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ALTERNATE SIGNS BANACH-SAKS PROPERTY AND REAL INTERPOLATION OF OPERATORS

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ABSTRACT. In the space of bounded linear operators acting between Banach spaces we define a seminorm vanishing on the subspace of operators having the alternate signs Banach-Saks property. We obtain logarithmically convex-type estimates of the seminorm for operators interpolated by the Lions-Peetre real method. In particular, the estimates show that the alternate signs Banach-Saks property is inherited from a space of an interpolation pair (A_0, A_1) to the real interpolation spaces $A_{\theta,p}$ with respect to (A_0, A_1) for all $0 < \theta < 1$ and $1 < p < \infty$.

1. INTRODUCTION

A bounded linear operator $T\colon X \to Y$ acting between Banach spaces is said to have the Banach-Saks (BS) property if every bounded sequence (x_n) in X contains a subsequence (x'_n) such that the Cesàro means of (Tx'_n) converge in Y. If we restrict this definition to all weakly null sequences (x_n) in X, we say that T has the weak Banach-Saks (WBS) property or the Banach-Saks-Rosenthal property. We say that T has the alternate signs Banach-Saks (ABS) property if every bounded sequence (x_n) in X contains a subsequence (x'_n) such that the Cesàro means of $((-1)^n Tx'_n)$ converge in Y.

A Banach space X is said to have the BS, WBS or ABS property if the corresponding property is possessed by the identity operator $I\colon X \to X$. The ABS property is weaker than the BS property and stronger than the WBS property. The relations are strict: c_0 has the ABS property but does not have the BS property; l_1 has the WBS property but does not have the ABS property. For a detailed study of these properties we refer the reader to [2].

A natural question is the behavior of Banach-Saks properties under interpolation. Beauzamy [2] proved that if (A_0, A_1) is an interpolation pair such that A_0 is continuously embedded in A_1 and the embedding has the ABS property, then the real interpolation spaces $A_{\theta,p}$ with respect to (A_0, A_1) have the ABS property for all $0 < \theta < 1$ and $1 < p < \infty$. This in turn served to show that every operator with the BS or ABS property factors through a space with the same property (see also [4]). Heinrich [9] proved that if the embedding $I\colon A_0 \cap A_1 \to A_0 + A_1$ has the BS

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property, then so has $A_{\theta,p}$ with respect to (A_0, A_1) for all $0 < \theta < 1$ and $1 < p < \infty$ (see also [3]).

Another line of investigation has a quantitative character. In this approach, we focus on inequalities which may also provide qualitative information. Astala and Tylli introduced the so-called outer [1] and inner [14] measures for bounded linear operators. These measures are related to certain operator ideals such as compact, weakly compact, Rosenthal operators and operators with the BS property. Roughly speaking, they measure the deviation of an operator from a given ideal. The interpolation properties of these uniform measures have been investigated in detail by Cobos, Manzano and Martínez in [6] and [7].

The aim of this paper is to find a measure of deviation from the ABS property with good interpolation properties. Our work is motivated by [10], where similar results for a measure of weak noncompactness were obtained. Using Beauzamy's geometric characterization of the ABS property, we define in the space of bounded linear operators a seminorm vanishing on the subspace of operators having the ABS property. As in Beauzamy's works, the key tool here is the spreading model of Brunel and Sucheston. We obtain logarithmically convex (up to a constant) estimates of the seminorm for operators interpolated by the real method. In particular, the inequalities show that the ABS property is hereditary under real interpolation. It is worth pointing out that the types of estimates obtained in this paper are significantly different from that of [6] and [7].

The cardinality of a subset $A \subset \mathbb{N}$ will be denoted by $|A|$. We write $\mathsf{B}(X)$ for the open unit ball of a Banach space X. By $\mathcal{L}(X,Y)$ we denote the space of all bounded linear operators acting between Banach spaces X and Y. The subspace of $\mathcal{L}(X,Y)$ consisting of all operators having the ABS property will be denoted by $\mathcal{ABS}(X,Y)$.

