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SELECTION THEOREMS AND TREEABILITY

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ABSTRACT. We show that domains of non-trivial Σ_1^1 trees have Δ_1^1 members. Using this, we show that smooth treeable equivalence relations have Borel transversals, and essentially countable treeable equivalence relations have Borel complete countable sections. We show also that treeable equivalence relations which are ccc idealistic, measured, or generated by a Borel action of a Polish group have Borel complete countable sections.

1. Introduction

Suppose that E and F are equivalence relations on Polish spaces X and Y. A *reduction of E to F* is a function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2)).$$

We say that E is *smooth* if there is a Polish space Y and a Borel reduction of E to the diagonal $\Delta(Y) = \{(y, y) : y \in Y\}$. It is clear that every smooth equivalence relation is Borel. We say that E is *countable* if its equivalence classes are all countable. We say that E is *essentially countable* if it is Borel reducible to a countable equivalence relation on a Polish space. A simple reflection argument can be used to show that every essentially countable Borel equivalence relation is Borel reducible to a countable Borel equivalence relation.

A *graph on X* is an irreflexive, symmetric set $\mathcal{G} \subseteq X \times X$. A *$\mathcal{G}$-path* is a sequence $\langle x_i \rangle_{i \leq n}$ such that $\forall i < n \ ((x_i, x_{i+1}) \in \mathcal{G})$. We say that $\mathcal{G}$ is *acyclic* if there is no $\mathcal{G}$-path $\langle x_i \rangle_{i \leq n+2}$ such that $x_0 = x_{n+2}$ and $\forall i \leq n \ (x_i \neq x_{i+2})$. We use $E_{\mathcal{G}}$ to denote the equivalence relation on X consisting of pairs (x, y) for which there is a $\mathcal{G}$-path from x to y, and we use $[x]_{\mathcal{G}}$ to denote the corresponding equivalence class.

A *treeing of E* is an acyclic graph $\mathcal{T}$ on X such that $E = E_{\mathcal{T}}$. We say that E is *treeable* if it has a $\mathbf{\Sigma_1^1}$ treeing. A simple boundedness argument can be used to show that every $\mathbf{\Sigma_1^1}$ treeing of a Borel equivalence relation is Borel. We say that E is *σ-treeable* if it is the union of countably many treeable subequivalence relations. A simple reflection argument can be used to show that every σ-treeable Borel equivalence relation is the union of countably many treeable Borel equivalence relations. While the notion of treeability has played an important role in the study

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of countable Borel equivalence relations, it has gone largely unstudied outside of the countable case.

An *E-transversal* is a set $B \subseteq X$ whose intersection with each E-class is a singleton. An *E-countable section* is a set $B \subseteq X$ whose intersection with each E-class is countable. An *E-complete section* is a set $B \subseteq X$ whose intersection with each E-class is non-empty.

Our first two results concern the circumstances under which $\boldsymbol{\Sigma}^1_1$ equivalence relations admit Borel transversals and complete countable sections.

Theorem. *Every σ-treeable smooth equivalence relation has a Borel transversal.*

Theorem. *Suppose that E is an essentially countable $\boldsymbol{\Sigma}^1_1$ equivalence relation on a Polish space. Then the following are equivalent:*

(1) *E has a Borel complete countable section;*
(2) *E is σ-treeable.*

In order to obtain these results, we employ the methods of effective descriptive set theory. We direct the reader unfamiliar with this area to [5] for an excellent introduction to the subject and to [6] for a more thorough treatment.

