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ON FOLIATIONS WITH MORSE SINGULARITIES

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ABSTRACT. We study codimension one smooth foliations with Morse type singularities on closed manifolds. We obtain a description of the manifold if there are more centers than saddles. This result relies on and extends previous results of Reeb for foliations having only centers, results of Wagneur for foliations with Morse singularities and results of Eells and Kuiper for manifolds admitting Morse functions with three singularities.

1. INTRODUCTION AND MAIN RESULTS

The interplay between the topology of a closed manifold and the combinatorics of the critical points of a real valued function of class C^2 defined on the manifold is a well known fact of Morse Theory ([6]). It is natural to expect a similar relationship for foliated manifolds. This became evident for the first time with the following result of G. Reeb ([8]), a consequence of his Stability Theorem ([1], [5], [7]):

Theorem 1.1. *Let M be a closed oriented and connected manifold of dimension $m \geq 2$. Assume that M admits a C^1 transversely oriented codimension one foliation $\mathcal{F}$ with a non-empty set of singularities, all of them centers. Then the singular set of $\mathcal{F}$ consists of two points and M is homeomorphic to the m-sphere.*

Later on Eells and Kuiper classified the closed manifolds admitting a C^3 function with exactly three non-degenerated singular points ([3], [4]):

Theorem 1.2. *Let M be a connected closed manifold (not necessarily orientable) of dimension m. Suppose M admits a Morse function $f\colon M \to \mathbb{R}$ of class C^3 with exactly three singular points. Then:*

(i) *$m \in \{2, 4, 8, 16\}$.*
(ii) *M is topologically a compactification of $\mathbb{R}^m$ by an $\frac{m}{2}$-sphere.*
(iii) *If $m = 2$, then M is diffeomorphic to $\mathbb{R}P(2)$. For $m \geq 4$, M is simply-connected and has the integral cohomology structure of the complex projective plane ($m = 4$) of the quaternionic projective plane ($m = 8$) on of the Cayley projective plane ($m = 16$).*

We will call these manifolds *Eells-Kuiper* manifolds. In both situations we have a closed manifold endowed with a foliation with Morse singularities where the number

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of centers is greater than the number of saddles. In [2] we proved that in the case that the manifold is orientable of dimension three, this implies it is homeomorphic to the 3-sphere. The aim of this paper is to consider the n-dimensional case. We proceed to define the main notions we use. A codimension one foliation with isolated singularities on a compact manifold M is a pair $\mathcal{F} = (\mathcal{F}_0, \operatorname{sing}\mathcal{F})$, where $\operatorname{sing}\mathcal{F} \subset M$ is a discrete subset and $\mathcal{F}_0$ is a regular foliation of codimension one on the open manifold $M\setminus \operatorname{sing}\mathcal{F}$. We say that $\mathcal{F}$ is *of class* C^k if $\mathcal{F}_0$ is of class C^k, $\operatorname{sing}\mathcal{F}$ is called the singular set of $\mathcal{F}$ and the *leaves* of $\mathcal{F}$ are the leaves of $\mathcal{F}_0$ on $M\setminus \operatorname{sing}\mathcal{F}$. A point $p \in \operatorname{sing}\mathcal{F}$ is a *Morse type* singularity if there is a function $f_p \colon U_p \subset M \to \mathbb{R}$ of class C^2 in a neighborhood of p such that $\operatorname{sing}\mathcal{F} \cap U_p = \{p\}$, f_p has a non-degenerate critical point at p and the levels of f_p are contained in leaves of $\mathcal{F}$. By the classical Morse Lemma ([6]) there are local coordinates $(y_1, \dots, y_m)$ in a neighborhood U_p of p such that $y_j(p) = 0$, $\forall\, j \in \{1, \dots, m\}$ and $f(y_1, \dots, y_m) = f(p) - (y_1^2 + \cdots + y_{r(p)}^2) + y_{r(p)+1}^2 + \cdots + y_m^2$. The number $r(p)$ is called the Morse *index* of p. The singularity p is a *center* if $r(p) \in \{0, m\}$ and it is a *saddle* otherwise. The leaves of $\mathcal{F}$ in a neighborhood of a center are diffeomorphic to the $(m-1)$-sphere. Given a saddle singular point $p \in \operatorname{sing}\mathcal{F}$ we have leaves of $\mathcal{F}\big|_{U_p}$ that accumulate on p. They are contained in the cone τ_p: $y_1^2 + \cdots + y_{r(p)}^2 = y_{r(p)+1}^2 + \cdots + y_m^2 \neq 0$, and there are two possibilities: either $r(p) = 1$ or $m-1$, and then τ_p is the union of *two* leaves of $\mathcal{F}\big|_{U_p}$, or $r(p) \neq 1$ and $m-1$ and τ_p is a leaf of $\mathcal{F}\big|_{U_p}$. Any leaf of $\mathcal{F}\big|_{U_p}$ contained in τ_p is called a *local separatrix of* $\mathcal{F}$ at p, or a *cone leaf* at p. Any leaf of $\mathcal{F}$ such that its restriction to U_p contains a local separatrix of $\mathcal{F}$ at p is called a *separatrix of* $\mathcal{F}$ *at* p. A *saddle connection* for $\mathcal{F}$ is a leaf which contains local separatrices of two *different* saddle points. A *saddle self-connection* for $\mathcal{F}$ at p is a leaf which contains two different local separatrices of $\mathcal{F}$ at p. A foliation $\mathcal{F}$ with Morse singularities is *transversely orientable* if there exists a vector field X on M, possibly with singularities at $\operatorname{sing}\mathcal{F}$, such that X is transverse to $\mathcal{F}$ outside $\operatorname{sing}\mathcal{F}$.