2. ABS property and spreading models

One of the basic results on Banach-Saks properties is the following one of Rosenthal [13]: if a Banach space X does not have the WBS property, then there exist a number $\delta > 0$ and a bounded sequence (x_n) in X such that for all $k \in \mathbb{N}$, all subsets $A \subset \mathbb{N}$ with $|A| = 2^k$ and $k \leq \min A$, and all sequences of scalars (c_n), we have $\left\|\sum_{n\in A} c_n x_n\right\| \geq \delta \sum_{n\in A} |c_n|$.

Beauzamy [2] proved that the above implication turns into an equivalence if we replace the WBS property by the ABS property. The starting point of our considerations is another of Beauzamy's results: a Banach space X does not have the ABS property if and only if there exist a number $\delta > 0$ and a bounded sequence (x_n) in X such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n), $\epsilon_n = \pm 1$ for all n, we have $\left\|\sum_{n\in A} \epsilon_n x_n\right\| \geq \delta\, |A|$.

We introduce the following relations: (y_n) is a sequence of *successive variable signs means* (svsm) for (x_n) if there exist $m \in \mathbb{N}$, a sequence (A_n) of finite subsets of $\mathbb{N}$ with $\max A_n < \min A_{n+1}$ and $|A_n| = m$ for all n, and a sequence of signs (ϵ_n), such that $y_n = m^{-1} \sum_{k\in A_n} \epsilon_k x_k$ for all n; if in this definition $\epsilon_n = 1$ for all n, then (y_n) is called a sequence of *successive arithmetic means* (sam) for (x_n). Since all sets A_n are equipollent, the relation svsm is transitive.

Definition 2.1. Let (x_n) be a bounded sequence in a Banach space X. Define

$$\phi_{vsm}(x_n) = \inf \left\| |A|^{-1} \sum_{n\in A} \epsilon_n x_n \right\|, \quad \phi_{am}(x_n) = \inf \left\| |A|^{-1} \sum_{n\in A} x_n \right\|,$$

the infimum for ϕ_{vsm} being taken over all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n), the infimum for ϕ_{am} being taken over all finite subsets $A \subset \mathbb{N}$.

If (y_n) is a sequence of svsm for (x_n), in particular, if (y_n) is a subsequence of (x_n) or (y_n) is a sequence of sam for (x_n), then $\phi_{vsm}(x_n) \le \phi_{vsm}(y_n)$.

We recall the result of Brunel and Sucheston [5] on extraction of 'good' subsequences, following [2]. We apply this result to construct 'good' sequences of svsm and sam for a given bounded sequence.

Proposition 2.2. *Let (x_n) be a bounded sequence in a Banach space X. There exist a subsequence (x'_n) of (x_n) and a seminorm L in the set S of all finite sequences of scalars (real or complex), with the following property: for every $\varepsilon > 0$ and every $a = (a_1, \dots, a_m) \in S$ there exists $v \in \mathbb{N}$ such that, if $v \le n_1 < \dots < n_m$, then $\left| \left\| \sum_{i=1}^m a_i x'_{n_i} \right\| - L(a) \right| < \varepsilon$.*

If (x_n) has no Cauchy subsequence, the formula $\|a_1 x'_1 + \dots + a_m x'_m\|_E = L(a)$, $a = (a_1, \dots, a_m)$, defines a norm in the space spanned by vectors x'_n. Let E be the completion of $\operatorname{span}\{x'_n\}$ under this norm. The space E is called the spreading model of X built on (x_n). The sequence (x'_n) is called the fundamental sequence of E. The norm of E is invariant under spreading; that is, $\|a_1 x'_1 + \dots + a_m x'_m\|_E = \left\|a_1 x'_{n_1} + \dots + a_m x'_{n_m}\right\|_E$ for all $n_1 < \dots < n_m$.

