orders due to Auslander and Goldman are the main tool used below.

Now we sketch a different approach for the main question of this article. It uses a result of J. T. Stafford (Theorem 2.10, [9]). In our context, this result says that a Noetherian, Auslander-regular and Cohen-Macaulay k-algebra over a field k which is stably free is a domain and a maximal order in its quotient division ring. For the notion of a maximal order in this theorem, refer to 5.1.1, [8]. Now by Lemma 4.2, [6], the down-up algebra A (when $\beta \neq 0$) is Auslander-regular and Cohen-Macaulay. Since a graded ring is stably free, the theorem cited earlier shows that the algebra $A = A(\alpha, \beta, 0)$ is a maximal order. From Proposition 3.10, [8], it follows that various notions of a maximal order agree in our situation. So the main theorem of this note is proved in this case (for algebras $A(\alpha, \beta, 0)$). It should be noted that finiteness of order of θ does not force $\gamma = 0$. For example, the algebra $A(-1, -1, \gamma)$ has a corresponding automorphism θ of order three for any γ. For prime rings, the hypothesis of stable freeness can be weakened by the hypothesis that the state space (see [9] for a definition) is trivial. Since orders in central algebras are prime (PI) rings (Corollary 13.6.6, [8]), this provides an alternate method to prove that the down-up algebras are maximal orders over their centers when θ is a finite order automorphism. However the purpose of this article to (re)introduce the method sketched above to questions related to maximality of orders. The method of the last paragraph is particularly useful in applications to crossed product algebras.

The results of this article complete the analysis in [7]. The main result of this article gives a family of explicit examples of maximal orders with three dimensional centers. Currently there is considerable activity in the area of maximal orders on algebraic surfaces, and so it is natural to consider similar questions for higher dimensional algebraic varieties. We hope these examples will be of interest in this context.

2. Down-up algebras at roots of unity

In this section, we consider the case when the automorphism θ is of finite order m. Then the center R of A is generated by $\{u^m, d^m\}$ over $(\mathbb{C}[\xi, \eta])^\theta$ (Corollary 2.0.2, [7]). In this case, we have two polynomials $w_i \in \mathbb{C}[\xi, \eta]$, $i = 1, 2$, such that

$$\theta(w_1) = \lambda_1 w_1, \ \theta(w_2) = \lambda_2 w_2,$$

where λ_1, λ_2 are l_1^{th} and l_2^{th} roots of unity resp. and m is the LCM of l_1 and l_2. It is easy to see that λ_i's are roots of the quadratic equation $q(t) = t^2 + \frac{\alpha}{\beta} t - \frac{1}{\beta}$. We consider the commutative ring

$$S = \frac{\mathbb{C}[\xi, \eta, \mu, \nu]}{(\xi\theta(\xi) \cdots \theta^{m-1}(\xi) - \mu\nu)}$$

and extend the action of θ to S by letting it act trivially on $\mu,\ \nu$. Define the homomorphism

$$\begin{array}{ccl} S & \longrightarrow & A, \\ \xi,\ \eta & \mapsto & ud,\ du \text{ resp.}, \\ \mu,\ \nu & \mapsto & u^m,\ d^m \text{ resp.} \end{array}$$

Note that this is an embedding of S into A (Lemma 6.1.1, [7]). We identify the image of this map and denote it by S again. We start with an easy lemma.

Lemma 2.1. *The algebra S is a maximal commutative subalgebra of A.*

Proof. Let R' denote the polynomial ring $\mathbb{C}[\xi,\eta]$ and let $R'[u;\theta]$ and $R'[d;\theta^{-1}]$ be skew polynomial rings. We first recall (Lemma 2.0.1, [7]) that the algebra A is isomorphic to

$$R' \oplus uR'[u;\theta] \oplus dR'[d;\theta^{-1}]$$

as an R'-module. In particular, A is a free R'-bimodule with a basis given by the decomposition above. Let $p \in A$ be an element that commutes with S. We decompose this element by using the basis above. Note that since $R' \subset S$, we may assume that

$$p = \sum_{i\geq 1} a_i u^i + \sum_{j\geq 1} b_j d^j \ \ a_i, b_j \in R'.$$

Then

$$\begin{aligned} p\xi &= \sum_{i\geq 1} a_i u^i \xi + \sum_{j\geq 1} b_j d^j \xi \\ &= \sum_{i\geq 1} a_i \theta^i(\xi) u^i + \sum_{j\geq 1} b_j \theta^{-j}(\xi) d^j \\ \text{and} \quad \xi p &= \sum_{i\geq 1} a_i \xi u^i + \sum_{j\geq 1} b_j \xi d^j. \end{aligned}$$