Definition 1.3. A *Morse foliation* $\mathcal{F}$ on a manifold M is a transversely oriented codimension one foliation of class C^2 with singularities such that: (i) each singularity of $\mathcal{F}$ is of Morse type and (ii) there are no saddle connections.

Basic examples of Morse foliations are given by the levels of Morse functions $f \colon M \to \mathbb{R}$ of class C^2. Therefore any manifold of class C^2 supports a Morse foliation; i.e., the existence of a Morse foliation imposes no restriction on the topology of the manifold. Nevertheless, there are restrictions which come from the nature of the singularities of a Morse foliation $\mathcal{F}$ on M. Indeed, our purpose in this paper is to show the following theorem:

Theorem 1.4. *Let M be a compact connected manifold and $\mathcal{F}$ a Morse foliation on M such that the number k of centers and the number ℓ of saddles in* $\operatorname{sing}\mathcal{F}$ *satisfy $k \geq \ell + 1$. Then we have two possibilities:*

(i) *$k = \ell + 2$, and M is homeomorphic to the m-sphere.*

(ii) *$k = \ell + 1$, and M is an Eells-Kuiper manifold.*

Our results above extend original results of E. Wagneur in [9].

2. Preliminaries

Let us first fix the notation we use. Let $\mathcal{F}$ be a Morse foliation on a manifold M of dimension $m \geq 3$. Given a center singularity $p \in \operatorname{sing} \mathcal{F}$ the nearby leaves of $\mathcal{F}$ are compact diffeomorphic to S^{m-1}. Since $m-1 \geq 2$ any such leaf L has trivial holonomy group, and therefore by the Local Stability theorem of Reeb ([1], [5]) there is a fundamental system of open neighborhoods V of L such that the restriction $\mathcal{F}|_V$ is equivalent to a product foliation $\mathcal{G}$ on $L \times (-1,1)$ whose leaves are of the form $L \times \{t\}$, $t \in (-1,1)$. We therefore introduce the *open* subset $\mathcal{C}(\mathcal{F})$ as the union of centers in $\operatorname{sing} \mathcal{F}$ and leaves of $\mathcal{F}$ diffeomorphic to S^{m-1}. The *basin* of a center $p \in \operatorname{sing} \mathcal{F}$ is the connected component $\mathcal{C}_p(\mathcal{F})$ of $\mathcal{C}(\mathcal{F})$ that contains p. We have the following basic lemma:

Lemma 2.1. *Given centers $p, q \in \operatorname{sing} \mathcal{F}$ the sets $\mathcal{C}_p(\mathcal{F})$ and $\mathcal{C}_q(\mathcal{F})$ are open in M and $\mathcal{C}_p(\mathcal{F}) \cap \mathcal{C}_q(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{C}_p(\mathcal{F}) = \mathcal{C}_q(\mathcal{F})$. Moreover we have $\mathcal{C}_p(\mathcal{F}) = M$ if and only if $\partial \overline{\mathcal{C}_p(\mathcal{F})} = \emptyset$, and in this case M is homeomorphic to S^m provided that M is orientable.*

In particular M is homeomorphic to S^m or $\partial \overline{\mathcal{C}_p(\mathcal{F})}$ contains some saddle singularity.