The next proposition will play a key role in our considerations. Its assertion is related to property (P'_1) of [2]. In the proof, we follow the main line of the proof of Theorem II.2 of [2].

Proposition 2.3. *Let (x_n) be a bounded sequence in a Banach space X. Then for every $\varepsilon > 0$ there exist a sequence (y_n) of svsm for (x_n) and a sequence (v_n) of sam for (x_n) such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),*

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_n \right\| \le \phi_{vsm}(y_n) + \varepsilon, \quad \left\| |A|^{-1} \sum_{n\in A} v_n \right\| \le \phi_{am}(v_n) + \varepsilon.$$

Proof. We prove the assertion for the relation svsm. The proof for the relation sam is almost the same. Fix $\varepsilon > 0$. First assume that (x_n) contains a Cauchy subsequence (x'_n). Put $y_n = (x'_{2n} - x'_{2n-1})/2$. Ignoring a finite number of terms of (y_n), we see that (y_n) satisfies the assertion.

Now assume that (x_n) has no Cauchy subsequence. Let a subsequence (x'_n) of (x_n) be the fundamental sequence of the spreading model E built on (x_n), given by Proposition 2.2. Taking (x'_n) in the norm $\|\cdot\|_E$, we put $K = \phi_{vsm}(x'_n)$. There exists $z = m^{-1} \sum_{i=1}^m \epsilon'_i x'_{n_i}$, where $n_1 < \dots < n_m$ and $\epsilon'_1, \dots, \epsilon'_m$ is a finite sequence of signs, such that $K \le \|z\|_E \le K + \varepsilon/4$. Put $z_n = m^{-1} \sum_{i=1}^m \epsilon'_i x'_{(n-1)m+i}$ for every $n \in \mathbb{N}$. Since $\|\cdot\|_E$ is invariant under spreading, $K \le \|z_n\|_E \le K + \varepsilon/4$. Clearly, for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$K \le \left\| |A|^{-1} \sum_{n\in A} \epsilon_n z_n \right\|_E \le K + \varepsilon/4.$$

Let $k \in \mathbb{N}$. Applying Proposition 2.2, we get n_k such that if $B \subset \mathbb{N}$ with $|B| \le 2^k$ and $n_k \le \min B$, then for all sequences of signs (ϵ_n),

$$\left| \left\| |B|^{-1} \sum_{n\in B} \epsilon_n z_n \right\| - \left\| |B|^{-1} \sum_{n\in B} \epsilon_n z_n \right\|_E \right| < \varepsilon/4.$$

We may assume that $n_k < n_{k+1}$ for all k. It follows that for the sequence (z'_k) with $z'_k = z_{n_k}$, all $B \subset \mathbb{N}$ with $|B| \le 2^k$ and $k \le \min B$, and all sequences of signs (ϵ_n),

$$K - \varepsilon/4 \le \left\| |B|^{-1} \sum_{n\in B} \epsilon_n z'_n \right\| \le K + \varepsilon/2.$$

Let $A \subset \mathbb{N}$ be finite and $A_0 = \{n \in A : n < \log_2 |A|\}$. Then

$$\left\| \sum_{n\in A_0} \epsilon_n z'_n \right\| \le |A_0| \, (K + \varepsilon/2) \quad \text{and} \quad \left\| \sum_{n\in A\setminus A_0} \epsilon_n z'_n \right\| \ge |A\setminus A_0| \, (K - \varepsilon/4) \, .$$

Of course, we assume that the sum over the empty set is 0. Consequently,

$$\begin{aligned} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n z'_n \right\| &\ge \left\| |A|^{-1} \sum_{n\in A\setminus A_0} \epsilon_n z'_n \right\| - \left\| |A|^{-1} \sum_{n\in A_0} \epsilon_n z'_n \right\| \\ &\ge K - \varepsilon/4 - |A_0| \, |A|^{-1} \, (2K + \varepsilon/4). \end{aligned}$$

