

A HODGE DECOMPOSITION INTERPRETATION FOR THE COEFFICIENTS OF THE CHROMATIC POLYNOMIAL

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ABSTRACT. Let G be a simple graph with n nodes. The coloring complex of G , as defined by Steingrímsson, has r -faces consisting of all ordered set partitions, (B_1, \dots, B_{r+2}) in which at least one B_i contains an edge of G . Jonsson proved that the homology $H_*(G)$ of the coloring complex is concentrated in the top degree. In addition, Jonsson showed that the dimension of the top homology is one less than the number of acyclic orientations of G .

In this paper, we show that the Eulerian idempotents give a decomposition of the top homology of G into $n - 1$ components $H_{n-3}^{(j)}(G)$. We go on to prove that the dimensions of the Hodge pieces of the homology are equal to the absolute values of the coefficients of the chromatic polynomial of G . Specifically, if we write $\chi_G(\lambda) = (\sum_{j=1}^{n-1} c_j (-1)^{n-j} \lambda^j) + \lambda^n$, then $\dim(H_{n-3}^{(j)}(G)) = c_j$.

1. THE COLORING COMPLEX

Let G be a simple graph with vertex set V . For each r , let $\Delta_r(G)$ be the collection of all ordered set partitions of V , $(B_1, B_2, \dots, B_{r+2})$, such that at least one of the B_i contains an edge of G . Let $C_r(G)$ be the vector space, over a field \mathbf{F} of characteristic 0, having $\Delta_r(G)$ as a basis.

Example 1. Let G be the tree on three vertices which has edges between 1 and 2 and between 2 and 3. Then $\Delta_0(G)$ contains the four ordered partitions

$$\begin{aligned} &(12, 3), \\ &(3, 12), \\ &(23, 1), \\ &(1, 23) \end{aligned}$$

whereas $\Delta_{-1}(G)$ is one dimensional with basis (123).

The *coloring complex* of G has $\Delta_r(G)$ as its r -faces with boundary map

$$\delta_r : C_r(G) \rightarrow C_{r-1}(G),$$

defined by

$$\delta_r(B_1, \dots, B_{r+2}) = \sum_{i=1}^{r+1} (-1)^i (B_1, \dots, B_i \cup B_{i+1}, \dots, B_{r+2}).$$

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It is straightforward to check that $\delta_r \circ \delta_{r+1} = 0$. So we can define the homology of the coloring complex by

$$H_r(G) = \frac{\ker(\delta_r)}{\operatorname{im}(\delta_{r+1})}.$$

Returning to Example 1, it is easy to see that the boundary map δ_0 has rank 1 and so we have $\dim(H_0(G)) = 3$ and $\dim(H_{-1}(G)) = 0$.

The coloring complex was introduced by Steingrímsson ([St]). In subsequent work, Jonsson proved the following interesting result.

Theorem 1.1 (Jonsson [J]). *Let G be a simple graph with n nodes. Then the homology vanishes except at the top dimension. Moreover, assuming G has at least one edge, so that the top dimension is $n - 3$, then*

$$\dim(H_{n-3}(G)) = a_G - 1$$

where a_G is the number of acyclic orientations of G . (Recall that an acyclic orientation of G is an assignment of direction to each of the edges of G so that the resulting directed graph has no directed cycles.)

Returning to the graph G in Example 1, note that any orientation of the two edges leads to an acyclic digraph so $a_G = 4$. Thus, $a_G - 1 = 4 - 1 = 3$ which is equal to the dimension of $H_0(\Delta_G)$ as Jonsson’s theorem predicts.

2. THE EULERIAN IDEMPOTENTS

The Eulerian idempotents, denoted $e_r^{(1)}, \dots, e_r^{(r)} \in \mathbf{F}S_r$, are remarkable elements within the group algebra of the symmetric groups. There is a significant literature on the Eulerian idempotents (see [H2] and [L] for references). We will use just a few of the properties of the $e_r^{(j)}$. It is known that $e_r^{(1)}, e_r^{(2)}, \dots, e_r^{(r)}$ form a set of pairwise orthogonal idempotents that decompose the identity in $\mathbf{F}S_r$. Therefore, if M is any S_r -module, then

$$(2.4) \quad M = \bigoplus_j M^{(j)}$$

where $M^{(j)}$ denotes $e_r^{(j)} \cdot M$.