In order to fix notation we shall now introduce the notion of a holonomy group of an invariant subset of codimension one. We will consider two notions of holonomy. When we refer to the *holonomy of a leaf* L of $\mathcal{F}$ we mean the holonomy group of L as a leaf of $\mathcal{F}_0$ on $M \setminus \operatorname{sing} \mathcal{F}$. On the other hand, the notion of holonomy can be extended to invariant subsets of the form $S = \tau \cup \{p\}$, $p \in \operatorname{sing} \mathcal{F}$, and τ is either a cone leaf or a union of two cone leaves. Notice that in a small neighborhood of p, τ can consist of two components τ_1 and τ_2 and that this can only happen if $r(p) = 1$ or $m-1$. In this case S locally divides the manifold into three connected components. One of them, say R_3, is the union of (regular) leaves which are hyperboloids of one sheet, and the others, say R_1 and R_2, are the union of one of the connected components of hyperboloids of two sheets (we can think of R_1 as the region surrounded by τ_1 and R_2 as the region surrounded by τ_2). Let $\gamma \colon [0,1] \to S$ be a path on S which passes through the singularity p (from τ_1 to τ_2). In this case the holonomy along γ can be defined in the usual manner (lifting paths to leaves) on R_3; however, there is no canonical extension of this holonomy to the other side in general. Thus we adopt the following notion of holonomy. Fix a neighborhood U of $p \in \operatorname{sing} \mathcal{F}$ where $\mathcal{F}$ is given by a Morse function f with a single singularity at p. Let $\gamma \colon [0,1] \to S$ be a piecewise smooth path (as a map $\gamma \colon [0,1] \to M$). Let T_0 and T_1 be local transversals to $\mathcal{F}$ at $\gamma(0)$ and $\gamma(1)$, respectively. The *holonomy* along γ will be the mapping which assigns $t \in T_0$ to $f^{-1}(f(t)) \cap T_1 \in T_1$. This holonomy map is well-defined even if γ is not contained in $\{p\} \cup \tau_1$.

Next we study the possible intersections for the boundaries of two basins.

Lemma 2.2. *Suppose $p_1, p_2 \in \operatorname{sing} \mathcal{F}$ are distinct centers such that $\partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})} \neq \emptyset$. Then we have the following mutually exclusive possibilities:*

- (i) *$\partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, and so $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$.*
- (ii) *$\partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} \neq \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, and there is a saddle point $q \in \partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})}$ with Morse index 1 or $m-1$, without self-connection.*

Proof. Since $\partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})} \neq \emptyset$, by Lemma 2.1 there is a saddle singular point $q \in \partial \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial \overline{\mathcal{C}_{p_2}(\mathcal{F})}$. If the Morse index of q is different from 1 and $m-1$,

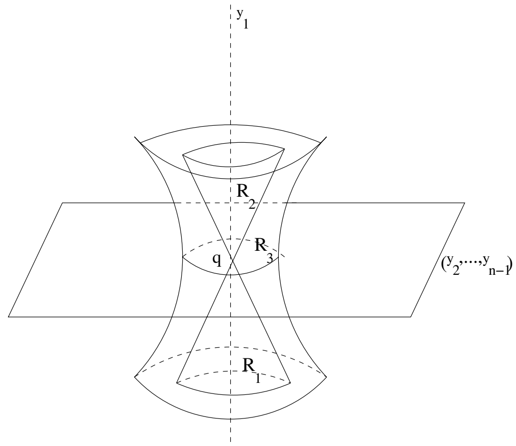

FIGURE 1

then in suitable local coordinates $(y_1, \ldots, y_m)$ we have $q = (0, \ldots, 0)$ and the local separatrix τ_q through q is given by $y_1^2 + \cdots + y_r^2 = y_{r+1}^2 + \cdots + y_m^2 \neq 0$ where $r \notin \{1, m-1\}$. In particular τ_q is connected.

