ON A CLASS OF IDEALS OF THE TOEPLITZ ALGEBRA
ON THE BERGMAN SPACE

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Abstract. Let $\mathcal{T}$ denote the full Toeplitz algebra on the Bergman space of the unit ball $B_n$. For each subset $G$ of $L^\infty$, let $\mathcal{C}(G)$ denote the closed two-sided ideal of $\mathcal{T}$ generated by all $T_f T_g - T_g T_f$ with $f, g \in G$. It is known that $\mathcal{C}(C(B_n)) = K$, the ideal of compact operators, and $\mathcal{C}(C(B_n) \cap L^\infty) = \mathcal{I}$. Despite these “extreme cases”, there are subsets $G$ of $L^\infty$ so that $K \subsetneq \mathcal{C}(G) \subsetneq \mathcal{I}$. This paper gives a construction of a class of such subsets.

1. Introduction

For any integer $n \geq 1$, let $\mathbb{C}^n$ denote the Cartesian product of $n$ copies of $\mathbb{C}$. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write $\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let $B_n$ denote the open unit ball which consists of all $z \in \mathbb{C}^n$ with $|z| < 1$. Let $S_n$ denote the unit sphere which consists of all $z \in \mathbb{C}^n$ with $|z| = 1$. For any subset $V$ of $B_n$ we write $cl(V)$ for the closure of $V$ as a subset of $\mathbb{C}^n$ with respect to the Euclidean metric. We write $\overline{B}_n$ for the closed unit ball which is also $cl(B_n)$. Let $C(B_n)$ (respectively, $C(\overline{B}_n)$) denote the space of all functions that are continuous in the open unit ball (respectively, the closed unit ball).

Let $\nu$ denote the Lebesgue measure on $B_n$ normalized so that $\nu(B_n) = 1$. Let $L^2 = L^2(B_n, d\nu)$ and $L^\infty = L^\infty(B_n, d\nu)$. The Bergman space $L^2_a$ is the subspace of $L^2$ which consists of all analytic functions. The normalized reproducing kernels for $L^2_a$ are of the form

$$k_z(w) = (1 - |z|^2)^{(n+1)/2} (1 - \langle w, z \rangle)^{-n-1}, \quad |z|, |w| < 1.$$ 

We have $\|k_z\| = 1$ and $\langle g, k_z \rangle = (1 - |z|^2)^{(n+1)/2} g(z)$ for all $g \in L^2_a, z \in B_n$.

The orthogonal projection from $L^2$ onto $L^2_a$ is given by

$$(Pg)(z) = \int_{\overline{B}_n} \frac{g(w)}{(1 - \langle z, w \rangle)^{n+1}} d\nu(w), \quad g \in L^2, \quad z \in B_n.$$ 

For any $f \in L^\infty$, the Toeplitz operator $T_f : L^2_a \rightarrow L^2_a$ is defined by $T_f h = P(fh)$ for $h \in L^2_a$. We have

$$T_f h(z) = \int_{\overline{B}_n} \frac{f(w) h(w)}{(1 - \langle z, w \rangle)^{n+1}} d\nu(w) \quad (1.1)$$.
for $h \in L^2_n$ and $z \in \mathbb{B}_n$.

For all $f \in L^\infty$, $\|T_f\| \leq \|f\|_\infty$ and $T_f^* = T_{f^*}$. In contrast with Toeplitz operators on the Hardy space of the unit sphere, there are functions $f \in L^\infty$ so that $\|T_f\| < \|f\|_\infty$. Since $T_f$ is an integral operator by equation (1.1), we see that $T_f$ is compact if $f$ vanishes almost everywhere in the complement of a compact subset of $\mathbb{B}_n$.