There is an action of S_{r+2} on $\Delta_r(G)$ given by letting a permutation σ act on a basis element B_1, \dots, B_{r+2} by permutation of blocks. We will need an important intertwining property of the action of the Eulerian idempotents with the boundary map. This lemma follows immediately from Theorem 1.7 in [H1] or can be derived from the earlier work of Gerstenhaber and Schack [GS].

Lemma 2.1. *Let G be any graph and let δ_r be the boundary map in the complex Δ_G as defined above. Then for each r, j*

$$\delta_r \circ e_{r+2}^{(j)} = e_{r+1}^{(j)} \circ \delta_r.$$

By Lemma 2.1, for fixed j , the set of $C_r^{(j)}(G) = e_r^{(j)} \cdot C_r(G)$ form a subcomplex of $(C_*(G), \delta_*)$. Let $H_*^{(j)}(G)$ be the homology of this subcomplex. This gives a direct sum decomposition of $H_*(G)$ into summands indexed by j ,

$$(2.5) \quad H_{n-3}(G) = \bigoplus_{j=1}^{n-1} H_{n-3}^{(j)}(G).$$

Example 2. To give an example of this decomposition, let G be the 4-cycle with vertex set $V = \{1, 2, 3, 4\}$ and edges joining 1 to 2, 2 to 3, 3 to 4, and 4 to 1. The maximal chains have one block of size two containing an edge of G and two blocks of size one. There are 24 such chains giving a basis for $C_1(G)$. The chains spanning $C_0(G)$ have either one block of size three and one block of size one (there are 8 of these) or else two blocks of size two (there are 4 of these). So $\dim(C_0(G)) = 12$. Finally, $\dim(C_{-1}(G)) = 1$ with a basis element consisting of the chain with a single block of size 4.

It is straightforward to compute the dimensions of the images of the Eulerian idempotents:

$$\begin{aligned} \dim(C_1^{(3)}(G)) &= 4, \\ \dim(C_1^{(2)}(G)) &= 12, & \dim(C_0^{(2)}(G)) &= 6, \\ \dim(C_1^{(1)}(G)) &= 8, & \dim(C_0^{(1)}(G)) &= 6, & \dim(C_{-1}^{(1)}(G)) &= 1. \end{aligned}$$

The map δ_1 has rank 6 when restricted to $C_1^{(2)}$ and rank 5 when restricted to $C_1^{(1)}$. The map δ_0 has rank 1 when restricted to $C_0^{(1)}$. It follows that the dimensions of the Hodge pieces of the homology of the coloring complex for this graph G are given by:

$$\begin{aligned} \dim(H_1^{(3)}(G)) &= 4, \\ \dim(H_1^{(2)}(G)) &= 6, \\ \dim(H_1^{(1)}(G)) &= 3. \end{aligned}$$

Following Gerstenhaber and Schack ([GS]), we call the decomposition in (2.5) the *Hodge decomposition* of $H(G)$. The main result in this paper will characterize the dimensions of the Hodge components $H_{n-3}^{(j)}(G)$.

3. CHROMATIC POLYNOMIALS

Let $\chi_G(\lambda)$ denote the chromatic polynomial of G , i.e., the polynomial whose value at k is the number of k -colorings of G . There has been much work done on chromatic polynomials including some work on interpretations of the coefficients of the chromatic polynomial. We record a few well-known results in what follows.

Theorem 3.1 (Birkhoff [B]). *Let G be a simple graph with n vertices. Then $\chi_G(\lambda)$ is a monic polynomial of degree n with no constant term. If we write*

$$(3.1) \quad \chi_G(\lambda) = \sum_{j=1}^{n-1} (-1)^{n-j} c_j(G) \lambda^j + \lambda^n,$$

then the c_j are non-negative integers.