We say that a subset A of a recursively presented Polish space is *$\sigma(x)$-treeable* if there are acyclic $\Sigma^1_1(x)$ graphs $\mathcal{T}_n$ on A such that $A \times A = \bigcup_{n \in \mathbb{N}} E_{\mathcal{T}_n}$. The above results are corollaries of the following basis theorem:

Theorem. *Let $x \in 2^{\mathbb{N}}$. Every non-empty $\sigma(x)$-treeable $\Sigma^1_1(x)$ set has a $\Delta^1_1(x)$ member.*

We say that a σ-ideal $\mathcal{I}$ on X is *ccc* if there is no sequence $\langle A_\alpha \rangle_{\alpha < \omega_1}$ of $\mathcal{I}$-positive $\boldsymbol{\Sigma}^1_1$ sets such that $\forall \alpha < \beta < \omega_1 \ (A_\alpha \cap A_\beta \in \mathcal{I})$. An assignment $C \mapsto \mathcal{I}_C$ of σ-ideals $\mathcal{I}_C$ on C to each equivalence class C of E is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ if for every Polish space Y and $\boldsymbol{\Sigma}^1_1$ set $R \subseteq (X \times Y) \times X$, the set

$$\bigcup_{C \in X/E} \{(x, y) \in C \times Y : R_{(x,y)} \in \mathcal{I}_C\}$$

is $\boldsymbol{\Pi}^1_1$. We say that E is *ccc idealistic* if there is a $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ assignment $C \mapsto \mathcal{I}_C$ of non-trivial ccc σ-ideals $\mathcal{I}_C$ on C to each equivalence class C of E. We say that E is *measured* if there is an assignment $C \mapsto \mu_C$ of non-zero σ-finite measures μ_C on C to each equivalence class C of E such that for every $r \in \mathbb{R}$, Polish space Y, and $\boldsymbol{\Sigma}^1_1$ set $R \subseteq (X \times Y) \times X$, the set

$$\bigcup_{C \in X/E} \{(x, y) \in C \times Y : \mu(R_{(x,y)}) \leq r\}$$

is $\boldsymbol{\Pi}^1_1$. Clearly, every measured equivalence relation is ccc idealistic. The following fact can be viewed as an analog of Theorem 1.5 of [3] for treeable equivalence relations and answers a question that was raised in [1].

Theorem. *Every ccc idealistic σ-treeable equivalence relation has a Borel complete countable section. In particular, this holds for σ-treeable equivalence relations which are either measured or generated by a Borel action of a Polish group.*

The primary new ingredient behind our results is the isolation of a non-empty set of critical points associated with any pair $(\mathcal{T}, \mathcal{I})$, where $\mathcal{T}$ is an acyclic graph and $\mathcal{I}$ is a σ-ideal on the domain of $\mathcal{T}$. In §2, we define this set and use some of its basic properties to obtain our first three theorems. In §3, we show that if $\mathcal{I}$ is ccc, then the corresponding set of critical points is necessarily countable. We then use this observation to obtain our final theorem.

2. Treeable sets and essentially countable equivalence relations

Suppose that X is a set, $\mathcal{T}$ is an acyclic graph on X, and $\mathcal{I}$ is a σ-ideal on X. We use the notation $\forall^{\mathcal{I}} x \in A\ \phi(x)$ to indicate that $\{x \in A : \neg\phi(x)\} \in \mathcal{I}$. We say that x *is $\mathcal{T}$-between* y, z if $yE_{\mathcal{T}}z$ and the injective $\mathcal{T}$-path from y to z passes through x. Let $d_{\mathcal{T}}$ denote the usual graph metric, and set $C_{\mathcal{T}}(y,r) = \{x \in X : d_{\mathcal{T}}(x,y) = r\}$; note then that $C_{\mathcal{T}}(y,1) = \mathcal{T}_y$, the points directly adjacent to y in $\mathcal{T}$. We say that y is $(\mathcal{T},\mathcal{I},r)$*-critical* if $C_{\mathcal{T}}(y,r) \notin \mathcal{I}$ and

$$\forall^{\mathcal{I}} v \in C_{\mathcal{T}}(y,r)\, \forall^{\mathcal{I}} w \in C_{\mathcal{T}}(y,r)\ (y \text{ is } \mathcal{T}\text{-between } v, w).$$