Let $\mathfrak{B}(L^2_n)$ be the $C^*$-algebra of all bounded linear operators on $L^2_n$. Let $\mathcal{C}$ denote the ideal of $\mathfrak{B}(L^2_n)$ that consists of all compact operators. The full Toeplitz algebra $\mathfrak{T}$ is the $C^*$-subalgebra of $\mathfrak{B}(L^2_n)$ generated by $\{T_f : f \in L^\infty\}$. For any subset $G$ of $L^\infty$, let $\mathcal{C}(G)$ denote the closed two-sided ideal of $\mathfrak{T}$ generated by all $T_f$ with $f \in G$. Let $\mathfrak{C}(G)$ denote the closed two-sided ideal of $\mathfrak{T}$ generated by all commutators $[T_f, T_g] = T_fT_g - T_g T_f$ with $f, g \in G$. A result of L. Coburn [3] in the 1970’s showed that $\mathfrak{C}(C(\mathbb{B}_n)) = \mathcal{K}$. In 2004, D. Suárez [5] showed that $\mathfrak{C}(L^\infty) = \mathfrak{I}$ for the case $n = 1$. This result has been generalized by the author [2] to all $n \geq 1$. In fact, we are able to show that $\mathfrak{C}(G) = \mathfrak{T}$ for certain subsets $G$ of $L^\infty$. We can take $G = \{f \in C(\mathbb{B}_n) \cap L^\infty : f$ vanishes on $\mathbb{B}_n \setminus E\}$, where $0 < \nu(E)$ can be as small as we please. We can also take $G = \{f \in L^\infty : f$ vanishes on $\mathbb{B}_n \setminus E\}$ where $E$ is a closed nowhere dense subset of $\mathbb{B}_n$ with $0 < \nu(E)$ as small as we please. From these results, one may be interested in the question: is there any subset $G$ of $L^\infty$ so that $\mathcal{K} \subseteq \mathfrak{C}(G) \subseteq \mathfrak{I}$? The purpose of this paper is to show that there are infinitely many such subsets. Our main result is the following theorem.

**Theorem 1.1.** To every closed subset $F$ of $S_n$, there is a subset $G_F$ of $L^\infty$ so that the following statements hold true:

1. $\mathcal{G}(G_\emptyset) = \mathcal{K}$ and $\mathfrak{C}(G_{\mathbb{B}_n} \cap C(\mathbb{B}_n)) = \mathfrak{T}$.
2. If $F_1, F_2$ are closed subsets of $S_n$ and $F_1 \subset F_2$, then $G_{F_1} \subset G_{F_2}$.
3. If $F_1, F_2$ are closed subsets of $S_n$ and $F_2 \setminus F_1 \neq \emptyset$, then we have $\mathfrak{C}(G_{F_2} \cap C(\mathbb{B}_n)) \setminus \mathcal{G}(G_{F_1}) \neq \emptyset$. In particular, if $\emptyset \neq F \subseteq S_n$, then $\mathcal{K} \subseteq \mathfrak{C}(G_F \cap C(\mathbb{B}_n)) \subset \mathcal{C}(G_F) \subseteq \mathfrak{I}$.

In Section 2 and Section 3 we provide some preliminaries and basic results. In Section 4 we give a proof for Theorem 1.1.

2. Preliminaries

For any $z \in \mathbb{B}_n$, let $\varphi_z$ denote the Mobius automorphism of $\mathbb{B}_n$ that interchanges 0 and $z$. For any $z, w \in \mathbb{B}_n$, let $\rho(z, w) = |\varphi_z(w)|$. Then $\rho$ is a metric on $\mathbb{B}_n$ (called the pseudo-hyperbolic metric) which is invariant under the action of the automorphism group $\text{Aut}(\mathbb{B}_n)$ of $\mathbb{B}_n$. For any $w \in \mathbb{B}_n$, we have $\rho(z, w) \to 1$ as $|z| \to 1$. These properties and the following inequality can be proved by using the identities in [4] Theorem 2.2.2. See [2] Section 2] for more details.

For any $z, w, u \in \mathbb{B}_n$,

\begin{equation}
\rho(z, w) \leq \frac{\rho(z, u) + \rho(u, w)}{1 + \rho(z, u)\rho(u, w)}.
\end{equation}

For any $a$ in $\mathbb{B}_n$ and any $0 < r < 1$, let $B(a, r) = \{z \in \mathbb{B}_n : |z - a| < r\}$ and $E(a, r) = \{z \in \mathbb{B}_n : \rho(z, a) < r\}$. 

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Inequality (2.1) shows that if \( z, w \in \mathbb{B}_n \) so that \( E(z, r_1) \cap E(w, r_2) \neq \emptyset \) for some \( 0 < r_1, r_2 < 1 \), then
\[
\rho(z, w) \leq \frac{\rho(z, u) + \rho(u, w)}{1 + \rho(z, u) \rho(u, w)} \quad \text{(where } u \text{ is any element in } E(z, r_1) \cap E(w, r_2))
\]
\[
< \frac{r_1 + r_2}{1 + r_1 r_2}.
\]

This implies that if \( \rho(z, w) \geq \frac{r_1 + r_2}{1 + r_1 r_2} \), then \( E(z, r_1) \cap E(w, r_2) = \emptyset \).