As an example of Theorem 3.1, let G be a tree. It is well-known that $\chi_G(\lambda) = \lambda \cdot (\lambda - 1)^{n-1}$. So in the case that G is a tree with n vertices, $c_j(G) = \binom{n-1}{j}$.

The next result is also considered a basic result about chromatic polynomials.

Theorem 3.2. *Let G be a graph and let e be an edge of G . Let $G - e$ denote the graph obtained from G by deleting e and let G/e denote the graph obtained from G by contracting e . Then,*

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda).$$

Example 3. Let G be the 4-cycle. By applying Theorem 3.2, with e being any one of the edges of G , we obtain

$$\chi_G(\lambda) = (\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda) - (\lambda^3 - 3\lambda^2 + 2\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

For this graph G , the $c_j(G)$ defined above are given by $c_1(G) = 3, c_2(G) = 6, c_3(G) = 4$. Comparing this to Example 2 from the previous section, we see that, for every j , $c_j(G)$ is equal to the dimension of the j^{th} component of the Hodge decomposition of the top homology of the coloring complex of G . This example anticipates the main theorem of this paper (Theorem 4.1).

The next result is proven in a classic paper by Richard Stanley.

Theorem 3.3 (Stanley [S1]). *Let G be a simple graph with n vertices. Then the number of acyclic orientations of G is given by $(-1)^n \chi_G(-1)$. In other words,*

$$a_G = 1 + \sum_{j=1}^{n-1} c_j(G).$$

4. THE MAIN RESULT

Note that by combining the results by Jonsson and Stanley (Theorems 1.1 and 3.3) we obtain

$$(4.1) \quad \sum_{j=1}^{n-1} \dim(H_{n-3}^{(j)}(G)) = \sum_{j=1}^{n-1} c_j(G).$$

The next result, which is the main theorem of this paper, provides a natural refinement of (4.1).

Theorem 4.1. *Let G be a simple graph with n vertices and one or more edges. For each j , the dimension of the j^{th} Hodge piece in the homology of Δ_G is equal to the j^{th} coefficient of the chromatic polynomial of G . Put in terms of our earlier notation,*

$$\dim(H_{n-3}^{(j)}(G)) = c_j(G).$$

Equivalently,

$$\sum_j \dim(H_{n-3}^{(j)}(G)) \lambda^j = (-1)^n (\chi_G(-\lambda) - (-\lambda)^n).$$

Proof. The proof will be by induction on the number of edges. We begin with the base case in which G has just one edge e , joining the points u and v . Note that $a_G = 2$ and so $\dim(H_{n-3}(\Delta_G)) = 1$.

Let $\gamma = (B_1, \dots, B_{n-1})$ be the chain where $B_1 = \{u, v\}$ and the other B_m are singletons containing the remaining vertices (the order is unimportant). It is easy to check that

$$\Gamma = \sum_{\sigma} \text{sgn}(\sigma) \sigma \cdot \gamma$$

is non-zero and $\delta \cdot \Gamma = 0$. Thus, Γ is a representative for $H_{n-3}(\Delta_G)$. Since

$$\frac{1}{(n-1)!} \sum_{\sigma} \text{sgn}(\sigma) \sigma = e_{n-1}^{(n-1)},$$

we have that $\dim(H_{n-3}^{(n-j)}(G))$ is equal to 1 for $j = 1$ and 0 for $j \geq 2$.

Note that all assignments of colors to the vertices of G lead to legal colorings, except those in which u and v are assigned the same color. So, $\chi_G(\lambda) = \lambda^n - \lambda^{n-1}$, i.e., $c_1(G) = 1$ whereas $c_j(G) = 0$ for $j \geq 2$. This completes the proof for G with exactly one edge.

We now proceed with the proof of Theorem 4.1 by induction on the number of edges. We have just completed the base case and move on to the induction step. Our proof of the induction step will mimic the proof of the main result from Jonsson’s paper.

For each d , let $X^{(d)}(G)$ denote the Euler characteristic of the d^{th} Hodge piece of Δ_G . In other words,

$$X^{(d)}(G) = \sum_{i=-1}^{n-3} (-1)^{n-3-i} \dim(C_i^{(d)}(G)).$$

We assume that G has $m \geq 2$ edges and that Theorem 4.1 has been proved for all graphs with fewer than m edges. Fix an edge e of G . Let E denote the graph with the same vertex set as G but with e being the only edge. For each r , let D_r denote the space spanned by chains in $C_r(G)$ that have no step containing the edge e .

