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UPPER BOUND FOR ISOMETRIC EMBEDDINGS $\ell_2^m \to \ell_p^n$

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ABSTRACT. The isometric embeddings $\ell_{2;\mathbb{K}}^m \to \ell_{p;\mathbb{K}}^n$ ($m \geq 2$, $p \in 2\mathbb{N}$) over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are considered, and an upper bound for the minimal n is proved. In the commutative case ($\mathbb{K} \neq \mathbb{H}$) the bound was obtained by Delbaen, Jarchow and Pełczyński (1998) in a different way.

Let $\mathbb{K}$ be one of three fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (real, complex or quaternion). Let $\mathbb{K}^n$ be the $\mathbb{K}$-linear space consisting of columns $x = [\xi_i]_1^n$, $\xi_i \in \mathbb{K}$, with the right (for definiteness) multiplication by scalars $\alpha \in \mathbb{K}$. The normed space $\ell_{p;\mathbb{K}}^n$ is $\mathbb{K}^n$ provided with the norm

$$\|x\|_p = \left(\sum_{k=1}^n |\xi_i|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

For $p = 2$ this space is Euclidean, $\|x\|_2 = \sqrt{\langle x, x \rangle}$, where the inner product $\langle x, y \rangle$ of x and a vector $y = [\eta_i]_1^n$ is

$$\langle x, y \rangle = \sum_{i=1}^n \overline{\xi_i} \eta_i.$$

An isometric embedding $\ell_{2;\mathbb{K}}^m \to \ell_{p;\mathbb{K}}^n$, $2 \leq m \leq n$, may exist only if $p \in 2\mathbb{N} = 2, 4, 6, \ldots$; see [3] for $\mathbb{K} = \mathbb{R}$ and [4] for any $\mathbb{K}$. Conversely, under these conditions for m and p, there exists an n such that $\ell_{2;\mathbb{K}}^m$ can be isometrically embedded into $\ell_{p;\mathbb{K}}^n$; see [6] (and also [5, 7]) for $\mathbb{K} = \mathbb{R}$, [2] for $\mathbb{K} = \mathbb{C}$, and [4] for $\mathbb{K} = \mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ simultaneously. The proofs of existence in these papers also yield some upper bounds for the minimal $n = N_{\mathbb{K}}(m, p)$. According to [4], these bounds can be joined in the inequality

$$N_{\mathbb{K}}(m, p) \leq \dim \Phi_{\mathbb{K}}(m, p), \tag{1}$$

where $\Phi_{\mathbb{K}}(m, p)$ is the space of homogeneous polynomials (forms) $\phi(x)$ over $\mathbb{R}$ of degree p in real coordinates on $\mathbb{K}^m$ such that $\phi(x\alpha) = \phi(x)$ for all $\alpha \in \mathbb{K}$, $|\alpha| = 1$. For $\mathbb{K} = \mathbb{R}$ the latter condition is fulfilled automatically since $p \in 2\mathbb{N}$, so $\Phi_{\mathbb{R}}(m, p)$ consists of all forms of degree p on $\mathbb{R}^m$. The space $\Phi_{\mathbb{C}}(m, p)$ coincides with that which was used in [2]. Note that in all cases $\dim \Phi_{\mathbb{K}}(m, p)$ can be explicitly expressed through binomial coefficients. (All the formulas are brought together in [4, Theorem 2].)

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In the present paper we prove that

$$N_{\mathbb{K}}(m,p) \le \dim \Phi_{\mathbb{K}}(m,p) - 1. \tag{2}$$

For $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ this result (in terms of binomial coefficients) was obtained by Delbaen, Jarchow and Pełczyński [1] as a by-product of the proof of their Theorem B. Their rather complicated technique essentially uses the commutativity of the field $\mathbb{K}$, so it is not applicable to $\mathbb{K} = \mathbb{H}$. Our proof of (2) is general and elementary. Let us start with two lemmas, the first of which is well known.

Lemma 1. *A linear mapping $f : \ell^m_{2;\mathbb{K}} \to \ell^n_{p;\mathbb{K}}$ is isometric if and only if there is a system of vectors $u_k \in \ell^m_{2;\mathbb{K}}$, $1 \le k \le n$, such that the identity*

$$\sum_{k=1}^{n} |\langle u_k, x\rangle|^p = \langle x, x\rangle^{p/2} \tag{3}$$

holds for $x \in \ell^m_{2;\mathbb{K}}$.