Proposition 1. *Suppose that X is a set, $\mathcal{T}$ is an acyclic graph on X, $\mathcal{I}$ is a σ-ideal on X, and $[x]_{\mathcal{T}} \notin \mathcal{I}$. Then $[x]_{\mathcal{T}}$ contains a $(\mathcal{T},\mathcal{I},r)$-critical point, for some $r \in \mathbb{N}$.*

Proof. Fix $r \in \mathbb{N}$ least for which there exists $y \in [x]_{\mathcal{T}}$ such that $C_{\mathcal{T}}(y,r) \notin \mathcal{I}$. We claim that y is $(\mathcal{T},\mathcal{I},r)$-critical. If $r = 0$, then this is clear. Otherwise, suppose that $v \in C_{\mathcal{T}}(y,r)$, and fix $z \in \mathcal{T}_y$ the first point along the injective $\mathcal{T}$-path from y to v. Our choice of r ensures that $C_{\mathcal{T}}(z,r-1) \in \mathcal{I}$, and it is clear that if $w \in C_{\mathcal{T}}(y,r) \setminus C_{\mathcal{T}}(z,r-1)$, then y is $\mathcal{T}$-between v, w. □

Given sets X, Y and σ-ideals $\mathcal{I}, \mathcal{J}$ on X, Y, let $\mathcal{I} * \mathcal{J}$ denote the σ-ideal of sets $R \subseteq X \times Y$ such that $\forall^{\mathcal{I}} x \in X\, \forall^{\mathcal{J}} y \in Y\ ((x,y) \notin R)$.

Proposition 2. *Suppose that X is a Polish space, $\mathcal{T}$ is an acyclic $\boldsymbol{\Sigma}^1_1$ graph on X, $\mathcal{I}$ is a σ-ideal on X, $x \in X$, $R \subseteq X \times Y$, and*

$$\{(y,z) \in R : x \textit{ is } \mathcal{T}\textit{-between } y, z\} \notin \mathcal{I} * \mathcal{I}.$$

Then there are $\boldsymbol{\Sigma}^1_1$ sets $A, B \subseteq X$ such that $(A \times B) \cap R \notin \mathcal{I} * \mathcal{I}$ and

$$\forall y \in A\, \forall z \in B\ (x \textit{ is } \mathcal{T}\textit{-between } y, z).$$

Proof. For each Borel set $U \subseteq X$, define

$$U' = \{y \in [x]_{\mathcal{T}} : x = y \text{ or } \exists z \in U \cap \mathcal{T}_x\ (z \text{ is } \mathcal{T}\text{-between } x, y)\},$$

and set $U'' = (X \setminus U)'$. Thus U' is the set of points y either equal to x or for which we have an injective path from y to x, the last point of which prior to x is in U.

Lemma 3. *There is an open set $U \subseteq X$ such that $(U' \times U'') \cap R \notin \mathcal{I} * \mathcal{I}$.*

Proof. The *notion of betweenness associated with* $\mathcal{T}$ is given by

$$B_{\mathcal{T}} = \{(x,(y,z)) \in X \times (X \times X) : x \text{ is } \mathcal{T}\text{-between } y, z\}.$$

It is clearly sufficient to show that $(B_{\mathcal{T}})_x$ is contained in the union of the sets of the form $U' \times U''$, where U varies over some fixed countable basis for X. Towards this end, suppose that $(y,z) \in (B_{\mathcal{T}})_x$. If $x = y = z$ and U is any basic open set, then $(y,z) \in U' \times U''$. If $x \neq y$ and $x = z$, then there exists $v \in \mathcal{T}_x$ such that v is $\mathcal{T}$-between x, y. If U is any basic open set containing v, then $(y,z) \in U' \times U''$. If $x = y$ and $x \neq z$, then there exists $w \in \mathcal{T}_x$ such that w is $\mathcal{T}$-between x, z. If U is any basic open set avoiding w, then $(y,z) \in U' \times U''$. If $x \neq y$ and $x \neq z$, then there exist $v, w \in \mathcal{T}_x$ such that v is $\mathcal{T}$-between x, y and w is $\mathcal{T}$-between x, z. If U is any basic open set containing v and avoiding w, then $(y,z) \in U' \times U''$. □