Note that

$$(4.1) \quad C_r(G) = D_r \oplus C_r(E)$$

and that the isomorphism between the left and right hand sides of (4.1) commutes with the action of S_{r+2} . Thus, for each j and r we have

$$(4.2) \quad C_r^{(j)}(G) = D_r^{(j)} \oplus C_r^{(j)}(E).$$

We now turn our attention to D_r . Note that

$$(4.3) \quad D_r \simeq C_r(G - e) / (C_r(G - e) \cap C_r(E)).$$

The isomorphism (4.3) follows because the chains that span D_r are chains in $C_r(G - e)$, and conversely, a chain in $C_r(G - e)$ is in the spanning set of D_r unless it is also in $C_r(E)$.

Next, we observe that there is also an isomorphism

$$(4.4) \quad C_r(G - e) \cap C_r(E) \simeq C_r(G/e).$$

The isomorphism (4.4) follows because any chain $\mathbf{B} = (B_1, B_2, \dots, B_{r+2})$ that is in $\Delta_r(E)$ will include i and j together in the same block B_m . Such a chain can be identified with a chain $\hat{\mathbf{B}} = (\hat{B}_1, \hat{B}_2, \dots, \hat{B}_{r+2})$ where i and j are identified in \hat{B}_m . If in addition, \mathbf{B} is a chain in $\Delta(G - e)$, then some block of $\hat{\mathbf{B}}$ contains an edge of $G \setminus e$. Equation (4.4) follows.

The isomorphisms in (4.3) and (4.4) both commute with the actions of S_{r+2} . So, for each r and j we have refinements of the isomorphisms in (4.3) and (4.4), namely refining those isomorphisms to the j^{th} Hodge components of each space. Combining this fact with the equations above yields the following identity that holds for every r and j :

$$(4.5) \quad \dim(C_r^{(j)}(G)) = \dim(C_r^{(j)}(G - e)) - \dim(C_r^{(j)}(G/e)) + \dim(C_r^{(j)}(E)).$$

Multiplying this by $(-1)^{n-3-r}\lambda^j$ and summing over r and j yields:

$$\begin{aligned} \sum_j \dim(H_{n-3}^{(j)}(G))\lambda^j &= \sum_{j,r} (-1)^{(n-3)-r} \dim(H_r^{(j)}(G))\lambda^j \\ &= \sum_{r,j} (-1)^{n-3-r} \lambda^j \dim(C_r^{(j)}(G)) \\ &= \sum_{r,j} (-1)^{n-3-r} \lambda^j (\dim(C_r^{(j)}(G - e)) - \dim(C_r^{(j)}(G/e)) + \dim(C_r^{(j)}(E))) \\ &= (-1)^n (\chi_{G-e}(-\lambda) - (-\lambda)^n) - (\chi_{G/e}(-\lambda) - (-\lambda)^{n-1}) + (\chi_E(-\lambda) - (-\lambda)^n) \\ &= (-1)^n (\chi_{G-e}(-\lambda) - \chi_{G/e}(-\lambda)) - (-\lambda)^n \\ &= (-1)^n (\chi_G(-\lambda) - (-\lambda)^n). \end{aligned}$$

In the string of equalities above, the third to last follows from our induction hypothesis, whereas the second to last follows from the well-known result about chromatic polynomials. The last step follows from Theorem 3.2. This completes the induction step and the proof of Theorem 4.1.

5. APPLICATIONS

Theorem 4.1 gives an algebraic interpretation for the coefficients of the chromatic polynomial of a graph. In what follows, we give two applications that make use of this interpretation.

Application 1. Connectivity of G and the Minimal Coefficient of $\chi_G(\lambda)$.

Recall the following well-known result about chromatic polynomials of graphs.