There is an $m_0 \in \mathbb{N}$ such that if $|A| \ge m_0$, then $|A_0| \, |A|^{-1} \, (2K + \varepsilon/4) \le \varepsilon/4$. Then

$$K - \varepsilon/2 \le \left\| |A|^{-1} \sum_{n\in A} \epsilon_n z'_n \right\| \le K + \varepsilon/2.$$

Let $y_n = m_0^{-1} \sum_{i=1}^{m_0} z'_{(n-1)m_0+i}$ for every $n \in \mathbb{N}$. Then for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$K + \varepsilon/2 \ge \left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_n \right\| \ge \left\| |A|^{-1} \, m_0^{-1} \sum_{n\in A} \sum_{i=1}^{m_0} \epsilon_n z'_{(n-1)m_0+i} \right\| \ge K - \varepsilon/2.$$

Thus $\left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_n \right\| \le \phi_{vsm}(y_n) + \varepsilon$ for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). Of course, (y_n) is a sequence of svsm for (x_n). □

Definition 2.4. Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Define

$$\Phi_{ABS}(T) = \sup \{\phi_{vsm}(Tx_n) : (x_n) \subset \mathsf{B}(X)\} \, .$$

Proposition 2.5. *Φ_{ABS} is a seminorm in $\mathcal{L}(X, Y)$. $\Phi_{ABS}(T) = 0$ if and only if $T \in \mathcal{ABS}(X, Y)$.*

Proof. Clearly, $\Phi_{ABS}(\lambda T) = |\lambda| \, \Phi_{ABS}(T)$ for all scalars λ. We show that for all $S, T \in \mathcal{L}(X, Y)$, $\Phi_{ABS}(S+T) \le \Phi_{ABS}(S) + \Phi_{ABS}(T)$. Let $\varepsilon > 0$ and $(x_n) \subset \mathsf{B}(X)$. By Proposition 2.3, there exists a sequence (x'_n) of svsm for (x_n) such that for the sequence (Sx'_n) of svsm for (Sx_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n S x'_n \right\| \le \phi_{vsm}(Sx'_n) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). Again applying Proposition 2.3, we get a sequence (x''_n) of svsm for (x'_n), such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n T x''_n \right\| \leq \phi_{vsm}(Tx''_n) + \varepsilon.$$

Since the relation svsm is transitive,

$$\begin{aligned}\phi_{vsm}((S+T)x_n) \leq \phi_{vsm}((S+T)x''_n) \leq \left\| |A|^{-1} \sum_{n\in A} \epsilon_n (S+T) x''_n \right\| \\ \leq \left\| |A|^{-1} \sum_{n\in A} \epsilon_n S x''_n \right\| + \left\| |A|^{-1} \sum_{n\in A} \epsilon_n T x''_n \right\| \\ \leq \phi_{vsm}(Sx'_n) + \phi_{vsm}(Tx''_n) + 2\varepsilon \leq \Phi_{ABS}(S) + \Phi_{ABS}(T) + 2\varepsilon.\end{aligned}$$

By an arbitrary choice of $\varepsilon > 0$ and $(x_n) \subset \mathsf{B}(X)$, we obtain the conclusion.

By Theorem IV.2 of [2], T has the ABS property if and only if for every bounded sequence (x_n) in X there exist a subsequence (x'_n) of (x_n) and a sequence of signs (ϵ_n) such that the Cesàro means of $(\epsilon_n T x'_n)$ converge to 0 in Y. From this and by Theorem III.1 of [2], T has the ABS property if and only if for every bounded sequence (x_n) in X, $\phi_{vsm}(Tx_n) = 0$. By positive homogeneity of Φ_{ABS}, T has the ABS property if and only if $\Phi_{ABS}(T) = 0$. □