Thus, if C is the separatrix of $\mathcal{F}$ at q, we have $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})} = \overline{C} = C \cup \{q\}$, and we are in case (i).

In case the index of q is 1 or $m-1$ and C is a self-connection at q, then $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})} = \overline{C}$ and we are again in case (i). The remaining case is when the index of q is 1 or $m-1$ and q has no self-saddle connection. Consider local coordinates $(y_1, \ldots, y_m)$ with $q = (0, \ldots, 0)$ and $\mathcal{F}$ given by the levels of the function $-y_1^2 + y_2^2 + \cdots + y_m^2$. The level zero of this function bounds the regions R_1: $y_1 < 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$, R_2: $y_1 > 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$ and $R_3 : y_1^2 < y_2^2 + \cdots + y_m^2$. Then $\mathcal{C}_{p_i}(\mathcal{F}) \cap R_3 = \emptyset$, $i = 1, 2$, because otherwise we would have a saddle self-connection at q. On the other hand, $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_1 \neq \emptyset$ implies $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_2 = \emptyset$ by the same reason. Therefore $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$. This proves (ii). □

Proposition 2.3. *Let $\mathcal{F}$ be a Morse foliation on a closed connected manifold M of dimension $m \geq 3$. Assume that $k = 2$ and $\ell = 1$; i.e., $\mathcal{F}$ has exactly two centers and one saddle singularity. Then M is homeomorphic to an Eells-Kuiper manifold.*

Proof. We shall first prove that the nonsingular foliation $\mathcal{F}_0 = \mathcal{F}\big|_{M_0}$ on $M_0 = M \setminus \operatorname{sing} \mathcal{F}$ is a proper stable foliation. There are several equivalent conditions that define a stable foliation ([5]). We shall prove that given any leaf L_0 of $\mathcal{F}_0$ there is a fundamental system of open neighborhoods of L_0 in M_0 saturated by $\mathcal{F}_0$. We claim that $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})} = \mathcal{C}_{p_1}(\mathcal{F}) \cup \mathcal{C}_{p_2}(\mathcal{F}) \cup C \cup \{q\}$ where $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})} = \overline{C}$. Indeed, according to Lemma 2.2 it is enough to show that case (ii) in this same lemma cannot occur. Suppose the contrary and take a saddle $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ with Morse index 1 or $m-1$ and without self connection. Denote by C the singular leaf (separatrix) that contains q and denote by C_1, C_2 the two components of $C \setminus \{q\}$. Let C^+ be the leaf on the positive side of C, so that C^+ is a sphere in $\mathcal{C}_{p_2}(\mathcal{F})$, and let C_1^-, C_2^- be leaves on the negative side, so that C_1^- is a sphere in $\mathcal{C}_{p_1}(\mathcal{F})$. Since C^+ is homeomorphic to the connect sum of C_1^- and C_2^- and since C_1^- is a sphere, then C_2^- is also a sphere. Therefore, by the Reeb stability theorem the leaf C_2^- descends to a fourth critical point, which contradicts our hypothesis. This shows that $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})} = \mathcal{C}_{p_1}(\mathcal{F}) \cup \mathcal{C}_{p_2}(\mathcal{F}) \cup C \cup \{q\}$ as in (i) of Lemma 2.2. If $r(q) = 1$ or $m-1$, then C is a self-connection. We proceed to show that this cannot occur. Suppose that $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_3 \neq \emptyset$; then $\mathcal{C}_{p_2}(\mathcal{F})$ will have non-empty intersection with R_1 and with R_2. Taking a small closed ball $\overline{B_q(t)}$ of radius $t > 0$ centered at q for leaves L_i of $\mathcal{F}$, $L_i \subset \mathcal{C}_{p_i}(\mathcal{F})$ for $i \in \{1, 2\}$, close enough to C, we have that the intersection $L_1 \cap (M \backslash \overline{B}_q(t))$ is a union of two disjoint $(m-1)$-discs. Moreover $L_2 \cap (M \backslash \overline{B}_q(t))$ is the complement of two disjoint $(m-1)$-discs in an $(m-1)$- sphere. Since both manifolds $L_1 \cap (M \backslash \overline{B}_q(t))$ and $L_2 \cap (M \backslash \overline{B}_q(t))$ are homeomorphic to $C \backslash (C \cap \overline{B}_q(t))$, we obtain a contradiction. Thus, $r \neq 1, m-1$ and $C \cap \overline{B}_q(t)$ is connected for t small. Given a leaf L_0 of $\mathcal{F}$ we have two possibilities: either $L_0 \subset \mathcal{C}_{p_i}(\mathcal{F})$ for some $i \in \{1, 2\}$ and L_0 is homeomorphic to S^{m-1} or $L_0 = C$. In the case L_0 is in $\mathcal{C}_{p_i}(\mathcal{F})$, by the Reeb stability theorem, L_0 has a fundamental system of saturated neighborhoods consisting of compact leaves. This shows that the leaves of $\mathcal{F}_0$ in $M_0 \backslash C$ are stable. It suffices to show that C is also a stable leaf.