Theorem 5.1. *Let G be a simple graph with m connected components. Then the coefficient of λ^j is 0 for $j < m$.*

Theorem 4.1 allows us to deduce the following corollary from Theorem 5.1.

Corollary. *Let G be a simple graph with m connected components. Then $H_*^{(j)}(G) = 0$ for $j < m$.*

From a combinatorial perspective, Theorem 5.1 follows immediately from the observation that G is a disjoint union of graphs H and K ; then $\chi_G(\lambda) = \chi_H(\lambda)\chi_K(\lambda)$. It would be interesting to give an algebraic derivation, using properties of Eulerian idempotents, of the Corollary as stated or even of the observation that the chromatic polynomial of a disjoint union factors.

Application 2. A generating function for the Hodge decomposition of all labelled graphs.

For each n , let L_n denote the set of labelled graphs with n nodes. Let $A(x, \lambda)$ denote the power series:

$$(6.1) \quad A(x, \lambda) = \sum_n \sum_{G \in L_n} (\lambda^n + \sum_j c_j(G)\lambda^j) \frac{x^n}{n!2^{\binom{n}{2}}}.$$

Note that

$$(6.2) \quad A(-x, -\lambda) = \sum_n \sum_{G \in L_n} \chi_G(\lambda) \frac{x^n}{n!2^{\binom{n}{2}}}.$$

Let $E(x)$ be the power series where

$$(6.3) \quad E(x) = \sum_n \frac{x^n}{n! \cdot 2^{\binom{n}{2}}}.$$

We will derive a simple expression for $A(x, \lambda)$ in terms of $E(x)$.

Theorem 5.2. *Let the notation be as above. Then*

$$(6.4) \quad A(x, \lambda) = E(-x)^{-\lambda}.$$

Proof. We will derive equation (6.4) using the definition of $A(x, \lambda)$ in (6.1) and the interpretation of c_j given in Theorem 4.1.

Using the fact that $e_r^{(j)}$ are idempotents, we have that

$$(6.5) \quad c_j(G) = \dim(H_{n-3}^{(j)}(G)) = \text{tr}(e_{n-1}^{(j)}|_{H_{n-3}(G)}).$$

Since the homology vanishes except at top degree, we have

$$(6.6) \quad c_j(G) = \sum_r (-1)^{n-3-r} \text{tr}(e_{r+2}^{(j)}|_{C_r(G)}).$$

Since S_{r+2} acts on $C_r(G)$ by permuting the blocks in each chain,

$$\text{tr}(\sigma|_{C_r(G)}) = \begin{cases} \dim(C_r(G)), & \text{if } \sigma = id; \\ 0, & \text{otherwise.} \end{cases}$$

Applying that observation to (6.6) yields:

$$(6.7) \quad c_j(G) = \sum_r (-1)^{n-3-r} \dim(C_r(G)) [e_{r+2}^{(j)}]_{id}$$

where $[e_{r+2}^{(j)}]_{id}$ denotes the coefficient of the identity in $e_{r+2}^{(j)}$.

Combining formula (6.1) on page 115 with Lemma 5.2 on page 113 in Hanlon [H2], we obtain

$$(6.8) \quad \sum_{j \geq 1} \sum_{p \geq 0} \sum_{\tau \in S_p} [e_p^{(j)}]_{\tau} Z(\tau) \lambda^j = \prod_{\ell} (1 + (-1)^{\ell} a_{\ell})^{-\frac{1}{\ell} \sum d/\ell \mu(d) \lambda^{\ell/d}}$$

where $Z(\tau)$ is the cycle indicator of τ in the variables a_1, a_2, \dots . Replacing a_i by 0 for $i \geq 2$ in (6.8) yields

$$\sum_{j \geq 1} \sum_{p \geq 0} [e_p^{(j)}]_{id} \lambda^j a_1^p = (1 - a_1)^{-\lambda},$$

which implies for fixed p that

$$(6.9) \quad \sum_{j=1}^p [e_p^{(j)}]_{id} \lambda^j = (-1)^p \cdot \binom{-\lambda}{p}.$$