Since R' is a domain, by uniqueness of decomposition

$$\theta^i(\xi) = \xi \text{ whenever } a_i \neq 0 \text{ or } b_i \neq 0.$$

Similarly

$$\theta^i(\eta) = \eta \text{ whenever } a_i \neq 0 \text{ or } b_i \neq 0.$$

So $\theta^i =$ identity whenever either a_i or b_i is nonzero. So i is divisible by m whenever either a_i or b_i is nonzero. So

$$p = \sum_{i\geq 1} a_{mi} u^{mi} + \sum_{j\geq 1} b_{mj} d^{mj},$$

and hence it belongs to S. So S is a maximal commutative subalgebra of A. □

Next we prove another useful proposition.

Proposition 2.2. *The algebra S and hence R are integrally closed in their respective fraction fields.*

Proof. Let $Q(R), Q(S)$ denote the fraction fields of R and S respectively. Also let S_μ and S_ν denote the localizations of S at the subsets $\{\mu^i\}_{i\geq 1}$ and $\{\nu^i\}_{i\geq 1}$. Then S_μ and S_ν are isomorphic to the localizations $\mathbb{C}[\xi,\eta,\mu]_\mu$ and $\mathbb{C}[\xi,\eta,\nu]_\nu$ resp. Hence they are integrally closed. If $\alpha \in Q(S)$ is a root of a monic polynomial

f with coefficients in S, then $\alpha \in S_\mu \cap S_\nu$. So it will be sufficient to prove that $S_\mu \cap S_\nu = S$.

To that end, let $\frac{\bar{f}}{\mu^i} = \frac{\bar{g}}{\nu^j}$ for some nonnegative integers i, j and $f, g \in \mathbb{C}[\xi, \eta, \mu, \nu]$. The bar denotes the image in S. We use induction on $i + j$. Clearly the claim is true for $i + j = 1$. Now let $i + j \geq 2$. We may assume that $i, j \neq 0$. Then

$$\begin{aligned} \nu^j \bar{f} - \mu^i \bar{g} &= 0 \text{ in } S \text{ and hence} \\ \nu^j f - \mu^i g &= h(\xi \cdots \theta^{m-1}(\xi) - \mu\nu) \text{ for some } h \in \mathbb{C}[\xi, \eta, \mu, \nu]. \end{aligned}$$

So it follows that $h = a\mu + b\nu$ for some $a, b \in \mathbb{C}[\xi, \eta, \mu, \nu]$. Expanding we get

$$(\nu^j f + b\mu\nu^2) - (\mu^i g - a\mu^2\nu) = (a\mu + b\nu)(\xi \cdots \theta^{m-1}(\xi)).$$

Considering this equation mod μ,

$$\nu^j f = b\nu\xi \cdots \theta^{m-1}(\xi) \bmod \mu.$$

So

$$f = f_1 \xi \cdots \theta^{m-1}(\xi) + f_1' \mu \text{ for some } f_1, f_1' \in \mathbb{C}[\xi, \eta, \mu, \nu].$$

Now in S this gives $\bar{f} = \bar{f}_1 \mu\nu + \bar{f}_1' \mu$. So $\bar{f}$ is divisible by μ. Similarly $\bar{g}$ is divisible by ν. By induction

$$\frac{\bar{f}}{\mu^i} = \frac{\bar{g}}{\nu^j} \in S.$$

This finishes the proof that S is integrally closed. Now consider the ring R. Since $R \subset S$ and S is integrally closed, it follows that the integral closure of R is equal to $S \cap Q(R)$. But since $S^\theta = R$, it follows that R is integrally closed in its fraction field. □

Let p be a height one prime ideal in R. For later use, we compute the inertia groups for the extensions $(S)_p$ over R_p. Note that the ring $(S)_p$ is the semilocal ring obtained by localizing S at the multiplicative subset $R - p$. Recall that we have polynomials $w_1,\ w_2 \in \mathbb{C}[\xi, \eta]$ that are eigenvectors with eigenvalues $\lambda_1,\ \lambda_2$ resp. for the action of θ on (inhomogeneous) linear polynomials in $\mathbb{C}[\xi, \eta]$.