3. ABS property and $l_p(X)$ spaces

Let X be a Banach space, $1 < p < \infty$ and let (e_i) be the unit vector basis of l_p. We denote by $l_p(X)$ the Banach space of all sequences $x = (x(i))$ such that $x(i) \in X$ for every $i \in \mathbb{N}$ and

$$\|x\|_{l_p(X)} = \left\| \sum_{i=1}^{\infty} \|x(i)\|_X \, e_i \right\|_{l_p} < \infty.$$

In the sequel, we also deal with $l_p(X)$ of the families $(x(i))_{i\in\mathbb{Z}}$ indexed by integers.

Partington [12] proved that $l_p(X)$, $1 < p < \infty$, has the BS property if and only if so has X (in fact, a more general setting of direct sums was used). We use similar arguments as in the proof of Theorem 3 of [12] to show the next lemma.

Lemma 3.1. *Let X be a Banach space and (x_n) a bounded sequence in $l_p(X)$, $1 < p < \infty$. Then for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and a sequence (y_n) of sam for (x_n) such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),*

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_n(i) \right\|_X e_i \right\|_{l_p} < \varepsilon.$$

Proof. For $x_n = (x_n(i)) \in l_p(X)$, let $t_n = \sum_{i=1}^{\infty} \|x_n(i)\|_X \, e_i \in l_p$. Since l_p has the BS property, by Erdös-Magidor's theorem in [8], there exists a subsequence (t'_n) of (t_n) such that the Cesàro means of all subsequences of (t'_n) converge to the same limit t in l_p. Then $\phi_{am}(s_n - t) = 0$ for every sequence (s_n) of sam for (t'_n). By

Proposition 2.3, there exists a sequence (s_n) of sam for (t'_n) such that for every finite subset $A \subset \mathbb{N}$,

$$\left\| |A|^{-1} \sum_{n\in A} s_n - t \right\|_{l_p} < \varepsilon/2.$$

There exist $k_0 \in \mathbb{N}$ and a sequence (A_n) of finite subsets of $\mathbb{N}$ with $\max A_n < \min A_{n+1}$ and $|A_n| = k_0$ for all n such that $s_n = k_0^{-1} \sum_{k\in A_n} t'_k$. Let (y_n) be the corresponding sequence of sam for (x_n). That is, first we take the subsequence (x'_n) of (x_n) such that $t'_n = \sum_{i=1}^{\infty} \|x'_n(i)\|_X \, e_i$, and then we put $y_n = k_0^{-1} \sum_{k\in A_n} x'_k$. Let $t = \sum_{i=1}^{\infty} \alpha_i e_i$ and let $m \in \mathbb{N}$ satisfy $\left\|\sum_{i=m+1}^{\infty} \alpha_i e_i\right\|_{l_p} < \varepsilon/2$. Then for every finite subset $A \subset \mathbb{N}$,

$$\left\| \sum_{i=m+1}^{\infty} \left(|A|^{-1} \sum_{n\in A} k_0^{-1} \sum_{k\in A_n} \|x'_k(i)\|_X - \alpha_i \right) e_i \right\|_{l_p} < \varepsilon/2.$$

It follows that

$$\left\| \sum_{i=m+1}^{\infty} \left(|A|^{-1} \sum_{n\in A} k_0^{-1} \sum_{k\in A_n} \|x'_k(i)\|_X \right) e_i \right\|_{l_p} < \varepsilon.$$

By hyperorthogonality of the basis (e_i), for all sequences of signs (ϵ_n),

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_n(i) \right\|_X e_i \right\|_{l_p} < \varepsilon. \qquad \square$$

The next result is crucial for the interpolation properties of Φ_{ABS}. This is a counterpart of Theorem 3.6 of [10] proved for a measure of weak noncompactness.

