A SUFFICIENT AND NECESSARY CONDITION FOR THE CONVERGENCE OF THE SEQUENCE OF SUCCESSIVE APPROXIMATIONS TO A UNIQUE FIXED POINT

TOMONARI SUZUKI

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ABSTRACT. If (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, then the conclusion of the Banach-Caccioppoli contraction principle is that the sequence of successive approximations of T starting from any point of the space converges to a unique fixed point. In this paper, we obtain a sufficient and necessary condition of the above conclusion in terms of the so-called strong Leader mappings.

1. INTRODUCTION

The following famous theorem is referred to as the *Banach-Caccioppoli contraction principle*. This theorem is very forceful and simple, and it became a classical tool in nonlinear analysis.

Theorem 1 (Banach [1] and Caccioppoli [2])**.** *Let (X, d) be a complete metric space and let T be a contraction mapping on X; i.e., there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq r \, d(x, y)$ for all $x, y \in X$. Then the following holds*:

(A) *T has a unique fixed point, and for any $x \in X$ the sequence of successive approximations $\{T^n x\}$ converges to the fixed point.*

There are thousands of fixed point theorems in metric spaces. However, the conclusion of most of theorems is (A). For example, the conclusion of Ćirić's [5], Edelstein's [6], Kannan's [8], Kirk's [10], Matkowski's [12], Meir-Keeler's [13] and Suzuki's [15, 16] fixed point theorems is (A). There are a few exceptions. For example, the conclusions of Caristi's [3, 4] and Subrahmanyam's [14] fixed point theorems and their generalizations are not (A). See Kirk's survey [9].

In 1983, Leader obtained a sufficient and necessary condition of some condition which is closely connected with (A). See also [7, 18].

Definition 1 (Leader [11])**.** Let T be a mapping on a metric space (X, d). Then T is called a *Leader mapping* if the following holds:

(B) For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that

$$d(T^i x, T^j y) < \varepsilon + \delta \quad \text{implies} \quad d(T^{i+\nu} x, T^{j+\nu} y) < \varepsilon$$

for all $i, j \in \mathbb{N} \cup \{0\}$, where T^0 is the identity mapping on X.

Theorem 2 (Leader [11]). *Let T be a mapping on a metric space (X, d). Then T is a Leader mapping if and only if the following holds*:

(C) $\{T^n x\}$ *is a Cauchy sequence and* $\lim_n d(T^n x, T^n y) = 0$ *for all* $x, y \in X$.

Conditions (A) and (C) are very close. However, Theorem 2 does not assure the existence of fixed points. So from this point of view, we might be able to consider that both conditions are quite different. The author thinks that Theorem 2 is excellent; however, it is not perfect. In this paper, we proceed with Leader's study and complete the study. That is, we obtain a sufficient and necessary condition of (A).

2. Main results

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers.

Since Condition (C) is weaker than (A), we have to introduce a condition which is stronger than (B).

Definition 2. Let T be a mapping on a metric space (X, d). Then T is called a *strong Leader mapping* if (B) and the following hold:

(D) For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j y) < \alpha_n \quad \text{implies} \quad d(T^{i+\nu} x, T^{j+\nu} y) < 1/n$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

We shall prove that Condition (A) and the conjunction of Conditions (B) and (D) are equivalent.

Theorem 3. *Let (X, d) be a complete metric space and let T be a strong Leader mapping on X. Then* (A) *holds.*

Proof. By Theorem 2, $\{T^n x\}$ is a Cauchy sequence and $\lim_n d(T^n x, T^n y) = 0$ for all $x, y \in X$. Since X is complete, there exists $z \in X$ such that $\lim_n T^n x = z$ for all $x \in X$. From this, there exists at most one fixed point of T. Fix $x \in X$. From (D), there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j z) < \alpha_n \quad \text{implies} \quad d(T^{i+\nu} x, T^{j+\nu} z) < 1/n$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. For $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that $\ell \geq k$ and $d(T^\ell x, z) < \alpha_k$ because $\lim_n T^n x = z$. So we have

$$d(T^{\ell+\nu} x, T^\nu z) < 1/k.$$

As k tends to ∞, we obtain $d(z, T^\nu z) = 0$; that is, $T^\nu z = z$. Since $\lim_n T^n z = z$, z is a fixed point of T. $\qquad\square$