Lemma 2.3. *Let q be a maximal ideal of $(S)_p$ lying over the ideal p of R_p. If the inertia group G_q^I of q is nontrivial, then either $q = (w_1)$ or $q = (w_2)$. In these cases, the inertia groups are generated by θ^{l_2} and θ^{l_1} resp. (Recall that λ_1 and λ_2 are l_1^{th} and l_2^{th} roots of unity.)*

Proof. Let $q = (w_1)$. Then q is totally ramified. Also S_q/qS_q is isomorphic to the field of rational functions $\mathbb{C}(\mu, w_2)$. It follows that the inertia group is as above, as the stabilizer subgroup of w_2 is generated by θ^{l_2} and θ acts trivially on μ. The other case follows as well. This finishes the second part.

Now we consider the first statement. We first compute the differents $\mathcal{D}(S_q/R_p)$ when $w_i \notin q$ for $i = 1, 2$. We use the following fact about differents. Let $L = K(\alpha)$ be a separable field extension of degree n, and let A be a Dedekind domain with fraction field K. Suppose that α is integral over A with minimal polynomial $f(x) \in A[x]$. Let B be the integral closure of A in L. Then the different of B/A divides the ideal $(f'(\alpha))$ where $f'(x)$ is the formal derivative of $f(x)$.

Choose a pair of integers (i, j) so that $\lambda_1^i \lambda_2^j$ is a primitive m^{th} root of unity. We use the fact above with $\alpha = w_1^i w_2^j$. A standard calculation shows that α generates $Q((S)_p)$ over $Q(R_p)$ and that the minimal polynomial of α is $f(x) = x^m - \alpha^m$. So $\mathcal{D}(S_q/R_p)$ divides the ideal $((w_1^i w_2^j)^{m-1})$. Since $w_i \notin q$ for $i = 1, 2$, it follows that

$\mathcal{D}(S_q/R_p) = S_q$. This means that the inertia group G_q^I is trivial in these cases. This finishes the lemma. □

Next we recall a criterion due to S. Williamson for maximality of orders ([10]). Let B be a tamely ramified extension of a rank one discrete valuation ring A such that the quotient field extension is finite Galois with Galois group G. We assume that B is also a discrete valuation ring, as this is the only case we will need below. Let G^I be the inertia group of B over A. Let K be the residue field of B and let k be the residue field of A. Let $\Delta(f, B, G)$ be the crossed product associated to a cocycle f with cohomology class in $H^2(G, B^\times)$. Define the *conductor group* H_f to be the maximal subgroup of G^I such that $[\bar{f}]$ is in the image of the inflation map $H^2(G/H_f, K^\times) \to H^2(G, K^\times)$. Then S. Williamson's criterion (Thm. 2.5, [10], together with Thm. 3.3, [4]), says that the order $\Delta(f, B, G)$ is maximal if and only if the conductor group is trivial.

We also define the group Γ_f to be the maximal subgroup of G^I such that the image of class of $[\bar{f}]$ under the restriction map $H^2(G, K^\times) \to H^2(\Gamma_f, K^\times)$ is trivial. Since the composition map

$$H^2(G/H_f, K^\times) \longrightarrow H^2(G, K^\times) \longrightarrow H^2(\Gamma_f, K^\times)$$

is trivial, $H_f \subset \Gamma_f$.

Now we apply these considerations to our case. First we recall that certain localizations of the down-up algebra are crossed products. Namely, consider localizations A_μ and A_ν of A at $\{\mu^i\}$ and $\{\nu^j\}$ respectively. Also we consider two crossed products

$$\Delta_\mu = \Delta(S_\mu, f_\mu, G),\ \Delta_\nu = \Delta(S_\nu, f_\nu, G),$$

where G is the cyclic group of automorphisms of S_μ and S_ν generated by θ and f_μ, f_ν are the cocycles

$$f_\mu(\theta^i, \theta^j) = \begin{cases} \mu & i+j \geq m \\ 1 & i+j < m \end{cases} \qquad f_\nu(\theta^{-i}, \theta^{-j}) = \begin{cases} \nu & i+j \geq m \\ 1 & i+j < m \end{cases} .$$

Then we have proved (Theorem 6.1.1, [7]) that the algebra A_μ (resp. A_ν) is isomorphic to Δ_μ (resp. Δ_ν). We use these crossed product structures in the following paragraphs.

Let p be a codimension one prime in R. Suppose q is the only prime in S_μ lying over p (so that $q = (w_1)$ or (w_2)). Let G_q be the decomposition subgroup of q in G and let

$$f_p : G \times G \longrightarrow (S_q)^\times$$

be the cocycle describing the crossed product structure of A_p (that is, $A_p = \Delta(S_q, f_p, G)$). Let K denote the field $Q(S_q/qS_q)$. Note that in our case the decomposition group of q is G. We construct a new cocycle $\bar{f}_p$,

$$\bar{f}_p : G \times G \longrightarrow K^\times,$$

in an obvious way. We have the following lemma.