Fix an open set $U \subseteq X$ as in the lemma, and observe that $A = U'$ and $B = U''$ are $\boldsymbol{\Sigma}^1_1$ sets, $R \cap (A \times B) \notin \mathcal{I}$, and $\forall y \in A\, \forall z \in B$ (x is $\mathcal{T}$-between y, z). □

We now prove two results for Σ^1_1 graphings. For any parameter $x \in 2^{\mathbb{N}}$, these proofs relativize in the usual way to $\Sigma^1_1(x)$ graphings for Polish spaces which are recursively presented in x.

We say that a subset of a recursively presented Polish space X is Σ^1_1-*(co)meager* if it is (co)meager in the *Gandy-Harrington topology* generated by Σ^1_1 subsets of X.

Proposition 4. *Suppose that X is a recursively presented Polish space, $\mathcal{T}$ is an acyclic Σ^1_1 graph on X, $\mathcal{I}$ is the σ-ideal of Σ^1_1-meager subsets of X, $r \in \mathbb{N}$, and x is $(\mathcal{T}, \mathcal{I}, r)$-critical. Then x is Δ^1_1.*

Proof. By Proposition 2, there are $\mathcal{I}$-positive $\boldsymbol{\Sigma}^1_1$ sets $A, B \subseteq C_{\mathcal{T}}(x, r)$ such that

$$\forall y \in A\, \forall z \in B \ (x \text{ is } \mathcal{T}\text{-between } y, z).$$

By localization (see, for example, Proposition 8.26 of [4]), there are non-empty Σ^1_1 sets $A', B' \subseteq X$ such that A and B are Σ^1_1-comeager in A' and B', respectively. Then $A \times B$ is Σ^1_1-comeager in $A' \times B'$ (see, for example, Lemma 9.3.2 of [5]).

Fix a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of Δ^1_1 subsets of X which separates points, and for each $n \in \mathbb{N}$, let R_n denote the set of pairs $(y, z) \in A' \times B'$ with the property that there is an injective $\mathcal{T}$-path $\langle x_i \rangle_{i \leq 2r}$ from y to z such that exactly one of x, x_r is in U_n. Then R_n is a Σ^1_1 subset of $A' \times B'$ which is disjoint from $A \times B$; thus $R_n = \emptyset$. It follows that x is the unique point for which there is an injective $\mathcal{T}$-path $\langle x_i \rangle_{i \leq 2r}$ such that $(x_0, x_{2r}) \in A' \times B'$ and $x = x_r$, so $\{x\}$ is Σ^1_1; thus x is Δ^1_1 (see, for example, Lemma 7.3.5 of [5]). □

We can now prove our basis theorem for σ-treeable sets:

Theorem 5. *Suppose that X is a recursively presented Polish space and A is a non-empty σ-treeable Σ^1_1 subset of X. Then A has a Δ^1_1 member.*

Proof. Let $\mathcal{I}$ denote the σ-ideal of Σ^1_1-meager subsets of X, and fix $x \in A$. As A is σ-treeable, there is an acyclic Σ^1_1 graph $\mathcal{T}$ on A such that $[x]_{\mathcal{T}} \notin \mathcal{I}$. Proposition 1 implies that there is a $(\mathcal{T}, \mathcal{I}, r)$-critical point $y \in [x]_{\mathcal{T}}$, for some $r \in \mathbb{N}$, and Proposition 4 ensures that y is the desired Δ^1_1 member of A. □

We can now characterize the existence of Borel complete countable sections:

Theorem 6. *Suppose that X is a Polish space and E is an essentially countable $\boldsymbol{\Sigma}^1_1$ equivalence relation on X. Then the following are equivalent:*