**Lemma 2.1.** For any \( 0 < r < 1 \) and \( \zeta \in \mathbb{S}_n \) there is an increasing sequence \( \{t_m\}_{m=1}^\infty \subset (0, 1) \) so that \( t_m \to 1 \) as \( m \to \infty \) and \( E(t_k \zeta, r) \cap E(t_l \zeta, r) = \emptyset \) for all \( 1 \leq k < l \leq m \).

**Proof.** We will construct the required sequence \( \{t_m\}_{m=1}^\infty \) by induction. We begin by taking any \( t_1 \) in \( (0, 1) \). Suppose we have chosen \( t_1 < \cdots < t_m \) so that \( 1 - j^{-1} < t_j < 1 \) for all \( 1 \leq j \leq m \) and \( E(t_k \zeta, r) \cap E(t_l \zeta, r) = \emptyset \) for all \( 1 \leq k < l \leq m \), where \( m \geq 1 \). Since \( \rho(t \zeta, t \zeta) \to 1 \) as \( t \to 1 \) for all \( 1 < j \leq m \), we can choose a \( t_{m+1} \) with \( \max(t_m, 1 - (m + 1)^{-1}) < t_{m+1} < 1 \) and \( \rho(t_{m+1} \zeta, t_j \zeta) > \frac{2r}{1 + r^2} \) for all \( 1 \leq j \leq m \).

It then follows that \( E(t_{m+1} \zeta, r) \cap E(t_j \zeta, r) = \emptyset \) for all \( 1 \leq j \leq m \). Also, since \( 1 - m^{-1} < t_m < 1 \) for all \( m \), \( t_m \to 1 \) as \( m \to \infty \). \( \square \)

**Lemma 2.2.** For any \( 0 < r < 1 \) and any \( \epsilon > 0 \), there is a \( \delta \) depending on \( r \) and \( \epsilon \) so that for all \( \zeta \in \mathbb{S}_n \) and all \( a \in \mathbb{B}_n \) with \( |a - \zeta| < \delta \), we have
\[
E(a, r) \subset \{ z \in \mathbb{B}_n : |z - \zeta| < \epsilon \}.
\]

As a consequence, if \( b \in \mathbb{B}_n \) and \( \zeta \in \mathbb{S}_n \) so that \( E(b, r) \cap \{ z \in \mathbb{B}_n : |z - \zeta| < \delta \} \neq \emptyset \), then \( |b - \zeta| < \epsilon \).

**Proof.** From [4] Section 2.2.7], for any \( a \neq 0 \),
\[
E(a, r) \subset \{ z \in \mathbb{B}_n : \frac{|Pz - c|^2}{r^2 s^2} + \frac{|Qz|^2}{r^2 s} < 1 \},
\]
where \( Pz = (z, a) \), \( Qz = z - Pz \), \( c = \frac{1 - r^2}{1 - r^2 |a|^2} a \) and \( s = \frac{1 - |a|^2}{1 - r^2 |a|^2} \). It then follows that \( E(a, r) \subset B(c, r \sqrt{s}) \).

Since \( |a - c| = r^2 s |a| \leq r \sqrt{s} \), we get \( B(c, r \sqrt{s}) \subset B(a, 2r \sqrt{s}) \). Hence \( E(a, r) \subset B(a, 2r \sqrt{s}) \). Note that the inclusion certainly holds true for \( a = 0 \) (in this case \( s = 1 \)).

Now suppose \( |a - \zeta| < \delta \). Then \( |a| \geq |\zeta| - |a - \zeta| > 1 - \delta \). Hence,
\[
s = \frac{1 - |a|^2}{1 - r^2 |a|^2} \leq \frac{1 - |a|^2}{1 - r^2} \leq \frac{2(1 - |a|)}{1 - r^2} < \frac{2\delta}{1 - r^2}.
\]
So for any \( z \in E(a, r) \),
\[
|z - \zeta| \leq |z - a| + |a - \zeta| \leq 2r \sqrt{s} + |a - \zeta| < 2r \sqrt{s} + \delta < 2r \sqrt{s} + \delta.
\]

Choosing \( \delta \) so that \( 2r \sqrt{s} + \delta < \epsilon \), we then have the first conclusion of the lemma.