Theorem 3.2. *Let X, Y be Banach spaces and $1 < p < \infty$. If $T \in \mathcal{L}(X,Y)$ and if $\widetilde{T} \in \mathcal{L}(l_p(X), l_p(Y))$ is given by $\widetilde{T}x = (Tx(i))$ for every $x = (x(i))$, then $\Phi_{ABS}(T) = \Phi_{ABS}(\widetilde{T})$.*

Proof. Since $l_p(X)$ contains isometric copies of X, $\Phi_{ABS}(T) \le \Phi_{ABS}(\widetilde{T})$. Fix $\varepsilon > 0$. There exists $(x_n) \subset \mathsf{B}(l_p(X))$ such that $\Phi_{ABS}(\widetilde{T}) - \varepsilon \le \phi_{vsm}(\widetilde{T}x_n)$. By Lemma 3.1, there exist $m \in \mathbb{N}$ and a sequence (x'_n) of sam for (x_n) such that for the sequence $(\widetilde{T}x'_n)$ of sam for $(\widetilde{T}x_n)$, and for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n T x'_n(i) \right\|_Y e_i \right\|_{l_p} < \varepsilon.$$

There exists a subsequence (x''_n) of (x'_n) such that for each $1 \le i \le m$ the limit $\beta_i = \lim_n \|x''_n(i)\|_X$ exists and $\|x''_n(i)\|_X < \beta_i + \varepsilon/m$ for every n. Putting $v_n(i) = (\beta_i + \varepsilon/m)^{-1} T x''_n(i)$, we have $(v_n(i)) \subset T(\mathsf{B}(X))$ for every $1 \le i \le m$.

By Proposition 2.3, there exists a sequence (x_n^1) of svsm for (x''_n) such that for the sequence $(v_n^1(1))$ of svsm for $(v_n(1))$, where $v_n^1(i) = (\beta_i + \varepsilon/m)^{-1} T x_n^1(i)$, $1 \le i \le m$, we have

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n v_n^1(1) \right\|_Y \le \phi_{vsm}\left(v_n^1(1)\right) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n).

Proceeding in this way consecutively for $i = 2, \ldots, m$, in the kth step, we obtain a sequence (x_n^k) of svsm for (x_n^{k-1}) such that for the sequence $(v_n^k(k))$ of svsm for $(v_n^{k-1}(k))$, where $v_n^k(i) = (\beta_i + \varepsilon/m)^{-1} T x_n^k(i)$, $1 \le i \le m$, we have

$$\left\| |A|^{-1} \sum_{n \in A} \epsilon_n v_n^k(k) \right\|_Y \le \phi_{vsm}\left(v_n^k(k)\right) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). In this way, all sequences $(v_n^m(i))$, $1 \le i \le m$, are built on the common sequence (x_n^m) of svsm for (x_n), and for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n \in A} \epsilon_n v_n^m(i) \right\|_Y \le \phi_{vsm}\left(v_n^m(i)\right) + \varepsilon, \quad 1 \le i \le m.$$

It follows that

$$\begin{aligned}
\phi_{vsm}(\widetilde{T} x_n) \le \phi_{vsm}(\widetilde{T} x_n^m) &\le \left\| \sum_{i=1}^m \left\| |A|^{-1} \sum_{n \in A} \epsilon_n T x_n^m(i) \right\|_Y e_i \right\|_{l_p} + \varepsilon \\
&= \left\| \sum_{i=1}^m \left\| (\beta_i + \varepsilon/m) \, |A|^{-1} \sum_{n \in A} \epsilon_n v_n^m(i) \right\|_Y e_i \right\|_{l_p} + \varepsilon \\
&\le \left\| \sum_{i=1}^m |\beta_i + \varepsilon/m| \, e_i \right\|_{l_p} \max_{1 \le i \le m} \left\| |A|^{-1} \sum_{n \in A} \epsilon_n v_n^m(i) \right\|_Y + \varepsilon \\
&\le \left(1 + \varepsilon m^{1/p-1}\right) \max_{1 \le i \le m} \left\{ \phi_{vsm}\left(v_n^m(i)\right) + \varepsilon \right\} + \varepsilon.
\end{aligned}$$