Putting together equations (6.1), (6.7) and (6.9) we obtain

$$(6.10) \quad A(x, \lambda) = \sum_n \sum_{G \in L_n} \frac{(\lambda x)^n}{n! 2^{\binom{n}{2}}} + \sum_n \sum_{G \in L_n} \sum_r (-1)^{n-3} \dim(C_r(G)) \binom{-\lambda}{r+2} \frac{x^n}{n! 2^{\binom{n}{2}}}.$$

Observe that $|L_n| = 2^{\binom{n}{2}}$. So,

$$(6.11) \quad A(x, \lambda) = e^{\lambda x} + \sum_n \sum_{\underline{B} = B_1, \dots, B_{r+2}} \binom{-\lambda}{r+2} \frac{(-1)^{n-1} G(\underline{B}) x^n}{n! 2^{\binom{n}{2}}}$$

where the sum over \underline{B} is over ordered set partitions of n and where $G(\underline{B})$ denotes the number of labeled graphs in L_n that have at least one edge contained in a block of \underline{B} . Let $|B_i| = b_i$. Note that

$$(6.12) \quad G(\underline{B}) = \left(\prod_i 2^{\binom{b_i}{2}} - 1 \right) \cdot \left(\prod_{i < j} 2^{b_i b_j} \right)$$

as the factor $2^{\binom{b_i}{2}}$ counts the number of ways to insert edges within the block B_i , the factor $2^{b_i b_j}$ counts the number of ways to insert edges between points in B_i and B_j , and -1 eliminates the case where no edges were inserted within the blocks B_i . It is straightforward to see that $(\prod_i 2^{\binom{b_i}{2}}) \cdot (\prod_{i < j} 2^{b_i b_j}) = 2^{\binom{n}{2}}$, so

$$(6.13) \quad G(\underline{B}) = 2^{\binom{n}{2}} \left(1 - \frac{1}{\prod_i 2^{\binom{b_i}{2}}} \right).$$

Inserting in (6.11) the formula for $G(\underline{B})$ given in (6.13) we obtain

$$(6.14) \quad A(x, \lambda) = e^{\lambda x} - \sum_n \sum_{\underline{B}=B_1, \dots, B_{r+2}} \binom{-\lambda}{r+2} \frac{(-x)^n}{n!} \left(1 - \prod_i 2^{-\binom{b_i}{2}} \right).$$

Given an ordered integer partition (b_1, \dots, b_{r+2}) of n , the number of ordered set partitions (B_1, \dots, B_{r+2}) where each B_i has cardinality b_i is given by $\frac{n!}{\prod_i b_i!}$. So we can rewrite (6.14) as:

$$(6.15) \quad A(x, \lambda) = e^{\lambda x} - \left(\sum_{t \geq 0} \left(\sum_{b=1}^{\infty} \frac{(-x)^b}{b!} \binom{-\lambda}{t} \right) \right) + \left(\sum_{t \geq 0} \left(\sum_{b=1}^{\infty} \frac{(-x)^b}{b! 2^{\binom{b}{2}}} \right)^t \binom{-\lambda}{t} \right).$$

It follows immediately that

$$(6.16) \quad A(x, \lambda) = e^{\lambda x} - (e^{-x})^{-\lambda} + E(-x)^{-\lambda} = E(-x)^{-\lambda},$$

which completes the derivation of (6.4).

Note that if we set x equal to $-x$ and λ equal to -1 in Theorem 5.2 and also apply Theorem 3.3, we obtain the following result that was proved independently by Stanley [S1] and Robinson [R].

Corollary (R. P. Stanley and R. W. Robinson). *For each n , let α_n be the number of labeled acyclic digraphs with n vertices. Then,*

$$\sum_n \alpha_n \frac{x^n}{2^{\binom{n}{2}} \cdot n!} = E(x)^{-1}.$$

The applications above give just two examples of a rich problem area that is suggested by Theorem 4.1. Namely, it would be interesting to use the interpretation of the chromatic polynomial given by Theorem 4.1 to obtain new algebraic results about the Hodge structure of the homology of the coloring complex from the many combinatorial results known about chromatic polynomials, or conversely to derive new combinatorial results about chromatic polynomials from properties of the Eulerian idempotents.

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