We claim that the holonomy group of $C \cup \{q\}$ is a finite group conjugated to a subgroup of $\text{Diff}(\mathbb{R}, 0)$. Indeed, $C \backslash \overline{B_q(t)}$ is a disc and therefore simply-connected; on the other hand in $\overline{B_q(t)}$ the foliation has a first integral as $f = -(y_1^2 + \cdots + y_r^2) + y_{r+1}^2 + \cdots + y_m^2$ so that the holonomy group of $\mathcal{F} \cap \overline{B_q(t)}$ is finite. Since $C \cup \{q\}$ has a finite holonomy group which is a subgroup of $\text{Diff}(\mathbb{R}, 0)$, the holonomy group of $C \cup \{q\}$ is either trivial or, in the case $\mathcal{F}$ is not transversely orientable, it has order 2. By the classical argument on stability of leaves we conclude that finite holonomy implies that the leaf C is stable and $\mathcal{F}_0$ is stable in M_0.

Since $\mathcal{F}_0$ is stable in M_0 , the leaf space $M_0 \big/ \mathcal{F}_0 =: \mathfrak{X}_{\mathcal{F}_0}$ is Hausdorff, and therefore it is a 1-manifold. The choice of a differentiable submersion $\mathfrak{X}_{\mathcal{F}_0} \to \mathbb{R}$ then gives a differentiable first integral $F_0 \colon M_0 \to \mathbb{R}$ for $\mathcal{F}_0$. Clearly F_0 can be modified in order to admit a differentiable (radial) extension to the center singularities $p_1, p_2 \in \operatorname{sing} \mathcal{F}$. It remains to show that F_0 can be modified in order to admit a differentiable extension to q. This is a consequence of the fact that by the triviality of the holonomy group of $C \cup q$ we can extend the local first integral $f = -\sum\limits_{j=1}^{r} y_j^2 + \sum\limits_{k=r+1}^{m} y_k^2$ from a ball $B_q(t)$ to a neighborhood T of $C \cup \{q\}$ in M in such a way that ∂T is a union of leaves of $\mathcal{F}$, each leaf diffeomorphic to an $(m-1)$-sphere and contained in some basin $\mathcal{C}_{p_i}(\mathcal{F})$.

Thus we have proved that $\mathcal{F}$ is given by a Morse function $F \colon M \to \mathbb{R}$ and therefore, by Eells-Kuiper Theorem 1.2, that M is an Eells-Kuiper manifold. □

3. Trivial pairings

In this section we recall and extend some notions from [2]. The aim is to introduce and characterize an elimination procedure for suitable pairings of singularities of a given Morse foliation. We begin in $\mathbb{R}^m$ with a foliation with an isolated center and an isolated saddle of type $x_m^2 - \sum\limits_{j=1}^{m-1} x_j^2 = 0$ as in Figure 2. A first example of a *trivial pairing* is obtained by rotating this figure with respect to an axis that passes through the center and the saddle. The arrow indicates the passage (surgery) from the trivial pairing to the trivial foliation.