Now suppose \( a, b \in \mathbb{B}_n \) so that \( |a - \zeta| < \delta \) and \( a \in E(b, r) \). Then since \( b \in E(a, r) \), the first conclusion of the lemma implies \( |b - \zeta| < \epsilon \). \( \square \)
For any $z \in \mathbb{B}_n$, the formula
\[ U_z(f) = (f \circ \varphi_z)k_z, \quad f \in L^2, \]
defines a bounded operator on $L^2$. It is well-known that $U_z$ is a unitary operator with $L^2_\alpha$ as a reducing subspace and $U_zT_jU_z^* = T_{f \circ \varphi_z}$ on $L^2_\alpha$ for all $z \in \mathbb{B}_n$ and all $f \in L^\infty$. See, for example, [3, Lemmas 7 and 8].

**Lemma 2.3.** For any sequence $\{z_m\}_{m=1}^\infty \subset \mathbb{B}_n$ with $|z_m| \to 1$, $U_{z_m} \to 0$ in the weak operator topology of $\mathfrak{B}(L^2_\alpha)$.

**Proof.** Since Span\($\{k_z : z \in \mathbb{B}_n\}$) is dense in $L^2_\alpha$, it suffices to show that for all $z, w$ in $\mathbb{B}_n$, we have $\lim_{m \to \infty} \langle U_{z_m}k_z, k_w \rangle = 0$.

Fix such $z$ and $w$. For each $m \geq 1$,
\[
\langle U_{z_m}k_z, k_w \rangle = (1 - |w|^2)^{(n+1)/2}(U_{z_m}k_z)(w) = (1 - |w|^2)^{(n+1)/2k_z(\varphi_{z_m}(w))k_{z_m}(w)} = \frac{((1 - |w|^2)(1 - |z|^2)(1 - |z_m|^2))^{(n+1)/2}}{(1 - \langle \varphi_{z_m}(w), z \rangle)(1 - \langle w, z_m \rangle)^{n+1}}.
\]
Since $|\langle \varphi_{z_m}(w), z \rangle| \leq |z|$ and $|\langle w, z_m \rangle| \leq |w|$, we obtain
\[
|\langle U_{z_m}k_z, k_w \rangle| \leq \frac{((1 - |w|^2)(1 - |z|^2)(1 - |z_m|^2))^{(n+1)/2}}{(1 - |w|)^{n+1}}.
\]
It then follows that $\lim_{m \to \infty} \langle U_{z_m}k_z, k_w \rangle = 0$. □

**Lemma 2.4.** Let $\{z_m\}_{m=1}^\infty \subset \mathbb{B}_n$ so that $|z_m| \to 1$ as $m \to \infty$. Let $S$ be any non-zero positive operator on $L^2_\alpha$. Suppose $A = \sum_{m=1}^\infty U_{z_m}SU_{z_m}^*$ exists in the strong operator topology and is a bounded operator on $L^2_\alpha$. Then there is a constant $c > 0$ and an $f \in L^2_\alpha$ so that $\|AU_{z_m}f\| \geq c > 0$ for all $m$.

**Proof.** Since $S$ is non-zero and positive, there is an $f \in L^2_\alpha$ with $\|f\| = 1$ so that $\langle Sf, f \rangle > 0$. For each $m \geq 1$,
\[
\langle AU_{z_m}f, U_{z_m}f \rangle \geq \langle U_{z_m}SU_{z_m}^*U_{z_m}f, U_{z_m}f \rangle \geq \langle SU_{z_m}U_{z_m}f, U_{z_m}f \rangle \geq \langle Sf, f \rangle.
\]
Since $\|U_{z_m}f\| = 1$ it follows that $\|AU_{z_m}f\| \geq \langle Sf, f \rangle > 0$ for all $m$. □

3. Basic results

The first result in this section shows that for a certain class of subsets $G$ of $L^\infty$, $\mathfrak{I}(G)$ possesses a special property. This property will later help us distinguish $\mathfrak{I}(G_1)$ and $\mathfrak{I}(G_2)$ for $G_1 \neq G_2$.