There exists $1 \le j \le m$ such that $\phi_{vsm}\left(v_n^m(j)\right) = \max_{1 \le i \le m} \phi_{vsm}\left(v_n^m(i)\right)$. Since $(v_n^m(j))$ is a sequence of svsm for $(v_n(j))$, we have $(v_n^m(j)) \subset T(\mathsf{B}(X))$ and consequently,

$$\Phi_{ABS}(\widetilde{T}) - 2\varepsilon \le \left(1 + \varepsilon m^{1/p-1}\right) \left(\Phi_{ABS}(T) + \varepsilon\right).$$

Letting $\varepsilon \to 0$, we get $\Phi_{ABS}(\widetilde{T}) \le \Phi_{ABS}(T)$. □

Corollary 3.3. *The space $l_p(X)$, $1 < p < \infty$, has the ABS property if and only if X has the ABS property.*

4. ABS property and real interpolation

We recall briefly some basic definitions and facts concerning real interpolation. For a thorough treatment we refer to [11].

If two Banach spaces A_0 and A_1 are linearly and continuously embedded in a common Hausdorff topological vector space V, we call $\vec{A} = (A_0, A_1)$ an interpolation pair. Then $\Delta(\vec{A}) = A_0 \cap A_1$, $\Sigma(\vec{A}) = A_0 + A_1$ are Banach spaces with norms

$$\|a\|_{\Delta(\vec{A})} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}, \quad \|a\|_{\Sigma(\vec{A})} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 + a_1 = a\}.$$

We consider a discrete method of construction of the real interpolation spaces of Lions and Peetre [11]. For $0 < \theta < 1$ and $1 < p < \infty$, let

$$A_{\theta,p} = \left\{ a \in \Sigma(\vec{A}) : \|a\|_{A_{\theta,p}} < \infty \right\},$$

where

$$\|a\|_{A_{\theta,p}} = \inf \max \left\{ \|(2^{i\theta} a_0(i))\|_{l_p(A_0)}, \|(2^{i(\theta-1)} a_1(i))\|_{l_p(A_1)} \right\},$$

the infimum being taken over all families $(a_0(i)) \subset A_0$ and $(a_1(i)) \subset A_1$ with $a_0(i) + a_1(i) = a$ for all $i \in \mathbb{Z}$. Then $\Delta(\vec{A}) \subset A_{\theta,p} \subset \Sigma(\vec{A})$ with continuous embeddings. The Banach space $A_{\theta,p}$ with norm $\|\cdot\|_{A_{\theta,p}}$ is called a real interpolation space with respect to $\vec{A} = (A_0, A_1)$. If $a \in A_{\theta,p}$, then

$$\|a\|_{A_{\theta,p}} \leq 2^{\theta(1-\theta)} \|(2^{i\theta} a_0(i))\|_{l_p(A_0)}^{1-\theta} \|(2^{i(\theta-1)} a_1(i))\|_{l_p(A_1)}^{\theta}$$

for all families $(a_0(i)) \subset A_0$ and $(a_1(i)) \subset A_1$ with $a_0(i) + a_1(i) = a$ for all $i \in \mathbb{Z}$ (see [11], p. 18).

Let $A_{\theta,p}$ and $B_{\theta,p}$ be two interpolation spaces with respect to the interpolation pairs $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$, and let $T\colon \Sigma(\vec{A}) \to \Sigma(\vec{B})$ be a linear operator. We write $T\colon \vec{A} \to \vec{B}$, if for $j = 0, 1$, the restriction $T|A_j$ is a bounded operator into B_j. For every $T\colon \vec{A} \to \vec{B}$,

$$\|T\colon A_{\theta,p} \to B_{\theta,p}\| \leq 2^{\theta(1-\theta)} \|T\colon A_0 \to B_0\|^{1-\theta} \|T\colon A_1 \to B_1\|^{\theta}.$$

In the main result of this paper, we show that this classical inequality concerning boundedness has its counterpart for the ABS property.