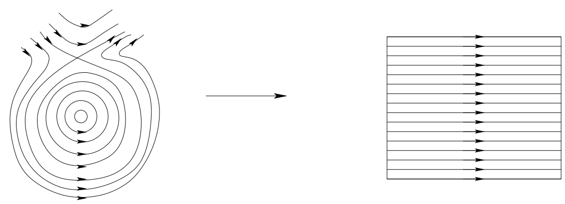

FIGURE 2

Other (center-saddle) pairings are depicted in Figure 3. The figure on the left shows a (m-dimensional) disc with a center-singularity which is replaced by a disc with three singularities: two of them are centers and one is a saddle which is in the boundary of the basin of both centers. We get $(m+1)$-dimensional versions of this example by rotating the disc with respect to a vertical axis as indicated. This construction gives two types of pairings, one trivial and one non-trivial. Indeed, choose spherical leaves L_1, L_2, L_3 as in the figure and consider only the annular region R bounded by L_1 and L_2 (i.e., deleting the open ball centered at the upper center singularity and bounded by L_1) as in the figure above. Then we have a pair of singularities $p_0 - q$ which are in a *non-trivial center saddle* pairing. On the other hand, if we consider the region bounded by L_3 and L_2, then we have a trivial pairing as in Figure 2. The left picture in Figure 3 shows two trivial pairings in a foliation that can be completed to an example on the sphere by adding a center at infinity.

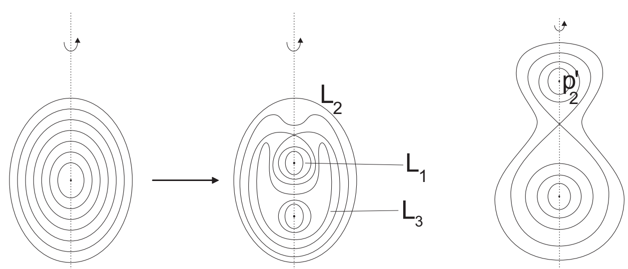

FIGURE 3

The above constructions can be performed in dimension $m \geq 3$ and can be completed to foliations on S^m. The resulting foliation satisfies $3 = k = \ell + 2$.

Definition 3.1 (Trivial pairing)**.** For any $r > 0$ we will write $B(r) = D(r) \times I(r) \subset \mathbb{R}^{m-1} \times \mathbb{R}$, where $D(r)$ and $I(r)$ are closed discs of radius r centered at zero. The foliation on $B(r)$ given by the submersion $(x,t) \mapsto t$ will be denoted by $\mathcal{H}$. Let $p \in M$ be a center and let $q \in \partial\overline{\mathcal{C}_p(\mathcal{F})}$ be a saddle point of $\mathcal{F}$. We will say that p, q *form a trivial pairing* if there are open neighborhoods $V \supset V' \supset \overline{\mathcal{C}_p(\mathcal{F})}, p, q \in V'$, and a diffeomorphism $\varphi\colon \overline{V} \to B(1)$, onto $B(1)$, such that $\varphi(\overline{V}') = B(1/2)$ and $\mathcal{F}\big|_{\overline{V}\setminus V'} = \varphi^*\mathcal{H}$.

The general concept of trivial pairing is illustrated by Figures 4 and 5, where Σ_1, Σ_2 represent transverse sections to the foliation and P_1, P_2 are plaques (discs) of leaves meeting Σ_1 and Σ_2.

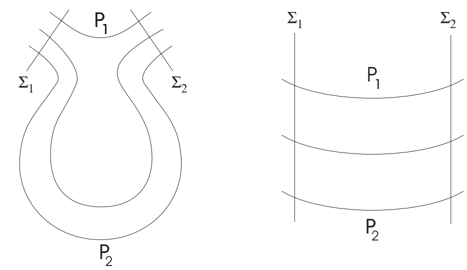

FIGURE 4

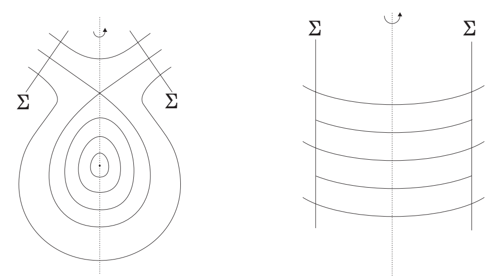

FIGURE 5

Lemma 3.2. *Suppose $p_1, p_2 \in M$ are two different centers such that $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})} \neq \emptyset$, and let q be the saddle point contained in this intersection. Assume that the index of q is one and that there is no saddle self-connection at q. Then, either p_1, q or p_2, q form a trivial pairing.*