(1) *E has a Borel complete countable section;*
(2) *E is σ-treeable.*

Proof. To see (1) ⇒ (2), suppose that $B \subseteq X$ is a Borel E-complete countable section. By Exercise 35.13 of [4], there are functions f_n with $\boldsymbol{\Sigma}^1_1$ graphs such that

$$E \cap ((X \setminus B) \times B) = \bigcup_{n \in \mathbb{N}} \text{graph}(f_n),$$

and functions g_n with $\boldsymbol{\Sigma}^1_1$ graphs such that $\text{dom}(g_n) \cap \text{range}(g_n) = \emptyset$ and

$$E \cap (B \times B) \setminus \Delta(B) = \bigcup_{n \in \mathbb{N}} \text{graph}(g_n).$$

Then $\mathcal{T}_{mn} = \mathrm{graph}(f_m^{\pm 1}) \cup \mathrm{graph}(g_n^{\pm 1})$ is an acyclic $\boldsymbol{\Sigma}^1_1$ graph, for all $m, n \in \mathbb{N}$, and $E = \bigcup_{m,n\in\mathbb{N}} E_{\mathcal{T}_{mn}}$; thus E is σ-treeable.

To see (2) $\Rightarrow$ (1) note that, by the usual relativization arguments, we can assume that X is recursively presented, E is Σ^1_1, there is a sequence $\langle \mathcal{T}_n \rangle_{n\in\mathbb{N}}$ of acyclic Σ^1_1 graphs on X such that $E = \bigcup_{n\in\mathbb{N}} E_{\mathcal{T}_n}$, and there is a recursively presented Polish space Y, a countable equivalence relation F on Y, and a Δ^1_1 reduction $\pi : X \to Y$ of E to F. Then for each $y \in \pi[X]$, the $\Delta^1_1(y)$ set $[\pi^{-1}(y)]_E$ is $\sigma(y)$-treeable; thus the obvious relativization of Theorem 5 ensures that it has a $\Delta^1_1(y)$ member. The uniformization criterion for Δ^1_1 sets (see, for example, §8.4 of [5]) then implies that there is a Δ^1_1 function $\phi : \pi[X] \to X$ such that $\forall x \in X\ (xE\phi \circ \pi(x))$, and it follows that $\phi \circ \pi[X]$, being the image of a countable-to-one Δ^1_1 function, is the desired Δ^1_1 countable complete section. ☐

Along similar lines, we can now characterize the existence of Borel transversals.

Theorem 7. *Suppose that X is a Polish space and E is a smooth σ-treeable equivalence relation on X. Then E has a Borel transversal.*

Proof. By the usual relativization arguments, we can assume that X is recursively presented, E is Σ^1_1, there is a sequence $\langle \mathcal{T}_n \rangle_{n\in\mathbb{N}}$ of acyclic Σ^1_1 graphs on X such that $E = \bigcup_{n\in\mathbb{N}} E_{\mathcal{T}_n}$, and there is a recursively presented Polish space Y and a Δ^1_1 reduction $\pi : X \to Y$ of E to $\Delta(Y)$. Then for each $y \in \pi[X]$, the $\Delta^1_1(y)$ set $[\pi^{-1}(y)]_E$ is $\sigma(y)$-treeable; thus the obvious relativization of Theorem 5 ensures that it has a $\Delta^1_1(y)$ member. The uniformization criterion for Δ^1_1 sets then implies that there is a Δ^1_1 function $\phi : \pi[X] \to X$ such that $\forall x \in X\ (xE\phi \circ \pi(x))$, and it follows that $\phi \circ \pi[X]$ is the desired Δ^1_1 transversal. ☐