Proof. A general form of f as a linear mapping is $fx = [\langle u_k, x\rangle]_1^n$, where $(u_k)_1^n$ is a system of vectors from $\ell^m_{2;\mathbb{K}}$ (called the *frame* of f [4, 5]). The identity (3) is nothing but $\|fx\|_p = \|x\|_2$. □

An isometric embedding $\ell^m_{2;\mathbb{K}} \to \ell^n_{p;\mathbb{K}}$ is called *minimal* if $n = N_{\mathbb{K}}(m,p)$.

Lemma 2. *If f is minimal and $(u_k)_1^n$ is its frame, then the functions $|\langle u_k, x\rangle|^p$ are linearly independent.*

Proof. Let

$$\sum_{k=1}^{n} \omega_k |\langle u_k, x\rangle|^p = 0 \tag{4}$$

with some real ω_k, $\max\limits_k \omega_k = 1$, and let $\omega_n = 1$ for definiteness. By subtraction of (4) from (3) we get

$$\sum_{k=1}^{n-1} (1-\omega_k)\, |\langle u_k, x\rangle|^p = \langle x, x\rangle^{p/2},$$

i.e.

$$\sum_{k=1}^{n-1} |\langle v_k, x\rangle|^p = \langle x, x\rangle^{p/2},$$

where $v_k = u_k\,(1-\omega_k)^{1/p}$. This contradicts the minimality of f. □

Remark 3. Since all functions $|\langle \cdot, x\rangle|^p$ belong to $\Phi_{\mathbb{K}}(m,p)$, the inequality (1) immediately follows from Lemma 2. However, the existence of an isometric embedding $\ell^m_{2;\mathbb{K}} \to \ell^n_{p;\mathbb{K}}$ is assumed in this context.

Now we proceed to the proof of (2).

Proof. Let $f : \ell^m_{2;\mathbb{K}} \to \ell^n_{p;\mathbb{K}}$ be a minimal isometric embedding. Then, according to (1), $n \le \dim \Phi_{\mathbb{K}}(m,p)$. We have to prove that the equality is impossible.

Suppose the contrary. Then the system $(|\langle u_k, x\rangle|^p)_1^n$ corresponding to the frame of f is a basis of $\Phi_{\mathbb{K}}(m,p)$ by Lemma 2. In particular, there is an expansion

$$\left(\sum_{i=1}^{m} \lambda_i |\xi_i|^2\right)^{p/2} = \sum_{k=1}^{n} a_k(\lambda_1,\ldots,\lambda_m) |\langle u_k, x\rangle|^p, \tag{5}$$

where $(\lambda_i)_1^m \in \mathbb{R}^m$ and a_k are some functions of these parameters.

Now we introduce the inner product

$$(\phi_1, \phi_2) = \int_S \phi_1(x)\phi_2(x)\, d\sigma(x) \quad (\phi_1, \phi_2 \in \Phi_{\mathbb{K}}(m,p)),$$

where σ is the standard measure on the unit sphere $S \subset \ell_{2;\mathbb{K}}^m$. In the Euclidean space $\Phi_{\mathbb{K}}(m,p)$ we have the basis $(\theta_k(x))_1^n$ dual to $(|\langle u_k, x\rangle|^p)_1^n$. This allows us to represent the coefficients a_k as

$$a_k(\lambda_1, \dots, \lambda_m) = \int_S \left(\sum_{i=1}^m \lambda_i |\xi_i|^2\right)^{p/2} \theta_k(x)\, d\sigma(x).$$

Hence, $a_k(\lambda_1, \dots, \lambda_m)$ are forms of degree $p/2$; a fortiori, they are continuous.