Lemma 2.4. *Let p be a codimension one prime in R. Then for the cocyle f_p the subgroups Γ_{f_p} and hence H_{f_p} are trivial.*

Proof. We first prove the lemma in the case when the codimension one prime $q = (w_1)$ or (w_2) lies over p. We may assume $q = (w_1)$, as the other case follows by a similar calculation. We show that the restriction of $\bar{f}_p$ in $H^2(G', K^\times)$ for any subgroup G' of G_q^I is trivial only if G' is the trivial subgroup. In this case, the inertia group $G_q^I = (\theta^{l_2})$ and so a subgroup G' is generated by $\theta^{l_2 n}$ for some integer n. Recall that the linear polynomials $w_1,\ w_2$ are the eigenvectors for the action of θ on the (affine) linear part of the polynomial ring $\mathbb{C}[\xi,\eta]$. In this notation, it is easy to see that the residue field K is the fraction field

$$\left(\frac{\mathbb{C}[w_2,\mu,\nu]}{\left(\prod_i\left(\frac{-\lambda_2^i w_2}{\lambda_1-\lambda_2}-\frac{\gamma}{\beta(\lambda_1-1)(\lambda_2-1)}\right)\right)-\mu\nu}\right).$$

Here we have assumed that the eigenvalues are distinct from 1. The only remaining possibility is when one of the eigenvalues is 1. This happens only if theta is of order two; that is, the other eigenvalue is -1 and $\gamma=0$. In this case, the algebra is $A(0,1,0)$ and the eigenvectors $w_1,\ w_2$ are

$$w_1=\xi+\eta,\quad w_2=\xi-\eta.$$

This gives that the field K in this case is $K=\left(\frac{\mathbb{C}[w_1,\mu,\nu]}{(w_1^2-4\mu\nu)}\right)$. Consider the restriction of the cocycle $\bar{f}_p$ to $G'\times G'$. If the corresponding cohomology class is trivial, then there exists a cochain $h_p : G'\to K^\times$ such that $dh_p=\bar{f}_p$. That is,

$$\frac{(\theta^{nl_2})^{i_1}(h_p((\theta^{nl_2})^{i_2}))h_p((\theta^{nl_2})^{i_1})}{h_p((\theta^{nl_2})^{i_1+i_2})}=\begin{cases}1 & i_1+i_2<\frac{m}{nl_2}\\ \mu & i_1+i_2\geq\frac{m}{nl_2}\end{cases}.$$

Note that θ^{l_2} acts trivially on K, and so this condition is equivalent to the condition that $\mu\in(K^\times)^{\frac{m}{nl_2}}$. The defining polynomial for the field K is

$$(-1)^m\left(\frac{{w_2}^{l_2}}{(\lambda_1-\lambda_2)^{l_2}}+\frac{\gamma^{l_2}}{\beta^{l_2}(\lambda_1-1)^{l_2}(\lambda_2-1)^{l_2}}\right)^{\frac{m}{l_2}}-\mu\nu.$$

It follows that μ is not an $(\frac{m}{nl_2})$-th root unless $nl_2=m$, which implies that G' is trivial. This shows that the group Γ_{f_p} is trivial. Since the conductor group $H_{f_p}\subset\Gamma_{f_p}$, it follows that it is also trivial. This finishes the lemma. □

Remark 2.5. By Lemma 2.3, when $q\neq(w_1)$ or (w_2), G_q^I is trivial. So in this case the conductor group for the corresponding cocycle is trivial as well. For the definition of conductor a group when the extension of the discrete valuation ring is not itself a discrete valuation ring, see the discussion preceding Thm. 2.5, [10].

Theorem 2.6. *Let the automorphism θ be of finite order m. Then A is a maximal R-order in $A\otimes_R Q(R)$.*

Proof. First note that

$$A\otimes_R Q(R)\cong A_\mu\otimes_{R_\mu}Q(R_\mu)\cong\Delta_\mu\otimes_{R_\mu}Q(R_\mu)\cong\Delta(S_\mu\otimes_{R\mu}Q(R_\mu),f_\mu,G).$$

Now S_μ is integral over R_μ, and so $S_\mu\otimes_{R_\mu}Q(R_\mu)$ is a field, isomorphic to $Q(S_\mu)$. So $A\otimes_R Q(R)$ is a central simple algebra. Since A is a finitely generated R-module, A is an R-order in $A\otimes_R Q(R)$.