Theorem 4. *Let (X, d) be a metric space and let T be a mapping on X. Assume that* (A) *holds. Then T is a strong Leader mapping.*

Proof. Let $z \in X$ be a unique fixed point of T. From the assumption, $\{T^n x\}$ is a Cauchy sequence and $\lim_n d(T^n x, T^n y) = d(z, z) = 0$ for all $x, y \in X$. So by Theorem 2, T satisfies (B). Let us prove (D). Fix $x, y \in X$ and $n \in \mathbb{N}$. We choose $L \in \mathbb{N}$ such that $d(T^m x, z) < 1/(2n)$ and $d(T^m y, z) < 1/(2n)$ for all $m \in \mathbb{N}$ with $m > L$. Put

$$\alpha_n := \min \Big\{ \min \big\{ \inf\{d(T^\ell x, T^m y) : m \in \mathbb{N} \cup \{0\}, T^m y \neq T^\ell x\}$$
$$: \ell \in \{0, 1, \cdots, L\}, T^\ell x \neq z \big\},$$
$$\min \big\{ \inf\{d(T^\ell y, T^m x) : m \in \mathbb{N} \cup \{0\}, T^m x \neq T^\ell y\}$$
$$: \ell \in \{0, 1, \cdots, L\}, T^\ell y \neq z \big\},$$
$$1 \Big\},$$

where $\min \varnothing = \infty$ and $\inf \varnothing = \infty$.

We shall show $\alpha_n > 0$. If not, then without loss of generality, we may assume there exists $\lambda \in \{0, 1, \cdots, L\}$ such that

$$T^\lambda x \neq z \quad \text{and} \quad \inf \big\{d(T^\lambda x, T^m y) : m \in \mathbb{N} \cup \{0\}, T^m y \neq T^\lambda x\big\} = 0.$$

Then we can choose a sequence $\{m_j\}$ in $\mathbb{N} \cup \{0\}$ such that

$$T^{m_j} y \neq T^\lambda x \quad \text{and} \quad \lim_{j \to \infty} d(T^\lambda x, T^{m_j} y) = 0.$$

Since $T^{m_j} y \neq T^\lambda x$ for all $j \in \mathbb{N}$, we have $\lim_j m_j = \infty$ and hence

$$d(T^\lambda x, z) = \lim_{j \to \infty} d(T^\lambda x, T^{m_j} y) = 0.$$

This is a contradiction. Therefore we obtain $\alpha_n > 0$.

Fix $i, j \in \mathbb{N} \cup \{0\}$ with $d(T^i x, T^j y) < \alpha_n$. We consider the following cases:

(i) $T^i x = T^j y$,
(ii) $i \leq L$ and $T^i x \neq z$,
(iii) $j \leq L$ and $T^j y \neq z$,
(iv) $T^i x = z$ and $T^j y = z$,
(v) $T^i x = z$ and $j > L$,
(vi) $T^j y = z$ and $i > L$,
(vii) $i > L$ and $j > L$.

	$i \leq L, T^i x \neq z$	$i \leq L, T^i x = z$	$i > L$
$j \leq L, T^j y \neq z$	(ii), (iii)	(iii)	(iii)
$j \leq L, T^j y = z$	(ii)	(iv)	(vi)
$j > L$	(ii)	(v)	(vii)