Proof. In a neighborhood $q \in U$ there are local coordinates $(y_1, ..., y_m)$ such that $q = (0, ..., 0)$ and the leaves of $\mathcal{F}\big|_U$ are given by the levels of the function $f(y_1, ..., y_m) = -y_1^2 + (y_2^2 + \cdots + y_m^2)$ on $\mathbb{R}^m$. As before the cone $-y_1^2 + (y_2^2 + \cdots + y_m^2) = 0$ divides U into three regions (see Figure 1). The regions R_1 and R_2 are defined by R_1: $y_1 < 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$, R_2: $y_1 > 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$ and the region R_3 by $y_1^2 < y_2^2 + \cdots + y_m^2$. The cone leaves τ_1 and τ_2 of $\mathcal{F}\big|_U$ bound R_1 and R_2 respectively, and since there is no self-connection at q there are different leaves of $\mathcal{F}$, ℓ_1 and ℓ_2 such that $\ell_1 \supset \tau_1$ and $\ell_2 \supset \tau_2$. Since we have two center basins $\mathcal{C}_{p_1}(\mathcal{F})$ and $\mathcal{C}_{p_2}(\mathcal{F})$ and three regions R_1, R_2, R_3, then some $\mathcal{C}_{p_i}(\mathcal{F})$ will intersect R_1 or R_2. Suppose that $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_1 \neq \emptyset$. Then $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_2 = \emptyset$ and $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_3 = \emptyset$, because both R_2 and R_3 have ℓ_2 in their boundary; and if either $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_2 \neq \emptyset$ or $\mathcal{C}_{p_2}(\mathcal{F}) \cap R_3 \neq \emptyset$, this would imply a saddle self-connection at q. Thus $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \overline{\ell}_1$. We claim that $\overline{\ell}_1 = \ell_1 \cup \{q\}$. Indeed, if, on the contrary, there is a regular point of $\mathcal{F}$, $s \in \overline{\ell}_1 \setminus \ell_1$, and S_s denotes an arbitrarily small cross section to $\mathcal{F}$ centered at s, then the number of points of intersection of ℓ_1 with S_s, $n(\ell_1, S_s)$, is infinite. On the other hand, by Reeb's theorem, given any local transverse section S to $\mathcal{F}$, the number of points of intersection $n(\ell, S)$, of a leaf $\ell \subset \mathcal{C}_{p_1}(\mathcal{F})$ with S, is locally constant. Since ℓ_1 is approached by leaves in $\mathcal{C}_{p_1}(\mathcal{F})$ we would obtain leaves $\ell(k) \subset \mathcal{C}_{p_1}(\mathcal{F})$ with $n(\ell(k), S_s) \to \infty$ as $k \to \infty$, which is a contradiction. Therefore $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \ell_1 \cup \{q\}$.

We take U small enough so that $\overline{\ell}_1 \cap U = \overline{\tau}_1$. Thus, for any leaf $L \subset \mathcal{C}_{p_1}(\mathcal{F})$ close enough to $\overline{\ell}_1$, $L \cap U$ is connected and $f\big|_{L\cap U} = \delta < 0$, a constant. Write $L = L_\delta$. Then, as $\delta \to 0$, $L_\delta \setminus U$ approaches $\overline{\ell}_1 \setminus \tau_1$, and this implies that $\overline{\ell}_1 \setminus \tau_1$ is homeomorphic to $L_\delta \setminus U$, i.e., to an $(m-1)-$disc. Therefore $\overline{\ell}_1$ is homeomorphic to S^{m-1}. Moreover for $\epsilon > 0$ small enough each leaf $f^{-1}(\delta)$ of $\mathcal{F}\big|_{R_1\cup R_3}$, $-2\epsilon \le \delta \le 2\epsilon$, bounds an $(m-1)-$disc D_δ close to $\ell_1 \setminus \tau_1$, with $D_0 = \ell_1 \setminus \tau_1$. The union $T_{2\epsilon} = \bigcup_{-2\epsilon\le\delta\le 2\epsilon} \overline{D}_\delta$ is a trivially foliated neighborhood of $\ell_1 \setminus \tau_1$. We can extend the function f to $U \cup T_{2\epsilon}$ by writing $f\big|_{\overline{D}_\delta} = \delta$. We define a saturated neighborhood V_0 of $\overline{\ell}_1 \cup \tau_2$ by $V_0 = f^{-1}([-\epsilon,\epsilon]) \cup \mathcal{C}_{p_1}(\mathcal{F})$ and define $g = f\big|_{V_0\setminus \mathcal{C}_{p_1}(\mathcal{F})}$. Now consider a Riemannian metric defined on M and a normal vector field to $\mathcal{F}$ that in U takes the form $\mathcal{N} = -y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + \cdots + y_m\frac{\partial}{\partial y_m}$. For $a > 0$ small enough the submanifold $e = (y_1 = a)\cap\tau_2$ is well defined and diffeomorphic to S^{m-2}. Let Σ be a cross-section to $\mathcal{F}$ over e, i.e. $\Sigma \cap \tau_2 = e$, with Σ contained in V_0 and invariant by $\mathcal{N}$. We take Σ diffeomorphic to $e \times [-\epsilon,\epsilon]$ by means of a map that takes each $e \times \delta$, $-\epsilon \le \delta \le \epsilon$, to the leaf $(f = \delta) \cap \Sigma$ of $\mathcal{F}\big|_\Sigma$.