**Proposition 3.1.** Let $W$ be a subset of $\mathbb{B}_n$ and let $F = \text{cl}(W) \cap S_n$. Let $f$ be in $L^\infty$ so that $f$ vanishes almost everywhere in $\mathbb{B}_n \setminus W$. Let $g_1, \ldots, g_l$ be any functions in $L^\infty$. Let $\{z_m\}_{m=1}^\infty$ be any sequence in $\mathbb{B}_n$ so that $|z_m| \to 1$ and $|z_m - w| \geq \epsilon > 0$ for all $w \in F$, all $m \geq 1$, where $\epsilon$ is a fixed constant. Then the sequence $\{T_fT_{g_1} \cdots T_{g_l}U_{z_m}\}_{m=1}^\infty$ converges to 0 in the strong operator topology of $\mathfrak{B}(L^2_\alpha)$. Consequently, if we set
\[
G = \{f \in L^\infty : f \text{ vanishes almost everywhere in } \mathbb{B}_n \setminus W\},
\]
then for any $T \in \mathcal{B}(G)$, $TU_{\zeta m} \to 0$ in the strong operator topology of $\mathcal{B}(L^2_a)$.

**Proof.** Let $V_1 = \{|z| \leq 1 : |z - w| < \epsilon/3 \text{ for some } w \in F\}$ and $V_2 = \{|z| \leq 1 : |z - w| < \epsilon/2 \text{ for some } w \in F\}$. Let $\eta$ be a continuous function on $\mathbb{B}_n$ so that $0 \leq \eta \leq 1$, $\eta(z) = 1$ if $z \in \text{cl}(V_1)$ and $\eta(z) = 0$ if $z \notin V_2$. Let $Z = \text{cl}(W) \cap (\mathbb{B}_n \setminus V_1)$. Then $Z \subset \mathbb{B}_n$ and $Z$ is compact with respect to the Euclidean metric. We have

$$Z \cap S_n = \text{cl}(W) \cap S_n \cap (\mathbb{B}_n \setminus V_1) = F \cap (\mathbb{B}_n \setminus V_1) = \emptyset.$$ 

Thus $Z$ is a compact subset of $\mathbb{B}_n$. Since the function $f(1 - \eta)$ vanishes almost everywhere in $(\mathbb{B}_n \setminus W) \cup \text{cl}(V_1)$ which contains $\mathbb{B}_n \setminus Z$, the operator $T_{f(1 - \eta)}$ is compact.

Since $\eta$ is continuous on $\mathbb{B}_n$, the operators $T_{fT_{g_1} \cdots T_{g_i} = T_{fT_{g_1}} \cdots T_{g_i}T_{\eta} + T_{fT_{g_1}} \cdots T_{g_i}T_{fT_{g_1}} \cdots T_{g_i} + K_1}$

(3.1)

where $K_1$ is a compact operator and $K = T_{f(1 - \eta)}T_{g_1} \cdots T_{g_i} + K_1$ is also a compact operator.

For any $h \in L^2_a \cap L^\infty$ and any $m \geq 1$ we have

$$\|T_\eta U_{\zeta m} h\|^2 \leq \|T_\eta U_{\zeta m} h\|^2$$

$$\leq \int_{V_2} |(U_{\zeta m} h)(z)|^2 \, d\nu(z)$$

$$= \int_{V_2} |h(\varphi_{\zeta m}(z))k_{\zeta m}(z)|^2 \, d\nu(z)$$

$$\leq \|h\|^2 \int_{V_2} |k_{\zeta m}(z)|^2 \, d\nu(z).$$

Let $V_3 = \{|z| \leq 1 : |z - w| < \epsilon \text{ for some } w \in F\}$. Since the map $(z, w) \mapsto |1 - \langle z, w \rangle|$ is continuous and does not vanish on the compact set $\text{cl}(V_2) \times (\mathbb{B}_n \setminus V_3)$, there is a $\delta > 0$ so that $|1 - \langle z, w \rangle| \geq \delta$ for all $z \in \text{cl}(V_2)$ and $w \in (\mathbb{B}_n \setminus V_3)$.