3. Idealistic equivalence relations

We say that a σ-ideal $\mathcal{I}$ on a Polish space X is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ if for every Polish space W and $\boldsymbol{\Sigma}^1_1$ set $R \subseteq W \times X$, the set $\{w \in W : R_w \in \mathcal{I}\}$ is $\boldsymbol{\Pi}^1_1$. We say that a set $A \subseteq W \times X$ is *pairwise disjoint* if

$$\forall v, w \in W\ (v \neq w \Rightarrow A_v \cap A_w = \emptyset).$$

We say that $\mathcal{I}$ is *weakly ccc* if for every Polish space W and $\boldsymbol{\Sigma}^1_1$ pairwise disjoint set $A \subseteq W \times X$, there are only countably many $w \in W$ such that $A_w \notin \mathcal{I}$. A routine boundedness or reflection argument gives that a $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ σ-ideal $\mathcal{I}$ is weakly ccc if and only if there is no Borel pairwise disjoint set $B \subseteq 2^{\mathbb{N}} \times X$ such that $\forall w \in 2^{\mathbb{N}}\ (B_w \notin \mathcal{I})$.

Proposition 8. *Suppose that X, Y are Polish spaces and $\mathcal{I}, \mathcal{J}$ are $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ weakly ccc σ-ideals on X, Y. Then $\mathcal{I} * \mathcal{J}$ is weakly ccc.*

Proof. Suppose, towards a contradiction, that there is a Polish space W and a $\boldsymbol{\Sigma}^1_1$ pairwise disjoint set $A \subseteq W \times (X \times Y)$ for which there are uncountably many $w \in W$ such that $A_w \notin \mathcal{I} * \mathcal{J}$. Then the set $B = \{(w, x) \in W \times X : (A_w)_x \notin \mathcal{J}\}$ is $\boldsymbol{\Sigma}^1_1$ and there are uncountably many $w \in W$ such that $B_w \notin \mathcal{I}$.

Lemma 9. $\forall x \in X\ (|B^x| \leq \aleph_0)$.

Proof. Simply note that if $x \in X$, then the set $C = \{(w, y) \in W \times Y : (w, x, y) \in A\}$ is a $\boldsymbol{\Sigma}^1_1$ pairwise disjoint set and $\forall w \in B^x\ (C_w \notin \mathcal{J})$, so the fact that $\mathcal{J}$ is weakly ccc ensures that B^x is countable. ☐

Fix functions f_n with $\mathbf{\Sigma^1_1}$ graphs such that $B = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n^{-1})$. For each $n \in \mathbb{N}$, set $B_n = \mathrm{graph}(f_n^{-1})$, and observe that if $B_w \notin \mathcal{I}$, then there exists $n \in \mathbb{N}$ such that $(B_n)_w \notin \mathcal{I}$. Fix $n \in \mathbb{N}$ for which there are uncountably many $w \in W$ such that $(B_n)_w \notin \mathcal{I}$. As B_n is clearly pairwise disjoint, this contradicts our assumption that $\mathcal{I}$ is weakly ccc. □

Let $B_{\mathcal{T}} = \{(x,(y,z)) \in X \times (X \times X) : x \text{ is } \mathcal{T}\text{-between } y, z\}$, and let

$$\mathrm{crit}(\mathcal{T},\mathcal{I}) = \{x \in X : (B_{\mathcal{T}})_x \notin \mathcal{I} * \mathcal{I}\}$$

denote the set of $(\mathcal{T},\mathcal{I})$-*critical points*.

Proposition 10. *Suppose that X is a Polish space, $\mathcal{T}$ is an acyclic $\mathbf{\Sigma^1_1}$ graph on X, and $\mathcal{I}$ is a $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ weakly ccc σ-ideal on X. Then $|\mathrm{crit}(\mathcal{T},\mathcal{I})| \leq \aleph_0$.*

Proof. For each $r \in \mathbb{N}$, let A_r denote the set of $(x,(y,z)) \in B_{\mathcal{T}}$ such that x is the r^{th} point along the injective $\mathcal{T}$-path from y to z. It is clear that A_r is $\mathbf{\Sigma^1_1}$, and if $v \neq w$, then $(A_r)_v \cap (A_r)_w = \emptyset$. Proposition 8 then ensures that there are only countably many x such that $(A_r)_x \notin \mathcal{I} * \mathcal{I}$, and it follows that there are only countably many $(\mathcal{T},\mathcal{I})$-critical points. □