Consider the region $V \subset V_0$ bounded by $(g = -\epsilon) \cup (g = \epsilon)$ and Σ and define a neighborhood $\partial V \subset W \subset V$ as $W = (-\epsilon \le g \le -\epsilon/2) \cup (\epsilon/2 \le g \le \epsilon) \cup N$, where N is a neighborhood of $\Sigma \subset V$ invariant by $\mathcal{N}$. The foliation $\mathcal{F}\big|_W$ is trivial in the sense that on $(-\epsilon \le g \le -\epsilon/2) \cup (\epsilon/2 \le g \le \epsilon)$ the leaves of $\mathcal{F}$ are levels of g diffeomorphic to $D(1)$, and on N the leaves of $\mathcal{F}$ are levels of g $(g = \delta)$, $-\epsilon/2 \le \delta \le \epsilon/2$ diffeomorphic to $D(1) \setminus D(1/2)$. Moreover in W the foliation $\mathcal{F}$ and the trajectories of $\mathcal{N}$ are everywhere transverse. Thus a diffeomorphism $\varphi\colon W \to B(1) \setminus B(1/2)$ can easily be constructed by sending leaves of $\mathcal{F}\big|_W$ at the level $(g = \alpha\epsilon)$ onto leaves of $\mathcal{H}\big|_{B(1)\setminus B(1/2)}$ at the level $(t = \alpha)$ and orbits of $\mathcal{N}\big|_W$ to orbits of $\frac{\partial}{\partial t}$. Then φ is extended to V. □

4. Proof of the theorem

Now we prove Theorem 1.4. We will proceed by induction on the number ℓ of saddle singularities. If $\ell = 0$, then Reeb's Theorem applies and M is homeomorphic to S^m. Assume now that $\ell \ge 1$ and the result has been proven for foliations with at most $\ell - 1$ singularities of saddle type. We have $k \ge \ell + 1$. Thus $k \ge 2$. Suppose that M is not homeomorphic to S^m. Then, by Lemmas 2.1 and 2.2, for each center $p \in \operatorname{sing}\mathcal{F}$ there must be a saddle $q(p) \in \partial\overline{\mathcal{C}_p(\mathcal{F})}$. Since $k \ge \ell + 1$ and $k \ge 2$ there are two centers p_1, p_2 such that $q(p_1) = q(p_2)$—i.e., there is a saddle q such that $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$—and by Lemma 2.2 we have either $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$ or q has index 1 or $m - 1$ and is not self-connected.

In the case $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, clearly $\mathcal{C}_{p_i}(\mathcal{F}) \cap \operatorname{sing}\mathcal{F} = \{p_i\}$, $i = 1, 2$. Thus $\operatorname{sing}\mathcal{F} = \{p_1, p_2, q\}$, and by Proposition 2.3 M is an Eells-Kuiper manifold.

In the case q has index 1 or $m - 1$ and no self-connection, then by Lemma 3.2 we can eliminate one saddle and one center replacing $\mathcal{F}$ by a Morse foliation $\mathcal{F}_1$ on M with a number k_1 of centers and ℓ_1 of saddles given by $k_1 = k - 1$ and $\ell_1 = \ell - 1$. Therefore $k_1 \ge \ell_1 + 1$ and $\ell > \ell_1 \ge 0$. By the induction hypothesis M is homeomorphic to S^m or to an Eells-Kuiper manifold. This proves the theorem. □

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