For each $m \geq 1$, $z_m \in (\mathbb{B}_n \setminus V_3)$, so for all $z \in V_2$,

$$|k_{\zeta m}(z)| \leq \frac{(1 - |z_m|^2)^{n+1/2}}{|1 - \langle z, z_m \rangle|^{n+1}} \leq \frac{(1 - |z_m|^2)^{n+1/2}}{\delta^{n+1}}.$$ 

Hence we have

$$\|T_\eta U_{\zeta m} h\| \leq \|h\| \sqrt{\nu(V_2)} \frac{(1 - |z_m|^2)^{n+1/2}}{\delta^{n+1}}.$$ 

This implies $\|T_\eta U_{\zeta m} h\| \to 0$ as $m \to \infty$. Since $L^2_a \cap L^\infty$ is dense in $L^2_a$ and $\|T_\eta U_{\zeta m}\| \leq \|T_\eta\| \leq 1$ for all $m$, we conclude that $T_\eta U_{\zeta m} \to 0$ in the strong operator topology of $\mathcal{B}(L^2_a)$. So $T_{fT_{g_1} \cdots T_{g_i} U_{\zeta m} \to 0}$ in the strong operator topology of $\mathcal{B}(L^2_a)$. Also by Lemma $\frac{2}{3}$, $U_{\zeta m} \to 0$ in the weak operator topology, so $KU_{\zeta m} \to 0$ in the strong operator topology for any compact operator $K$. Combining these facts with (3.1), we conclude that $T_{fT_{g_1} \cdots T_{g_i} U_{\zeta m} \to 0}$ in the strong operator topology of $\mathcal{B}(L^2_a)$. \qed
The following proposition was proved by Suárez for the case \( n = 1 \) (see [3 Proposition 2.9]). The case \( n \geq 2 \) is similar and can be proved with the same method. The point is that for all \( n \geq 2 \), the metric \( \rho \) and the reproducing kernel functions have all the properties needed for Suárez’s proof. See [2, Section 2] for more details.

**Proposition 3.2.** Let \( 0 < r < 1 \) and \( \{w_m\}_{m=1}^{\infty} \) be a sequence in \( \mathbb{B}_n \) so that \( E(w_k, r) \cap E(w_l, r) = \emptyset \) for all \( k \neq l \). For each \( m \in \mathbb{N} \), let \( c_m^1, \ldots, c_m^l, a_m, b_m, d_m^1, \ldots, d_m^k \in L^\infty \) be functions of norm \( 1 \) that vanish almost everywhere on \( \mathbb{B}_n \setminus E(w_m, r) \). Then

\[
\sum_{m \in \mathbb{N}} T_{c_m^1} \cdots T_{c_m^l} (T_{a_m} T_{b_m} - T_{b_m} T_{a_m}) T_{d_m^1} \cdots T_{d_m^k}
\]

belongs to \( \mathcal{J}(L^\infty) \).

**Remark 3.3.** In the proof of Proposition 3.2, we work only with Toeplitz operators with symbols in the set \( G \) which consists of functions of the form \( \sum_{m \in \mathbb{N}} f_m \), where \( F \) is a subset of \( \mathbb{N} \) and \( f \) is one of the symbols \( c^1, \ldots, c^l, a, b, d^1, \ldots, d^k \). So in the conclusion of the proposition, we may replace \( \mathcal{J}(L^\infty) \) by \( \mathcal{J}(G) \).

4. **Proof of the main theorem**

We are now ready for the proof of Theorem 1.1.

Fix \( 0 < r < 1 \). Let \( W_0 = E(0, r) \). For any closed non-empty subset \( F \) of \( \mathbb{S}_n \), let

\[
W_F = \bigcup_{0 < t < 1} \bigcup_{\zeta \in F} E(t\zeta, r).
\]

It is clear that \( W_{\mathbb{S}_n} = \mathbb{B}_n \). We always have \( F \subset \text{cl}(W_F) \cap \mathbb{S}_n \). We will show that in fact \( F = \text{cl}(W_F) \cap \mathbb{S}_n \). Suppose \( \zeta \in \text{cl}(W_F) \cap \mathbb{S}_n \). For any \( m \geq 1 \), applying Lemma 2.2 with \( \epsilon = m^{-1} \), we get a \( \delta_m > 0 \) so that if \( E(b, r) \cap V_m \neq \emptyset \), then \( |b - \zeta| < \delta_m \). Since \( W_F \cap V_m \neq \emptyset \), there is \( 0 < t_m < 1 \) and \( \zeta_m \in F \) so that \( E(t_m \zeta_m, r) \cap V_m \neq \emptyset \). Hence \( |t_m \zeta_m - \zeta| < \delta_m \).