We can now establish the promised sufficient conditions for essential countability:

Theorem 11. *Suppose that X is a Polish space and E is a ccc idealistic σ-treeable equivalence relation on X. Then E has a Borel complete countable section.*

Proof. Fix a $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ assignment $C \mapsto \mathcal{I}_C$ of non-trivial ccc σ-ideals $\mathcal{I}_C$ on C to each equivalence class C of E, as well as acyclic $\mathbf{\Sigma^1_1}$ graphs $\mathcal{T}_n$ such that $E = \bigcup_{n \in \mathbb{N}} E_{\mathcal{T}_n}$. For each $n \in \mathbb{N}$, set $A_n = \bigcup_{C \in X/E} \mathrm{crit}(\mathcal{T}_n, \mathcal{I}_C)$, and observe that

$$A_n = \bigcup_{C \in X/E} \{x \in C : \{y \in C : \{z \in C : (x,(y,z)) \in B_{\mathcal{T}_n}\} \notin \mathcal{I}\} \notin \mathcal{I}\};$$

thus A_n is $\mathbf{\Sigma^1_1}$. Proposition 8 ensures that A_n is an E-countable section. For each $C \in X/E$ and $x \in C$, the fact that $C = \bigcup_{n \in \mathbb{N}} [x]_{\mathcal{T}_n}$ implies that there exists $n \in \mathbb{N}$ such that $[x]_{\mathcal{T}_n} \notin \mathcal{I}_C$; thus Proposition 1 ensures that $A_n \cap [x]_{\mathcal{T}_n} \neq \emptyset$. It follows that $A = \bigcup_{n \in \mathbb{N}} A_n$ is an E-complete countable section. As A is $\mathbf{\Sigma^1_1}$ and the Mazurkiewicz-Sierpiński Theorem (see, for example, Theorem 29.19 of [4]) implies that the property of being an E-countable section is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$, it follows from the first reflection theorem (see, for example, Theorem 35.10 of [4]) that A is contained in a Borel E-complete countable section. □

As a corollary, we obtain the following:

Theorem 12. *Suppose that X is a Polish space and E is a σ-treeable equivalence relation on X which is either generated by a Borel action of a Polish group or measured. Then E has a Borel complete countable section.*

Proof. In light of Theorem 11, it is enough to show that if E is generated by a Borel action of a Polish group, then E is ccc idealistic. Towards this end, suppose that G is a Polish group which acts in a Borel fashion on X so as to generate E. For each $x \in X$, let $\mathcal{I}_x$ denote the σ-ideal of sets $A \subseteq [x]_E$ such that $\{g \in G : g \cdot x \in A\}$ is meager. Localization easily implies that $\mathcal{I}_x$ is ccc. As meagerness is preserved under right multiplication, it follows that if xEy, then $\mathcal{I}_x = \mathcal{I}_y$; thus we obtain an assignment $C \mapsto \mathcal{I}_C$ of σ-ideals to the equivalence classes of E by setting $\mathcal{I}_C = \mathcal{I}_x$, for $x \in C$. Novikov's Theorem (see, for example, Theorem 29.22 of [4]) implies that $C \mapsto \mathcal{I}_C$ is $\mathbf{\Sigma^1_1}$ on $\mathbf{\Pi^1_1}$, and it follows that E is ccc idealistic. □

Recall that E_1 is the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by

$$xE_1y \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

In light of [2] and our work here, it is natural to ask the following:

Question 13. Suppose that E is a σ-treeable Borel equivalence relation on a Polish space. Must it be the case that exactly one of the following holds:

(1) E is essentially countable, or
(2) $E_1 \leq_B E$?

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