In the first case, it is obvious that $d(T^{i+1} x, T^{j+1} y) = 0 < 1/n$. In the second case, we have

$$d(T^i x, T^j y) < \alpha_n \leq \inf \big\{ d(T^i x, T^m y) : m \in \mathbb{N} \cup \{0\}, T^m y \neq T^i x \big\}$$

and hence $T^i x = T^j y$. Thus the first case includes the second case. Similarly the first case includes the third case. Also, it is obvious that the first case includes the fourth case. In the fifth case, we have

$$d(T^{i+1} x, T^{j+1} y) = d(Tz, T^{j+1} y) = d(z, T^{j+1} y) < 1/(2n) < 1/n.$$

In the sixth case, we can prove $d(T^{i+1}x, T^{j+1}y) < 1/n$ as in the fifth case. In the seventh case, we obtain

$$d(T^{i+1}x, T^{j+1}y) \leq d(T^{i+1}x, z) + d(T^{j+1}y, z) < 1/(2\,n) + 1/(2\,n) = 1/n.$$

Thus we obtain (D).   □

From the proof Theorem 4, we obtain the following.

Theorem 5. *Let T be a mapping on a complete metric space (X, d). Then the following are equivalent*:

 (i) (A) *holds.*
 (ii) T *is a strong Leader mapping.*
 (iii) (B) *and the following hold*:
   (E) *For $x, y \in X$, there exists a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that*

$$d(T^i x, T^j y) < \alpha_n \quad implies \quad d(T^{i+1}x, T^{j+1}y) < 1/n$$

   for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

3. Additional result

We finally give an alternative proof of Lemma 2 in [11] (Lemma 1 below). The lemma played a very important role in the proof of Theorem 2. Since Leader proved Lemma 1 by reductio ad absurdum, we will give a proof without reductio ad absurdum. See also the proof of Theorem 1 in [17].

Lemma 1 (Leader [11]). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d). Let $\varepsilon \in (0, \infty)$, $\delta \in (0, \infty]$ and $\nu \in \mathbb{N}$ satisfy*

$$d(x_i, y_j) < \varepsilon + \delta \quad implies \quad d(x_{i+\nu}, y_{j+\nu}) < \varepsilon$$

for all $i, j \in \mathbb{N}$. Let $n \in \mathbb{N}$ satisfy

$$\max\big\{d(x_i, y_j) : i, j \in \{n, n+1, \cdots, n+\nu\}\big\} < \min\{\varepsilon, \delta/2\}.$$

Then $d(x_i, y_j) < 3\,\varepsilon$ for all $i, j \in \mathbb{N}$ with $i \geq n$ and $j \geq n$.

Proof. We first show by induction

$$(1) \qquad\qquad\qquad\qquad d(x_{n+\nu}, y_{n+m}) < \varepsilon$$

for all $m \in \mathbb{N} \cup \{0\}$. From the assumption, (1) holds for $m = 0, 1, \cdots, \nu$. We assume (1) holds for some $m \in \mathbb{N}$. That is, $d(x_{n+\nu}, y_{n+m}) < \varepsilon$, which implies

$$d(x_n, y_{n+m}) \leq d(x_n, y_n) + d(y_n, x_{n+\nu}) + d(x_{n+\nu}, y_{n+m}) < \delta/2 + \delta/2 + \varepsilon = \varepsilon + \delta.$$

From the assumption, we have $d(x_{n+\nu}, y_{n+m+\nu}) < \varepsilon$. Thus, (1) holds when $m := m + \nu$. By induction, (1) holds for all $m \in \mathbb{N} \cup \{0\}$. Similarly we can prove $d(y_{n+\nu}, x_{n+m}) < \varepsilon$ for all $m \in \mathbb{N} \cup \{0\}$. Therefore we obtain

$$d(x_i, y_j) \leq d(x_i, y_{n+\nu}) + d(y_{n+\nu}, x_{n+\nu}) + d(x_{n+\nu}, y_j) < \varepsilon + \varepsilon + \varepsilon = 3\,\varepsilon$$

for $i, j \in \mathbb{N}$ with $i \geq n$ and $j \geq n$.   □

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Department of Mathematics, Kyushu Institute of Technology, Sensuicho, Tobata, Kitakyushu 804-8550, Japan

E-mail address: suzuki-t@mns.kyutech.ac.jp