We recall a theorem of Auslander-Goldman (Thm. 1.5, [1]) which gives a characterization of maximal orders. It says that an R-order Λ is maximal if and only if it is a reflexive R-order such that for any codimension 1 prime p of R, Λ_p is a maximal R_p-order. Recall that an R-order Λ is reflexive if $\Lambda^{**}=\Lambda$ where $\Lambda^*=\mathrm{Hom}_R(\Lambda,R)$. If R is integrally closed, then $\Lambda^{**}=\bigcap\Lambda_p$, where the intersection is taken over all codimension 1 primes of R.

In order to prove that A is a reflexive R-order, we first show that $A_\mu \cap A_\nu = A$ in $A \otimes_R Q(R)$. Since A is a domain, $A \subset A_\mu \cap A_\nu$. Let

$$\frac{\sum a_{ijk} u^i (du)^j d^k}{u^{m\alpha}} = \frac{\sum b_{i'j'k'} u^{i'} (du)^{j'} d^{k'}}{d^{m\beta}} \tag{2.1}$$

for some positive integers α, β and $a_{ijk}, b_{i'j'k'} \in \mathbb{C}$. Note that the numerators of these expressions are uniquely determined by the PBW basis theorem. Comparing coefficients, the same theorem gives

$$a_{ijk} = b_{i'j'k'} \text{ if and only if } i = i' + m\alpha,\ j = j',\ k = k' - m\beta, \text{ and}$$
$$i \geq m\alpha,\ k' \geq m\beta \text{ for all nonzero coefficients } a_{ijk}, b_{i'j'k'}.$$

So the element in Eq. (2.1) is in A. Now we show that $\bigcap A_p = A_\mu \cap A_\nu$, where the intersection is taken over all codimension one primes in R. Observe that a codimension one prime p of R cannot contain both μ and ν. So it will be sufficient to show that

$$A_\mu = \bigcap A_p \text{ and } A_\nu = \bigcap A_q,$$

where the first (resp. second) intersection is taken over all codimension one primes not containing μ (resp. ν). Note that proving these equalities is the same as proving that A_μ and A_ν are reflexive orders. We show one of these equalities, as the other is similar.

Recall that A_μ is isomorphic to the crossed product algebra Δ_μ described above. So it will be sufficient to show that Δ_μ is a reflexive order. First we note that as an S_μ module, Δ_μ is free of rank equal to the order of θ. So the reflexivity of Δ_μ follows from that of S_μ. Since R_μ is integrally closed, it is sufficient to show that

$$S_\mu = \bigcap (S_\mu)_p = (S)_p,$$

where the intersection is taken over codimension one primes of R not containing μ. Note that the multiplicative subset in each of these localizations is $R - p$. So

$$(S)_p \subset \bigcap_i S_{p_i},$$

where the p_i's are primes in S lying over the prime p of R. Note that the multiplicative subset in localization S_{p_i} is $S - p_i$. Thus it follows that

$$\bigcap (S_\mu)_p \subset \bigcap_p \bigcap_i S_{p_i},$$

where on the right side the first intersection is over codimension one primes p of R not containing μ and the second intersection is over all primes p_i of S lying over p. Now since S_μ is normal and S_μ is a finite module over R_μ, it follows that

$$\bigcap_p \bigcap_i S_{p_i} = S_\mu.$$

So we conclude that S_μ is reflexive over R_μ, and hence so is Δ_μ. This finishes the proof that A is a reflexive R-order.

It remains to consider maximality of A_p as an R_p-order for any codimension one prime p of R. Since a codimension one prime cannot contain both μ and ν, p is a codimension one prime in R_μ or R_ν. Thus A_p is also a crossed product order. We assume that μ is not in p, as the other case is similar. By S. Williamson's criterion above, A_p is a maximal order if and only if the corresponding conductor group H_{f_p}

is trivial. If $q = (w_1)$ or (w_2) does not lie over p, then by Remark 2.5, the conductor group is trivial. So A_p is a maximal order for such a prime p.

If $q = (w_1)$ or (w_2) lies over p, then by Lemma 2.4, the conductor group H_{f_p} is trivial. So the order A_p is maximal by the criterion of S. Williamson. Since localizations of A at codimension one primes are maximal and A is a reflexive order, it follows that the algebra A is a maximal order in $A \otimes_R Q(R)$. This finishes the proof of the theorem. □

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Department of Mathematics, Wells Hall, Michigan State University, East Lansing, Michigan 48824

E-mail address: kulkarni@math.msu.edu