Denote by $\mathbb{R}_+^m$ the open coordinate cone in $\mathbb{R}^m$, so $\mathbb{R}_+^m = \{(\lambda_i)_1^m \subset \mathbb{R}^m : \lambda_1 > 0, ..., \lambda_m > 0\}$. We prove that *on* $\mathbb{R}_+^m$ *all* $a_k(\lambda_1, \dots, \lambda_m) > 0$ or equivalently, $\hat{a}(\lambda_1, \dots, \lambda_m) \equiv \min_k a_k(\lambda_1, \dots, \lambda_m) > 0$. Suppose the contrary: let $\hat{a}(\gamma_1, \dots, \gamma_m) \le 0$ for some $(\gamma_i)_1^m \subset \mathbb{R}_+^m$. On the other hand, $\hat{a}(1, \dots, 1) = 1$ by comparing (3) to (5) with all $\lambda_i = 1$. Since $\hat{a}$ is continuous, we have $\hat{a}(\mu_1, \dots, \mu_m) = 0$ for some $(\mu_i)_1^m \subset \mathbb{R}_+^m$. But the latter means that all $a_k(\mu_1, \dots, \mu_m) \ge 0$ and at least one of them is zero, say $a_n(\mu_1, \dots, \mu_m) = 0$. Therefore,

$$\left(\sum_{i=1}^m \mu_i |\xi_i|^2\right)^{p/2} = \sum_{k=1}^{n-1} a_k(\mu_1, \dots, \mu_m) |\langle u_k, x\rangle|^p,$$

whence

$$\langle z, z\rangle^{p/2} = \sum_{k=1}^{n-1} |\langle v_k, z\rangle|^p, \tag{6}$$

where

$$z = \mathcal{D}x, \quad v_k = (a_k(\mu_1, \dots, \mu_m))^{1/p}\, \mathcal{D}^{-1} u_k$$

and $\mathcal{D}$ is the diagonal matrix with entries $\mu_1^{1/2}, \dots, \mu_m^{1/2}$. By Lemma 1 the identity (6) means that the system $(v_k)_1^{n-1}$ is the frame of an isometric embedding $\ell_{2;\mathbb{K}}^m \to \ell_{p;\mathbb{K}}^{n-1}$. This contradicts the minimality of n. As a result, all $a_k(\lambda_1, \dots, \lambda_m) \ge 0$ for $\lambda_1 \ge 0, \dots, \lambda_m \ge 0$, i.e. on the closed coordinate cone.

Now we take $\xi_1 = 1$ and $\xi_i = 0$ for all $i \ge 2$, so $x = e_1$, the first vector from the canonical basis of $\ell_{2;\mathbb{K}}^m$. In this setting (5) reduces to

$$\lambda_1^{p/2} = \sum_{k=1}^n a_k(\lambda_1, \dots, \lambda_m) |\langle u_k, e_1\rangle|^p.$$

(Recall that $m \ge 2$.) This yields

$$\sum_{k=1}^n a_k(0, \lambda_2, \dots, \lambda_m) |\langle u_k, e_1\rangle|^p = 0.$$

Assume all $\langle u_k, e_1\rangle \ne 0$. Since for $\lambda_2 > 0, \dots, \lambda_m > 0$ all $a_k(0, \lambda_2, \dots, \lambda_m) \ge 0$, all of them are equal to zero. Hence, the right side of the identity (5) vanishes as long as $\lambda_1 = 0$, in contrast to the function on the left side, a contradiction. To finish the proof we only have to show that the assumption $\langle u_k, e_1\rangle \ne 0$, $1 \le k \le n$, is not essential.

First, note that all $u_k \neq 0$; otherwise, the number n in (3) would be reduced. Therefore, the sets $\{x : \langle u_k, x\rangle = 0\}, 1 \leq k \leq n$, are hyperplanes in $\ell^m_{2;\mathbb{K}}$. Their union is different from the whole space. Hence, there is a vector e such that all $\langle u_k, e\rangle \neq 0$, $\|e\|_2 = 1$. This e can be represented as $e = ge_1$ where g is an isometry of the space $\ell^m_{2;\mathbb{K}}$. Indeed, this space is Euclidean, so its isometry group is transitive on the unit sphere. Thus, all $\langle g^{-1}u_k, e_1\rangle \neq 0$. On the other hand, $(g^{-1}u_k)^n_1$ is the frame of the isometric embedding $fg : \ell^m_{2;\mathbb{K}} \to \ell^n_{p;\mathbb{K}}$. □

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