So \( |t_m \zeta_m - \zeta| \to 0 \) as \( m \to \infty \). Since \( t_m = |t_m \zeta_m| \to |\zeta| = 1 \) as \( m \to \infty \), we get \( \zeta_m = t_m^{-1} (t_m \zeta_m) \to \zeta \) in the Euclidean metric as \( m \to \infty \). This implies that \( \zeta \in F \). Thus, \( \text{cl}(W_F) \cap \mathbb{S}_n \subset F \) and hence, \( \text{cl}(W_F) \cap \mathbb{S}_n = F \).

Now define

\[
G_F = \{ f \in L^\infty : f \text{ vanishes almost everywhere in } \mathbb{B}_n \setminus W_F \}.
\]

It is clear that if \( F_1 \subset F_2 \), then \( W_{F_1} \subset W_{F_2} \); hence \( G_{F_1} \subset G_{F_2} \).

Since \( T_f \) is compact for all \( f \in G_{\emptyset} \), \( \mathfrak{S}(G_{\emptyset}) = \mathfrak{K} \). Since \( G_{\mathbb{S}_n} = L^\infty \), we have \( \mathfrak{S}(G_{\mathbb{S}_n} \cap C(\mathbb{B}_n)) = \mathfrak{S}(L^\infty \cap C(\mathbb{B}_n)) = \mathfrak{K} \).

Now suppose \( F_1 \) and \( F_2 \) are two closed subsets of \( \mathbb{S}_n \) so that \( F_2 \setminus F_1 \neq \emptyset \). Let \( \zeta \in F_2 \setminus F_1 \). From Lemma 2.4 there is a sequence \( \{t_m\}_{m=1}^{\infty} \subset (0, 1) \) with \( t_m \uparrow 1 \) and \( E(t_k \zeta, r) \cap E(t_l \zeta, r) = \emptyset \) for all \( k \neq l \). Let \( z_m = t_m \zeta \) for all \( m \geq 1 \). Since \( |z_m - \zeta| \to 0 \) and \( \zeta \notin F_1 \), which is a closed subset of \( \mathbb{S}_n \), there is an \( \epsilon > 0 \) so that \( |z_m - w| \geq \epsilon \) for all \( w \in F_1 \), all \( m \geq 1 \). Since \( \text{cl}(W_{F_1}) \cap \mathbb{S}_n = F_1 \), Proposition 3.3 shows that \( TU_{z_m} \to 0 \) in the strong operator topology for all \( T \in \mathfrak{K}(G_{F_1}) \).

Take \( f \) to be any continuous function supported in \( E(0, r) \) such that \( [T_f, T_f] \neq 0 \). Any function of the form \( f(z) = z_1 \eta(|z|/r) \) where \( \eta \) is non-negative, continuous and
supported in \([0, 1]\) with \(\|\eta\|_\infty > 0\) will work. Let \(S = [T_f, T_f^2]\), then \(S\) is a non-zero, positive operator on \(L^2_a\). Define
\[
T = \sum_{m=1}^{\infty} U_{zm} S U_{zm}^*.
\]

By Lemma 2.4, there is a constant \(c > 0\) and an \(h \in L^2_a\) so that \(\|TU_{zm} h\| \geq c\) for all \(m\). This implies that \(T\) is not in \(\mathcal{J}(G_{F_1})\).

For each \(m\),
\[
U_{zm} [T_f, T_f^* U_{zm}^*] = U_{zm} T_f T_f^* U_{zm} - U_{zm} T_f T_f^* U_{zm} = T_f \varphi_{zm} T_f \varphi_{zm} - T_f \varphi_{zm} T_f \varphi_{zm} = [T_f \varphi_{zm}, T_f \varphi_{zm}].
\]

So \(U_{zm} S U_{zm}^* = [T_f \varphi_{zm}, T_f \varphi_{zm}]^2\). Hence \(T = \sum_{m=1}^{\infty} [T_f \varphi_{zm}, T_f \varphi_{zm}]^2\).

Since each \(f \circ \varphi_{zm}\) is continuous and supported in \(\{ w \in \mathbb{B}_n : |\varphi_{zm}(w)| < r \}\) = \(E(z_m, r)\) and \(E(z_k, r) \cap E(z_l, r) = \emptyset\) for all \(k \neq l\), Proposition 3.2 and Remark 3.3 show that \(T \in \mathcal{C}(G_{F_2} \cap C(\mathbb{B}_n))\). So \(T \in \mathcal{C}(G_{F_2} \cap C(\mathbb{B}_n)) \setminus \mathcal{J}(G_{F_1})\).

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References


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