Theorem 4.1. *Let $A_{\theta,p}$ and $B_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ be real interpolation spaces with respect to interpolation pairs $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$. Then for every $T\colon \vec{A} \to \vec{B}$,*

$$\Phi_{ABS}(T\colon A_{\theta,p} \to B_{\theta,p}) \leq 2^{\theta(1-\theta)} \Phi_{ABS}^{1-\theta}(T\colon A_0 \to B_0) \Phi_{ABS}^{\theta}(T\colon A_1 \to B_1).$$

Proof. Fix $\varepsilon > 0$. Let (a_n) be a sequence in $\mathsf{B}(A_{\theta,p})$. For each a_n there exist $x_{jn} = (2^{i(\theta-j)} a_{jn}(i))_{i\in\mathbb{Z}} \in \mathsf{B}(l_p(A_j))$, $j = 0, 1$, such that $a_{0n}(i) + a_{1n}(i) = a_n$ for all $i \in \mathbb{Z}$. Set $y_{jn} = (2^{i(\theta-j)} T a_{jn}(i))_{i\in\mathbb{Z}}$ for $j = 0, 1$ and every $n \in \mathbb{N}$. As in the proof of subadditivity of Φ_{ABS}, by Proposition 2.3, passing to a sequence of svsm built on a common sequence of svsm for (a_n), we may assume that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_{jn} \right\|_{l_p(B_j)} \leq \phi_{vsm}(y_{jn}) + \varepsilon, \quad j = 0, 1.$$

Let $\widetilde{T}_j\colon l_p(A_j) \to l_p(B_j)$, $j = 0, 1$, be defined as the operator $\widetilde{T}$ in Theorem 3.2. Then $y_{jn} = \widetilde{T} x_{jn}$. It follows that

$$\begin{aligned}
\phi_{vsm}(Ta_n) &\leq \left\| |A|^{-1} \sum_{n\in A} \epsilon_n T a_n \right\|_{B_{\theta,p}} \\
&\leq 2^{\theta(1-\theta)} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_{0n} \right\|_{l_p(B_0)}^{1-\theta} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n y_{1n} \right\|_{l_p(B_1)}^{\theta} \\
&\leq 2^{\theta(1-\theta)} \left(\phi_{vsm}(y_{0n}) + \varepsilon\right)^{1-\theta} \left(\phi_{vsm}(y_{1n}) + \varepsilon\right)^{\theta} \\
&\leq 2^{\theta(1-\theta)} \left(\Phi_{ABS}(\widetilde{T}_0) + \varepsilon\right)^{1-\theta} \left(\Phi_{ABS}(\widetilde{T}_1) + \varepsilon\right)^{\theta}.
\end{aligned}$$

Since $l_p(X)$ with families indexed by integers is isometrically isomorphic to $l_p(X)$ with sequences indexed by $\mathbb{N}$, and ϕ_{vsm} is invariant under linear isometries, by

Theorem 3.2, $\Phi_{ABS}(\widetilde{T}_j) = \Phi_{ABS}(T\colon A_j \to B_j)$, $j = 0, 1$. By an arbitrary choice of ε and (a_n), we obtain the conclusion. □

As a corollary we obtain the following generalization of Proposition IV.1 of [2] without restrictions on interpolation pairs.

Corollary 4.2. *If $T\colon A_0 \to B_0$ or $T\colon A_1 \to B_1$ has the ABS property, then so has $T\colon A_{\theta,p} \to B_{\theta,p}$ for all $0 < \theta < 1$ and $1 < p < \infty$. In particular, if A_0 or A_1 has the ABS property, then so has $A_{\theta,p}$.*

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